ЖЭТФ, 2013, том 143, вып. 1, стр. 198–204

DYNAMICS OF AN N-VORTEX STATE AT SMALL DISTANCES

Yu. N. Ovchinnikov [*]

Max-Planck Institute for Physics of Complex Systems
01187, Dresden, Germany

Landau Institute for Theoretical Physics, Russian Academy of Sciences
142432, Chernogolovka, Moscow Region, Russia

Received July 18, 2012

We investigate the dynamics of a state of N vortices, placed at the initial instant at small distances from some point, close to the "weight center" of vortices. The general solution of the time-dependent Ginsburg–Landau equation for N vortices in a large time interval is found. For $N = 2$, the position of the "weight center" of two vortices is time independent. For $N \geq 3$, the position of the "weight center" weakly depends on time and is located in the range of the order of a^3, where a is a characteristic distance of a single vortex from the "weight center". For $N = 3$, the time evolution of the N-vortex state is fixed by the position of vortices at any time instant and by the values of two small parameters. For $N \geq 4$, a new parameter arises in the problem, connected with relative increases in the number of decay modes.

DOI: 10.7868/S0044451013010198

1. INTRODUCTION

The nonlinear Schrödinger equation

$$i\frac{\partial \Psi}{\partial t} = -\left\{\frac{\partial^2 \Psi}{\partial \mathbf{r}^2} + (1 - |\Psi|^2\Psi\right\} \quad (1)$$

has a solution of particle-like excitations, vortices. At large distances between vortices, the energy E can be represented in the leading approximation in the form (see, e. g., Ref. [1])

$$E = \sum_i E_i + \sum_{i \neq j} E_{ij}, \quad E_{ij} = \pi n_i n_j \ln \frac{R}{|\mathbf{r}_i - \mathbf{r}_j|}, \quad (2)$$

where E_i is the self energy of a separate vortex, E_{ij} is the pair interaction energy, and R is the cut-off distance. As a result, the equation of motion of such a system of vortices can be taken in the form

$$n_j \frac{\partial \mathbf{r}_j}{\partial t} = -\frac{1}{\pi} J \frac{\partial E}{\partial \mathbf{r}_j}, \quad (3)$$

where J is the simplectic matrix

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (4)$$

and n_j is the "charge" of a vortex.

In the next approximation, the emission of sound-like excitations should be taken into account [2, 3]. System of equations (3) can be integrated analytically only in simplest symmetric cases.

If the distances between vortices are small, the approximation of pair interaction is incorrect even in the leading approximation. Nevertheless, the N-vortex solution of Eq. (1) can be found in an explicit form if the distances between all vortices are small. As a result, the evolution of the N-vortex state can be estimated in a large time scale. The initial state is determined by the positions of all N vortices and by any number of free parameters. The number of free parameters depends on the number of "decay" modes, generated by Eq. (1).

The solution of the eigenvalue problem for $N = 2$ was given in Refs. [2, 3]. Here, we investigate the general case for any value of N.

We note that for $N = 2$, the position of the "weight center" of vortices ($\sum_i \mathbf{r}_i/N$) is conserved in time. For $N \geq 3$, the weight-center position depends on time. Its time dependence is investigated below.

2. SPLITTING OF N VORTICES

We seek an N-vortex solution of Eq. (1) in the form

$$\psi = \psi_N e^{iN\phi} + \sum_j \sum_{k=0}^{N-1} A^j_{(N,k)} \times \\ \times \left\{\exp\left(-i\lambda^j_k t + ik\phi\right) f_k(\rho) + \right. \\ \left. + \exp\left[i(2N-k)\phi\right] \exp\left[i(\lambda^j_k)^* t\right] f^*_{2N-k}(\rho)\right\}, \quad (5)$$

[*] E-mail: ovc@itp.ac.ru

where $A^j_{(N,k)}$ are any complex constants, $\psi_N \exp(iN\phi)$ is a static solution of Eq. (1), $f \equiv f^j_k$, $\lambda^j_k \equiv \lambda^j_k(N)$, and (ρ, ϕ) are polar coordinates.

In the linear approximation, system of equations (5) for the functions $\{f_k, f_{2N-k}\}_{k=0,\dots,N-1}$ decouples into a system of equations for the pair $\{f_k, f_{2N-k}\}$ only. Inserting expression (5) in Eq. (1), we obtain

$$\begin{gathered}\lambda^j_k f_k = -\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f_k}{\partial\rho}\right) - \frac{k^2}{\rho^2} f_k\right] - \\ - f_k(1-2|\psi_N|^2) + \psi_N^2 f_{2N-k}, \\ -\lambda^j_k f_{2N-k} = -\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f_{2N-k}}{\partial\rho}\right) - \right. \\ \left. - \frac{(2N-k)^2}{\rho^2} f_{2N-k}\right] - (1-2|\psi_N|^2) f_{2N-k} + (\psi^*_N)^2 f_k, \\ 0 \le k \le N-1.\end{gathered} \tag{6}$$

The “last” eigenvalue λ_{N-1} is equal to zero ($\lambda_{N-1} = 0$). It corresponds to the shift mode. In expression (5), the coefficient $A_{(N,N-1)}$ should be set equal to zero. This means that the “initial” state was created from the state at $(+t \to -\infty)$ with all zeros placed at $\rho = 0$.

Below, we take the function Ψ_N to be real. It is a solution of the equation

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Psi_N}{\partial\rho}\right) - \frac{N^2}{\rho^2}\Psi_N + (1-\Psi_N^2)\Psi_N = 0. \tag{7}$$

For $\rho \ll 1$, Ψ_N has the Taylor expansion:

$$\begin{gathered}\Psi_N = B_N\rho^N\left\{1 - \frac{\rho^2}{4(N+1)} + \frac{\rho^4}{32(N+1)(N+2)} - \right. \\ \left. - \frac{\rho^6}{12(N+3)}\left[\frac{1}{32(N+1)(N+2)} - B_N^2\delta_{N,2}\right] + \dots\right\}.\end{gathered} \tag{8}$$

For $\rho \gg 1$, we obtain from Eq. (7) that

$$\begin{gathered}\Psi_N = \left\{1 - \frac{N^2}{2\rho^2} - \frac{N^2}{\rho^4}\left(1 + \frac{N^2}{8}\right) - \right. \\ - \frac{N^2}{\rho^6}\left(8 + 2N^2 + \frac{N^4}{16}\right) - \\ \left. - \frac{N^2}{\rho^8}\left(144 + \frac{91}{2}N^2 + \frac{25}{8}N^4 + \frac{5}{128}N^6\right) + \dots\right\} + \\ + \frac{\tilde{C}_N}{\sqrt{\rho}}\exp\left(-\sqrt{2}\rho\right).\end{gathered} \tag{9}$$

The values of the coefficients B_N and $\tilde{C}_N$ can be found numerically, starting from expression (8) for the function Ψ_N in the range $\rho \ll 1$ and matching the solution thus found with expression (9) in the range $\rho \gg N$. As a result, we obtain the value of the coefficient B_N with high accuracy. To obtain an accurate value of $\tilde{C}_N$, we solve Eq. (7), starting from expression (9) for $\rho \gg N$ and matching this solution with the previously found solution at $\rho \sim 1$. As a result, for $N = 2, 3, 4$, we obtain

$$\begin{gathered}\{B_2 = 0.15289, \quad \tilde{C}_2 = -16.69\}, \\ \{B_3 = 0.03093519, \quad \tilde{C}_3 = -520\}, \\ \{B_4 = 0.004864699, \quad \tilde{C}_4 = -1290\}.\end{gathered} \tag{10}$$

We now consider the behavior of the functions $\{f_k, f_{2N-k}\}$ in the range of small ($\rho \ll 1$) and large ($\rho \gg 1$) distances. From system of equations (6), we obtain the Taylor expansion ($\rho \ll 1$)

$$\begin{gathered}\begin{pmatrix} f_k \\ f_{2N-k}\end{pmatrix} = \\ = \rho^k \begin{pmatrix} 1 - \dfrac{1+\lambda^j_k}{4(k+1)}\rho^2 + \dfrac{(1+\lambda^j_k)^2}{32(k+1)(k+2)}\rho^4 - \dfrac{\rho^6}{12(k+3)}\left[\dfrac{(1+\lambda^j_k)^3}{32(k+1)(k+2)} - 2B_N^2\delta_{N,2}\right] + \dots \\ \dfrac{B_N^2}{4(k+1)(2N+1)}\rho^{2N+2} + \dots \end{pmatrix} + C_k\rho^{2N-k} \times \\ \times \begin{pmatrix} \dfrac{B_N^2}{4(2N+1)(2N+1-k)}\rho^{2N+2} + \dots \\ 1 - \dfrac{(1-\lambda^j_k)\rho^2}{4(2N+1-k)} + \dfrac{(1-\lambda^j_k)^2\rho^4}{32(2N+2-k)(2N+1-k)} - \dfrac{\rho^6}{12(2N+3-k)} \times \\ \times\left[\dfrac{(1-\lambda^j_k)^3}{32(2N+2-k)(2N+1-k)} - 2B_N^2\delta_{N,2}\right] + \dots \end{pmatrix}.\end{gathered} \tag{11}$$

At large distances $\rho \gg 1$, we have the solution of Eq. (6) in the form of a sum of terms

$$\begin{pmatrix} f_k \\ f_{2N-k} \end{pmatrix} = \frac{\exp(-S_k)}{\sqrt{\rho}} \begin{pmatrix} \alpha_k \\ \beta_k \end{pmatrix}, \qquad (12)$$

where the quantities $\{\alpha_k, \beta_k, S_k\}$ are functions of ρ.

Inserting expression (12) in Eq. (6), we obtain

$$\lambda \begin{pmatrix} \alpha_k \\ -\beta_k \end{pmatrix} = -\left(\frac{\partial S_k}{\partial \rho}\right)^2 \begin{pmatrix} \alpha_k \\ \beta_k \end{pmatrix} + 2\frac{\partial S_k}{\partial \rho} \begin{pmatrix} \dfrac{\partial \alpha_k}{\partial \rho} \\ \dfrac{\partial \beta_k}{\partial \rho} \end{pmatrix} + \\ + \left(-\frac{1}{4\rho^2} + \frac{\partial^2 S_k}{\partial \rho^2} - \frac{\partial^2}{\partial \rho^2}\right) \begin{pmatrix} \alpha_k \\ \beta_k \end{pmatrix} + \frac{1}{\rho^2} \begin{pmatrix} k^2 \alpha_k \\ (2N-k)^2 \beta_k \end{pmatrix} + \\ + (\alpha_k + \beta_k) \begin{pmatrix} 1 \\ 1 \end{pmatrix} - (1 - \Psi_N^2) \begin{pmatrix} 2\alpha_k + \beta_k \\ 2\beta_k + \alpha_k \end{pmatrix}. \qquad (13)$$

At the infinity, the solutions of Eq. (6) should decrease exponentially or should have the form of an outgoing wave. For each set $\{N \geq 2, 0 \leq k < N-1, j\}$, there are two linearly independent solutions of such a type. From Eq. (8), we obtain the following expansion for the function Ψ_N^2 $(\rho \gg 0)$:

$$\Psi_N^2 = 1 - \frac{N^2}{\rho^2} - \frac{2N^2}{\rho^4} - \ldots \qquad (14)$$

Inserting this in Eq. (13), we find

$$\begin{pmatrix} f_k \\ f_{2N-k} \end{pmatrix} = D_k \frac{\exp(-S_k)}{\sqrt{\rho}} \begin{pmatrix} \alpha_k \\ \beta_k \end{pmatrix} + \\ + \frac{\tilde{D}_k}{\sqrt{\rho}} \exp(-\tilde{S}_k) \begin{pmatrix} \tilde{\alpha}_k \\ \tilde{\beta}_k \end{pmatrix}, \qquad (15)$$

where $\{D_k, \tilde{D}_k\}$ are any constants. The values of the functions $\{S_k, \alpha_k, \beta_k\}$ and $\{\tilde{S}_k, \tilde{\alpha}_k, \tilde{\beta}_k\}$ are found in the Appendix. For the first term in (15), we have

$$\begin{pmatrix} \alpha_k \\ \beta_k \end{pmatrix} = \begin{pmatrix} 1 \\ \lambda_k^j + \sqrt{1 + (\lambda_k^j)^2} \end{pmatrix} + \frac{N}{1 + (\lambda_k^j)^2} \times \\ \times \left[(N-k) + \frac{\lambda_k^j N}{2}\right] \frac{1}{\rho^2} \times \\ \times \left[1 - \frac{2}{\rho\sqrt{1 + (\lambda_k^j)^2}} \sqrt{1 + \sqrt{1 + (\lambda_k^j)^2}}\right] \times \\ \times \begin{pmatrix} -\left(\lambda_k^j + \sqrt{1 + (\lambda_k^j)^2}\right) \\ 1 \end{pmatrix}, \\ S_k = \sqrt{1 + \sqrt{1 + (\lambda_k^j)^2}}\,\rho + \\ + \frac{1}{\rho\sqrt{1 + \sqrt{1 + (\lambda_k^j)^2}}} \times \\ \times \left[\frac{1}{8} - \frac{k^2}{2} + Nk\left(1 + \frac{\lambda_k^j}{\sqrt{1 + (\lambda_k^j)^2}}\right) - \right. \qquad (16) \\ \left. - \frac{N^2\left(\lambda_k^j - \frac{1}{2}\right)}{\sqrt{1 + (\lambda_k^j)^2}}\right] \times \\ \times \left(1 - \frac{1}{2\rho\sqrt{1 + \sqrt{1 + (\lambda_k^j)^2}}}\right).$$

For the second term in (15), we obtain

$$\begin{pmatrix} \tilde{\alpha}_k \\ \tilde{\beta}_k \end{pmatrix} = \begin{pmatrix} -\left(\lambda_k^j + \sqrt{1 + (\lambda_k^j)^2}\right) \\ 1 \end{pmatrix} - \\ - \frac{N}{1 + (\lambda_k^j)^2} \left[(N-k) + \frac{N\lambda_k^j}{2}\right] \times \\ \times \frac{1}{\rho^2} \left(1 + \frac{2i}{\sqrt{1 + (\lambda_k^j)^2}} \frac{\sqrt{\sqrt{1 + (\lambda_k^j)^2} - 1}}{\rho}\right) \times \\ \times \begin{pmatrix} 1 \\ \lambda_k^j + \sqrt{1 + (\lambda_k^j)^2} \end{pmatrix}, \\ \tilde{S}_k = i\left\{\sqrt{\sqrt{1 + (\lambda_k^j)^2} - 1}\,\rho + \right. \qquad (17) \\ + \frac{1}{\rho\sqrt{\sqrt{1 + (\lambda_k^j)^2} - 1}} \left[-\frac{1}{8} + \frac{k^2}{2} - \right. \\ - \frac{Nk}{\sqrt{1 + (\lambda_k^j)^2}\left(\lambda_k^j + \sqrt{1 + (\lambda_k^j)^2}\right)} - \\ \left.\left. - \frac{N^2\left(\lambda_k^j - \frac{1}{2}\right)}{\sqrt{1 + (\lambda_k^j)^2}}\right] \left(1 + \frac{i}{2\rho\sqrt{\sqrt{1 + (\lambda_k^j)^2} - 1}}\right)\right\}.$$

The eigenvalues $\lambda_k^j(N)$ of the operator given by system of equations (6) are close to the eigenvalues $\tilde{\lambda}_k^j(N)$ of the much simpler operator $\hat{L}$ given by

$$\hat{L} = -\frac{1}{\rho}\left(\rho\frac{\partial}{\partial\rho}\right) - (1-2|\Psi|^2) + \frac{k^2}{\rho^2}, \quad \hat{L}\tilde{f}_k = \tilde{\lambda}_k^j(N)\tilde{f}_k. \tag{18}$$

In the range $\rho \ll 1$, the function $\tilde{f}_k$ is given by the expansion

$$\tilde{f}_k = \rho^k\left\{1 - \frac{1+\tilde{\lambda}_k^i}{4(k+1)}\rho^2 + \frac{(1+\tilde{\lambda}_k^j)^2}{32(k+1)(k+2)}\rho^4 - \right.$$
$$\left. - \frac{\rho^6}{12(k+3)}\left[\frac{(1+\tilde{\lambda}_k^j)^3}{32(k+1)(k+2)} - 2B_N^2\delta_{N,2}\right] + \ldots\right\}. \tag{19}$$

For $\rho \gg 1$, it follows from Eq. (18) that

$$\tilde{f}_k = \frac{\tilde{G}}{\sqrt{\rho}}\exp\left\{-\left[\sqrt{1-\tilde{\lambda}_k^j}\rho + \frac{1}{\rho\sqrt{1-\tilde{\lambda}_k^j}} \times \right.\right.$$
$$\left.\left. \times \left(N^2 + \frac{1}{8} - \frac{k^2}{2}\right)\left(1 - \frac{1}{2\sqrt{1-\tilde{\lambda}_k^j}\rho}\right)\right] + \ldots\right\}, \tag{20}$$

where $\tilde{G}$ is a constant.

Matching the numerical solution of Eq. (18) starting from the value given by Eq. (19) for $\rho \ll 1$, with the values given by Eq. (20) for $\rho \gg 1$, we obtain the following values for the quantities $\tilde{\lambda}_k^j(N)$:

$$\begin{aligned}
N=2: \quad & \tilde{\lambda}_0 = -0.399689 \text{ (the index } j \text{ has only one value)},\\
N=3: \quad & \tilde{\lambda}_0 = -0.65496595, \quad \tilde{\lambda}_1 = -0.225866\\
& \text{(the index } j \text{ for both values of } k=\{0,1\} \text{ has only one value)},\\
N=4: \quad & \tilde{\lambda}_0^1 = -0.777134, \quad \tilde{\lambda}_0^2 = -0.0888206\\
& \text{(for } k=0, \text{ the index } j \text{ has two values } \{1,2\}),\\
N=4: \quad & \tilde{\lambda}_1 = -0.47814, \quad \tilde{\lambda}_2 = -0.1367221\\
& \text{(for } k=\{1,2\}, \text{ the index } j \text{ has only one value)}.
\end{aligned} \tag{21}$$

The values of $\tilde{\lambda}_{N,k}^j$ thus found are used as the initial data for numerical calculations of the eigenvalues $\lambda_k^j(N)$.

Matching the numerical solution of Eq. (6) starting from the values given by Eq. (11) for $\rho \ll 1$ with the values given by Eqs. (15), (16), and (17) for $\rho \gg 1$, we obtain the coefficients λ_k^j and C_k. As a result of numerical calculations, we obtain

$$\begin{aligned}
N=2: \quad & \lambda_0 = -0.443673 + i0.004937, \quad C_0 = -0.00734 + i0.0001494 \text{ [1, 2]},\\
N=3: \quad & \lambda_0 = -0.4452 + i0.0787, \quad C_0 = 0.000472 - i0.000095,\\
& \lambda_1 = -0.18 + i0.0672, \quad C_1 = 0.002711 + i0\\
& \text{(the index } j \text{ has only one value for } k=\{0,1\}),\\
N=4: \quad & \lambda_0^1 = -0.63494 + i0.07203, \quad C_0^1 = 3\cdot 10^{-6} - i5.7\cdot 10^{-5},\\
& \lambda_0^2 = -0.0458 + i0.01, \quad C_0^2 = 1.1\cdot 10^{-6} - i1.4\cdot 10^{-6},\\
& \lambda_1 = -0.4563 + i0.0154, \quad C_1 = -i0.00216,\\
& \lambda_2 = -0.08735 + i0.0225, \quad C_2 = 0.00289 + i0.00058\\
& \text{(the index } j \text{ takes two values for } k=0 \text{ and one value for } k=\{1,2\}).
\end{aligned} \tag{22}$$

3. TIME EVOLUTION OF THE POSITION OF VORTICES IN AN N-VORTEX STATE

The equation for the position of vortices is

$$\Psi = 0. \tag{23}$$

Using Eqs. (5), (9), (11), (13), and (15), we reduce Eq. (23) to the form

$$z^N + \sum_j \sum_{k=0}^{N-2} A^j_{(N,k)} \left\{ \exp\left[-i\lambda^j_k(N)t\right] \times \right.$$
$$\times z^k \left[1 - \frac{1+\lambda^j_k(N)}{4(k+1)}\rho^2 + \frac{(1+\lambda^j_k(N))^2}{32(k+1)(k+2)}\rho^4 + \right.$$
$$\left. + C^j_k(N)\rho^{4N+2-2k}\frac{B_N^2}{4(2N+1)(2N+1-k)} + \ldots \right] +$$
$$+ \exp\left[i(\lambda^j_k(N))^* t\right] z^{2N-k}\left[\rho^{2k+2}\frac{B_N^2}{4(k+1)(2N+1)} + \right.$$
$$\left.\left. + C^j_k(N)\left(1 - \frac{(1-\lambda^j_k(N))}{4(2N+1-k)}\rho^2\right) + \ldots\right]^* \right\} = 0, \quad (24)$$

where $z = x + iy$ and $\rho = \sqrt{x^2+y^2}$.

We note that nonlinear corrections to the solution of Eq. (1), in the first approximation given by expansion (5), leads only to a renormalization of the coefficients in Eq. (24). This statement implies an important conclusion: for $N = 2$ (double vortex), the position of the weight center of two vortices is independent of time: $\frac{1}{2}(z_1 + z_2) = 0$, where $z_{1,2}$ are positions of the vortices. But for $N \geq 3$, the position of the weight center of N vortices is time independent only in the leading approximation. For $N \geq 3$, we seek the zeros of Eq. (24) in the form

$$z_l = z_l^0 + z_l^{(1)}, \quad (25)$$

where $\{z_l^0\}$ are N solutions of the equation

$$z^N + \sum_j \sum_{k=0}^{N-2} A^j_{(N,k)} \left\{ z^k e^{-i\lambda^j_k(N)t} \right\} = 0. \quad (26)$$

From Eq. (26), we find

$$\sum_{l=1}^{N} z_l^0 = 0.$$

In the next approximation, we obtain

$$\sum_{l=1}^{N} z_l = \sum_{l=1}^{N} z_l^{(1)} \quad (27)$$

for a point of general position, where

$$z_l^{(1)} = \sum_j \sum_{k=0}^{N-2} A^j_{(N,k)} \exp(-i\lambda^j_k(N)t)(1+\lambda^j_k(N)) \times$$
$$\times \frac{|z_l^0|^2 (z_l^0)^k}{4(k+1)} \left\{ N(z_l^0)^{N-1} + \right.$$
$$\left. + \sum_j \sum_{k=1}^{N-1} k A^j_{(N,k)} (z_l^0)^{k-1} \exp\left[-i\lambda^j_k(N)t\right] \right\}^{-1}. \quad (28)$$

We thus obtain that the time evolution of the two-vortex state is determined only by the position of zeros at the initial instant. For $N \geq 3$, the unique prediction of the N-vortex state evolution in the time requires the knowledge of exact positions of all the vortices at the initial instant and the knowledge of two parameters: shift center of the N-vortex state at $t \to -\infty$ relative to the weight center at the current instant and $2\sum_{k=0}^{N-2}(j^{max}_{(N,k)} - 1)$ additional parameters entering Eq. (5).

4. CONCLUSION

The trajectories of N interacting vortices are found in a wide time interval under the assumption that at some initial instant, the distances between all vortices are small. In addition to the positions of the vortices at the initial instant, two additional small parameters should be given to fix unique dynamics of such a vortex state. The full number of unstable modes is $\sum_{k=0}^{N-2} j^{max}_{(N,k)}$. As a result, the coefficients at the "extra" unstable modes are additional free parameters in the problem. The full number of these "extra" modes is equal to $2\sum_{k=0}^{N-2}(j^{max}_{(N,k)} - 1)$.

In the considered approximation, the positions of vortices are given as roots of an algebraic equation. The time dependence of the coefficients in this equation is determined by $(N-1) + \sum_{k=0}^{N-2}(j^{max}_{(N,k)} - 1)$ frequencies. All these frequencies can be found as a solution of an eigenvalue problem. They are found numerically in the particular cases $N = \{3, 4\}$. The emission of sound-like excitations can lead to some self-organization of the position of vortices.

We thank I. M. Sigal for the numerous stimulating discussions. This paper is supported by the EOARD grant № 097006 and the program SIMTECH grant № 246937.

APPENDIX

System of equations (13) has two linearly independent solutions, which satisfy boundary conditions on infinity. In the leading approximation, we obtain

$$\lambda \begin{pmatrix} \alpha_k \\ \beta_k \end{pmatrix} = -\left(\frac{\partial S_k}{\partial \rho}\right)^2 \begin{pmatrix} \alpha_k \\ \beta_k \end{pmatrix} + (\alpha + \beta)\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (29)$$

from Eq. (13). Multiplying both sides of Eq. (29) by $(-\beta_k, \alpha_k)$, we obtain

$$\alpha_k^2 - \beta_k^2 + 2\alpha_k\beta_k = 0. \qquad (30)$$

From Eqs. (29) and (30), we obtain both solutions in the leading approximation: the first solution

$$\begin{pmatrix}\alpha_k\\ \beta_k\end{pmatrix} = \begin{pmatrix}1\\ \lambda_k + \sqrt{1+\lambda_k^2}\end{pmatrix}, \quad \frac{\partial S_k}{\partial\rho} = \left(1+\sqrt{1+\lambda_k^2}\right)^{1/2} \qquad (31a)$$

and the second solution

$$\begin{pmatrix}\tilde\alpha_k\\ \tilde\beta_k\end{pmatrix} = \begin{pmatrix}-\left(\lambda_k + \sqrt{1+\lambda_k^2}\right)\\ 1\end{pmatrix}, \quad \frac{\partial \tilde S_k}{\partial\rho} = i\left(\sqrt{1+\lambda_k^2}-1\right)^{1/2}. \qquad (31b)$$

We now seek the first solution in the form

$$\begin{gathered}\begin{pmatrix}\alpha_k\\ \beta_k\end{pmatrix} = \begin{pmatrix}1\\ \lambda_k + \sqrt{1+\lambda_k^2}\end{pmatrix} + \frac{\gamma_1}{\rho^2}\times\\ \times\left(1+\frac{\gamma_2}{\rho}\right)\begin{pmatrix}-\left(\sqrt{1+\lambda_k^2}+\lambda_k\right)\\ 1\end{pmatrix},\\ S_k = \left(1+\sqrt{1+\lambda_k^2}\right)^{1/2}\rho + \frac{\gamma_3}{\rho}\left(1+\frac{\gamma_4}{\rho}\right).\end{gathered} \qquad (32)$$

From the first two equations in (32), we obtain the useful relations

$$\begin{gathered}\alpha_k^2-\beta_k^2 = -2\left(\lambda_k+\sqrt{1+\lambda_k^2}\right)\times\\ \times\left(\lambda_k + \frac{2\gamma_1}{\rho^2}\left(1+\frac{\gamma_2}{\rho}\right)\right),\\ \alpha_k\beta_k = \left(\lambda_k+\sqrt{1+\lambda_k^2}\right)\times\\ \times\left(1-\frac{2\lambda_k\gamma_1}{\rho^2}\left(1+\frac{\gamma_2}{\rho}\right)\right),\\ \alpha_k^2+\beta_k^2 = 2\sqrt{1+\lambda_k^2}\left(\sqrt{1+\lambda_k^2}+\lambda_k\right).\end{gathered} \qquad (33)$$

Multiplying both sides of Eq. (13) by $(-\beta_k, \alpha_k)$ and using Eq. (33), we immediately obtain

$$\begin{gathered}\gamma_1 = \frac{N}{1+\lambda_k^2}\left[(N-k)+\frac{\lambda_k N}{2}\right],\\ \gamma_2 = -\frac{2}{\sqrt{1+\lambda_k^2}}\left(1+\sqrt{1+\lambda_k^2}\right)^{1/2}.\end{gathered} \qquad (34)$$

Multiplying both sides of Eq. (13) by (α_k, β_k), we obtain

$$\begin{gathered}\lambda_k(\alpha_k^2-\beta_k^2) = -\left(\frac{\partial S_k}{\partial\rho}\right)^2(\alpha_k^2+\beta_k^2)+\\ +\left(-\frac{1}{4\rho^2}+\frac{\partial^2 S_k}{\partial\rho^2}\right)(\alpha_k^2+\beta_k^2)+\\ +\frac{1}{\rho^2}(k^2(\alpha^2+\beta_k^2)-4Nk\beta_k^2)+(\alpha+\beta)^2-\\ -\frac{2N^2}{\rho^2}(\alpha_k^2-\beta_k^2+\alpha_k\beta_k).\end{gathered} \qquad (35)$$

Inserting the value of S_k in (32) into Eq. (35) and using Eq. (33), we obtain the coefficients

$$\begin{gathered}\gamma_3 = \frac{1}{\left(1+\sqrt{1+\lambda_k^2}\right)^{1/2}}\times\\ \times\left\{\frac{1}{8}-\frac{k^2}{2}+Nk\left(1+\frac{\lambda_k}{\sqrt{1+\lambda_k^2}}\right)-\right.\\ \left.-\frac{N^2(\lambda_k-1/2)}{\sqrt{1+\lambda_k^2}}\right\},\\ \gamma_4 = -\frac{1}{2\left(1+\sqrt{1+\lambda_k^2}\right)^{1/2}}.\end{gathered} \qquad (36)$$

In similar way, we seek the second solution of Eq. (13) in the form

$$\begin{gathered}\begin{pmatrix}\tilde\alpha_k\\ \tilde\beta_k\end{pmatrix} = \begin{pmatrix}-\left(\lambda_k+\sqrt{1+\lambda_k^2}\right)\\ 1\end{pmatrix}+\\ +\frac{\tilde\gamma_1}{\rho^2}\left(1+\frac{\tilde\gamma_2}{\rho}\right)\begin{pmatrix}1\\ \lambda_k+\sqrt{1+\lambda_k^2}\end{pmatrix},\\ \tilde S_k = i\left\{\left(\sqrt{1+\lambda_k^2}-1\right)^{1/2}\rho+\frac{\tilde\gamma_3}{\rho}\left(1+\frac{\tilde\gamma_4}{\rho}\right)\right\}.\end{gathered} \qquad (37)$$

As before, we obtain useful relations from the first two equations in (37):

$$\begin{gathered}\tilde\alpha_k^2-\tilde\beta_k^2 = 2\left(\lambda_k+\sqrt{1+\lambda_k^2}\right)\times\\ \times\left(\lambda_k-2\frac{\tilde\gamma_1}{\rho^2}\left(1+\frac{\tilde\gamma_2}{\rho}\right)\right),\\ \tilde\alpha_k\tilde\beta_k = -\left(\lambda_k+\sqrt{1+\lambda_k^2}\right)\times\\ \times\left(1+2\lambda_k\frac{\tilde\gamma_1}{\rho^2}\left(1+\frac{\tilde\gamma_2}{\rho}\right)\right),\\ \tilde\alpha_k^2+\tilde\beta_k^2 = 2\left(\lambda_k+\sqrt{1+\lambda_k^2}\right)\sqrt{1+\lambda_k^2}.\end{gathered} \qquad (38)$$

Multiplying both sides of Eq. (13) by $(-\tilde{\beta}_k, \tilde{\alpha}_k)$, we obtain the coefficients

$$\tilde{\gamma}_1 = -\frac{N}{1+\lambda_k^2}\left[(N-k) + \frac{\lambda_k N}{2}\right], \quad \tilde{\gamma}_2 = \frac{2i}{\sqrt{1+\lambda_k^2}}\left(\sqrt{1+\lambda_k^2}-1\right)^{1/2}. \tag{39}$$

Multiplying both sides of Eq. (13) by $(\tilde{\alpha}_k, \tilde{\beta}_k)$ gives

$$\lambda_k(\tilde{\alpha}_k^2 - \tilde{\beta}_k^2) = -\left(\frac{\partial \tilde{S}_k}{\partial \rho}\right)^2 (\tilde{\alpha}_k^2 + \tilde{\beta}_k^2) + \left(-\frac{1}{4\rho^2} + \frac{\partial^2 \tilde{S}_k}{\partial \rho^2}\right)(\tilde{\alpha}_k^2 + \tilde{\beta}_k^2) + \frac{1}{\rho^2}[k^2(\tilde{\alpha}_k^2 + \tilde{\beta}_k^2) - 4Nk\tilde{\beta}_k^2] + (\tilde{\alpha}_k^2 + \tilde{\beta}_k^2 + 2\tilde{\alpha}_k\tilde{\beta}_k) - \frac{2N^2}{\rho^2}(\tilde{\alpha}_k^2 - \tilde{\beta}_k^2 + \tilde{\alpha}_k\tilde{\beta}_k). \tag{40}$$

Inserting the value of $\tilde{S}_k$ given by Eq. (37) in this equation and using Eq. (38), we obtain

$$\tilde{\gamma}_3 = \frac{1}{\left(\sqrt{1+\lambda_k^2}-1\right)^{1/2}}\left\{-\frac{1}{8} + \frac{k^2}{2} - \frac{Nk}{\sqrt{1+\lambda_k^2}\left(\lambda_k + \sqrt{1+\lambda_k^2}\right)} - \frac{N^2(\lambda_k - 1/2)}{\sqrt{1+\lambda_k^2}}\right\}, \quad \tilde{\gamma}_4 = \frac{i}{2\left(\sqrt{1+\lambda_k^2}-1\right)^{1/2}}. \tag{41}$$

REFERENCES

1. Yu. N. Ovchinnikov and I. M. Sigal, Nonlinearity **11**, 1277 (1998).
2. Yu. N. Ovchinnikov and I. M. Sigal, Zh. Eksp. Teor. Fiz. **139**, 539 (2011).
3. B. L. G. Jonsson, Yu. N. Ovchinnikov, I. M. Sigal, and F. S. T. Ting, J. Math. Phys. **52**, 093505 (2011).