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FERMIONIC SCREENINGS AND LINE BUNDLE TWISTED CHIRAL DE RHAM COMPLEX ON CY MANIFOLDS

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We present a generalization of Borisov's construction of the chiral de Rham complex in the case of the line-bundle-twisted chiral de Rham complex on a Calabi–Yau hypersurface in a projective space. We generalize the differential associated with a polytope Δ of the projective space $\mathbb{P}^{d-1}$ by allowing nonzero modes for the screening currents forming this differential. It is shown that the numbers of screening current modes define the support function of the toric divisor of a line bundle on $\mathbb{P}^{d-1}$ that twists the chiral de Rham complex on the Calabi–Yau hypersurface.

1. INTRODUCTION

The Calabi–Yau manifolds with bundles appear in various types of compactifications of string theory. The first example is the heterotic string compactification on Calabi–Yau (CY) manifolds. This is currently the most successful approach to the problem of string model construction relevant to 4-dimensional particle physics. The main ingredients of the heterotic models is a CY three-fold and two holomorphic vector bundles on it. In the simplest case of the "standard embedding", one of the bundles is taken to be trivial and the other coincides with the tangent bundle of the CY manifold. But explicit constructions of the bundles are difficult to obtain in general. Nevertheless, in the series of papers [1], the monad bundles approach has been developed for the systematic construction of a large class of vector bundles over the CY manifolds defined as complete intersections in products of projective spaces.

Although the monad construction in [1] is quite general, it is purely classical, and the Gepner models [2] are the only known models of the quantum string compactification. This leads to the question: what is the quantum version of the monad bundle construction?

The second example is the type-IIA compactification with D-branes wrapping the CY manifold. In this case, the Chan–Paton vector bundle appears [3], and hence a similar question make sense: what object describes the quantum strings on a CY manifold with Chan–Paton bundles?

In this paper, we analyze the simplest version of these questions when we have only a line bundle on the CY manifold, and present the construction of a vertex operator algebra starting from a CY hypersurace in a projective space and a line bundle defined on this space.

Our approach is based essentially on the work of Borisov [4], where a certain sheaf of vertex operator algebras endowed with the $N=2$ Virasoro superalgebra action has been constructed for each pair of dual reflexive polytopes Δ and Δ^* defining a CY hypersurface in the toric manifold $\mathbb{P}_{\Delta}$. Borisov thus directly constructed the holomorphic sector of the CFT from toric data of a CY manifold. The main object of his construction is a set of fermionic screening currents associated with the points of that pair of polytopes. Zero modes of these currents are used to construct a differential $D_{\Delta}+D_{\Delta^*}$ whose cohomology calculated in some lattice vertex algebra gives the global sections of a sheaf known as the chiral de Rham complex due to [6]. On the local sections of the chiral de Rham complex, the $N=2$ Virasoro superalgebra acts [6]. In the CY case, this algebra survives the cohomology, and hence global sections of the chiral de Rham complex can be considered a holomorphic sector of the space of states of an $N=2$ superconformal sigma-model on the CY manifold. The question how the costruction in [4] is related to the Gepner models has been clarified to some extent in [7] and [8].

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Borisov's construction can also be generalized by allowing nonzero modes of the screening currents forming the differential. We consider a CY hypersurface in the projective space $\mathbb{P}^{d-1}$ and generalize Borisov's differential by allowing nonzero modes only for screening currents associated with the points of the $\mathbb{P}^{d-1}$ polytope Δ. We thus generalize the differential D_Δ, leaving the differential D_{Δ^*} defining the CY hypersurface unchanged. We show that the numbers of screening current modes from D_Δ define the support function of the toric divisor [9, 10] of a line bundle on $\mathbb{P}^{d-1}$. By this means, the chiral de Rham complex on $\mathbb{P}^{d-1}$ appears to be twisted by the line bundle.

The paper is organized as follows. In Sec. 2, we briefly review the construction in [4] for a CY hypersurface in $\mathbb{P}^{d-1}$. In Sec. 3, we first calculate the cohomology with respect to the generalized differential D_Δ and relate the result to sections of the chiral de Rham complex twisted by the sheaf $O(N)$. To do that, we find a generalized $bc\beta\gamma$ system of fields generating the cohomology. The generalization appears only for the modes of vector fields operators. They are replaced by the covariant derivative operators with a $U(1)$ connection. Then we define the trivialization maps of the modules generated by these generalized $bc\beta\gamma$ fields to the modules of sections of the usual chiral de Rham complex over the affine space and find the transition functions for different trivializations. These turn out to be the transition functions of an $O(N)$ bundle on $\mathbb{P}^{d-1}$, where N is determined by the number of modes of the screening currents composing the differential D_Δ. Moreover, we establish the relation between the number of screening current modes and the toric divisor support function for the line bundle $O(N)$ on $\mathbb{P}^{d-1}$. The support function and trivialization maps are consistent with the localization maps determined by the fan structure, which allows calculating the cohomology of the twisted chiral de Rham complex by a Čech complex of the covering. In complete analogy with [4], the second differential D_{Δ^*} is used to restrict the sheaf to the CY hypersurface.

In Sec. 4, we calculate the elliptic genus of the twisted chiral de Rham complex and represent it in terms of theta functions. For a torus in $\mathbb{P}^2$ and $K3$ in $\mathbb{P}^3$, we find the limit as $q \to 0$ and relate the results to the Hodge numbers of the sheaf $O(N)$ on the torus and $K3$. Section 5 contains concluding remarks.

2. CHIRAL DE RHAM COMPLEX ON A CY HYPERSURFACE IN $\mathbb{P}^{d-1}$

In this section, we review the construction in [4] of the chiral de Rham complex and its cohomology for a CY hypersurface in the projective space $\mathbb{P}^{d-1}$.

Let $\{e_1, \ldots, e_d\}$ be the standard basis in $\mathbb{R}^d$ and $\Lambda \subset \mathbb{R}^d$ be the lattice generated by the vectors $e_0 = \frac{1}{d}(e_1 + \ldots + e_d)$, $e_1, \ldots, e_d$:

$$\Lambda = \mathbb{Z}e_0 \oplus \mathbb{Z}e_1 \oplus \ldots \oplus \mathbb{Z}e_d. \tag{1}$$

We then consider the fan $\Sigma \subset \Lambda$ [9, 10] encoding the toric data of the total space E of an $O(d)$-bundle over $\mathbb{P}^{d-1}$. The maximal-dimension cones of the fan are d-dimensional cones $C_I \subset \Lambda$, $I = 1, 2, \ldots, d$, spanned by the vectors $e_0, \ldots, \hat{e_I}, \ldots, e_d$, where the vector e_I is omitted. The intersection of the maximal-dimension cones is also a cone in Σ:

$$C_I \cap C_J \cap \ldots \cap C_K = C_{IJ\ldots K} \in \Sigma. \tag{2}$$

All faces of the cone from Σ are the cones from Σ (see [9, 10] for a more detailed definition of a fan.)

Let $\{e_1^*, \ldots, e_d^*\}$ be the dual basis to the standard one $\{e_1, \ldots, e_d\}$ and let Λ^* be the dual lattice to Λ. For every cone $C \in \Sigma$, we consider the dual cone $C^* \in \Lambda^*$ defined by

$$C^* = \{p^* \in \Lambda^* | p^*(C) \geq 0\} \tag{3}$$

and the affine variety $A_C = \mathrm{Spec}(\mathbb{C}[C^*])$. If C^* is a face of $\tilde{C}^*$, then $\mathbb{C}[C^*]$ is a localization of $\mathbb{C}[\tilde{C}^*]$ by the monomials $a^{p^*} \in \mathbb{C}[C^*]$, where $p^* \in \tilde{C}^*$ and $p^*(C) = 0$. This allows gluing A_C to form the space E.

The polytope Δ of $\mathbb{P}^{d-1}$ is given by the points from Σ satisfying the equation

$$\deg^*(\Sigma) = 1, \tag{4}$$

where

$$\deg^* = e_1^* + \ldots + e_d^*. \tag{5}$$

Let $X_i(z)$, $X_i^*(z)$, $i = 1, 2, \ldots, d$, be free bosonic fields and $\psi_i(z)$, $\psi_i^*(z)$, $i = 1, 2, \ldots, d$, be free fermionic fields with the OPEs given by

$$\begin{aligned} X_i^*(z_1) X_j(z_2) &= \ln(z_{12})\delta_{i,j} + \text{reg}, \\ \psi_i^*(z_1)\psi_j(z_2) &= z_{12}^{-1}\delta_{i,j} + \text{reg}, \end{aligned} \tag{6}$$

where $z_{12} = z_1 - z_2$.

The fields are expanded into the integer modes

$$\begin{aligned} \partial X_i^*(z) &= \sum_{n\in\mathbb{Z}} X_i^*[n] z^{-n-1}, \\ \partial X_i(z) &= \sum_{n\in\mathbb{Z}} X_i[n] z^{-n-1}, \\ \psi_i^*(z) &= \sum_{n\in\mathbb{Z}} \psi_i^*[n] z^{-n-\frac{1}{2}}, \\ \psi_i(z) &= \sum_{n\in\mathbb{Z}} \psi_i[n] z^{-n-\frac{1}{2}}. \end{aligned} \tag{7}$$

We therefore consider the Ramond sector.

To the lattice $\Gamma = \Lambda \oplus \Lambda^*$, we associate the direct sum of Fock spaces

$$\Phi_\Gamma = \oplus_{(p,p^*)\in\Gamma} F_{(p,p^*)}, \tag{8}$$

where $F_{(p,p^*)}$ is the Fock module generated by $X_i[n]$, $X_i^*[n]$, $\psi_i[n]$, $\psi_i^*[n]$ from the vacuum $|p,p^*\rangle$ defined by

$$\begin{gathered} X_i^*[n]|p,p^*\rangle = X_i[n]|p, \\ p^*\rangle = \psi_i[n]|p,p^*\rangle = \psi_i^*[n-1]|p, \\ p^*\rangle = 0, \quad n>0, \\ X_i^*[0]|p,p^*\rangle = p_i^*|p,p^*\rangle, \\ X_i[0]|p,p^*\rangle = p_i|p,p^*\rangle. \end{gathered} \tag{9}$$

For each vector e_i, $i = 0,1,\ldots,d$, generating a 1-dimensional cone from Σ, we define the fermionic screening current and the screening charge

$$\begin{gathered} S_i^*(z) = e_i\cdot\psi^*\exp(e_i\cdot X^*)(z), \\ Q_i^* = \oint dz\, S_i^*(z). \end{gathered} \tag{10}$$

We form the BRST operators for each maximal-dimension cone C_I:

$$D_I^* = Q_0^* + \ldots + \hat{Q}^*{}_I + \ldots + Q_d^*, \tag{11}$$

where Q_I^* is omitted. We then consider the space

$$\Phi_{C_I\otimes\Lambda^*} = \oplus_{(p,p^*)\in C_I\otimes\Lambda^*} F_{(p,p^*)}. \tag{12}$$

The space of sections M_I of the chiral de Rham complex over the A_{C_I} is given by the cohomology of $\Phi_{C_I\otimes\Lambda^*}$ with respect to the operator D_I^*. It is generated by the fields [4]

$$\begin{aligned} &a_{I\mu}(z) = \exp[w_{I\mu}^*\cdot X](z), \\ &\alpha_{I\mu}(z) = w_{I\mu}^*\cdot\psi\exp[w_{I\mu}^*\cdot X](z), \\ &a_{I\mu}^*(z) = (e_\mu\cdot\partial X^* - w_{I\mu}^*\cdot\psi_i e_\mu\cdot\psi_i^*)\times \\ &\qquad\times\exp[-w_{I\mu}^*\cdot X](z), \\ &\alpha_{I\mu}^*(z) = e_\mu\cdot\psi^*\exp[-w_{I\mu}^*\cdot X](z), \end{aligned} \tag{13}$$

where $w_{I\mu}^*$ are the dual vectors to the basis of vectors $\left\{e_\mu, \mu = 0,\ldots,\hat{I},\ldots d\right\}$ generating the cone C_I:

$$\langle w_{I\mu}^*, e_\nu\rangle = \delta_{\mu\nu}. \tag{14}$$

The singular operator product expansions of these fields are

$$\begin{gathered} a_{I\mu}^*(z_1)a_{I\nu}(z_2) = z_{12}^{-1}\delta_{\mu\nu} + \ldots, \\ \alpha_{I\mu}^*(z_1)\alpha_{I\nu}(z_2) = z_{12}^{-1}\delta_{\mu\nu} + \ldots \end{gathered} \tag{15}$$

An important property is the behavior of the $bc\beta\gamma$ system under the local change of coordinates on A_{C_I} [6]. For each new set of coordinates

$$\begin{aligned} b_{I\mu} &= g_\mu(a_{I1},\ldots,a_{Id}), \\ a_{I\mu} &= f_\mu(b_{I1},\ldots,b_{Id}), \end{aligned} \tag{16}$$

the isomorphic $bc\beta\gamma$ system of fields is given by

$$\begin{aligned} &b_{I\mu}(z) = g_\mu(a_{I1}(z),\ldots,a_{Id}(z)), \\ &\beta_{I\mu}(z) = \frac{\partial g_\mu}{\partial a_{I\nu}}(a_{I1}(z),\ldots,a_{Id}(z))\alpha_{I\nu}(z), \\ &\beta_{I\mu}^*(z) = \frac{\partial f_\nu}{\partial b_{I\mu}}(a_{I1}(z),\ldots,a_{Id}(z))\alpha_{I\nu}^*(z), \\ &b_{I\mu}^*(z) = \frac{\partial f_\nu}{\partial b_{I\mu}}(a_{I1}(z),\ldots,a_{Id}(z))a_{I\nu}^*(z) + \\ &\qquad + \frac{\partial^2 f_\lambda}{\partial b_{I\mu}\partial b_{I\nu}}\frac{\partial g_\nu}{\partial a_{I\rho}}(a_{I1}(z),\ldots,a_{Id}(z))\times \\ &\qquad\times\alpha_{I\lambda}^*(z)\alpha_{I\rho}(z). \end{aligned} \tag{17}$$

Here, the normal ordering of operators is implied. It is also understood whenever necessary in what follows.

On the space M_{C_I}, the $N=2$ Virasoro superalgebra acts by the currents

$$\begin{aligned} &G^- = \sum_\mu \alpha_{I\mu}a_{I\mu}^*, \quad G^+ = -a_{I0}\partial\alpha_{I0}^* - \\ &- \sum_{\mu\neq 0,I}\alpha_{I\mu}^*\partial a_{I\mu}, \quad J = a_{I0}a_{I0}^* + \sum_{\mu\neq 0,I}\alpha_{I\mu}^*\alpha_{I\mu}, \\ &T = \frac{1}{2}(a_{I0}^*\partial a_{I0} - \partial a_{I0}^* a_{I0}) - \alpha_{I0}\partial\alpha_{I0}^* + \\ &+ \sum_{\mu\neq 0,I}\left(a_{I\mu}^*\partial a_{I\mu} + \frac{1}{2}(\partial\alpha_{I\mu}^*\alpha_{I\mu} - \alpha_{I\mu}^*\partial\alpha_{I\mu})\right). \end{aligned} \tag{18}$$

This algebra defines the mode expansion of the fields in the Ramond sector

$$\begin{aligned} a_{I0}(z) &= \sum_n a_{I0}[n]z^{-n-\frac{1}{2}}, \\ a_{I0}^*(z) &= \sum_n a_{I0}^*[n]z^{-n-\frac{1}{2}}, \\ \alpha_{I0}(z) &= \sum_n \alpha_{I0}[n]z^{-n-1}, \\ \alpha_{I0}^*(z) &= \sum_n \alpha_{I0}[n]z^{-n}, \\ a_{I\mu}(z) &= \sum_n a_{I\mu}[n]z^{-n}, \\ a_{I\mu}^*(z) &= \sum_n a_{I\mu}^*[n]z^{-n-1}, \\ \alpha_{I\mu}(z) &= \sum_n \alpha_{I\mu}[n]z^{-n-\frac{1}{2}}, \\ \alpha_{I\mu}^*(z) &= \sum_n \alpha_{I\mu}[n]z^{-n-\frac{1}{2}}, \quad \mu\neq I. \end{aligned} \tag{19}$$

Then M_{C_I} is generated by the creation operators acting on the Ramond vacuum state $|0\rangle$ defined by the conditions

$$a_{I\mu}[n]|0\rangle = a^*_{I\mu}[n-1]|0\rangle = \alpha_{I\mu}[n]|0\rangle = = \alpha^*_{I\mu}[n-1]|0\rangle = 0, \quad n > 0. \quad (20)$$

If the cone C_{KJ} is a face of the cones C_K and C_J, it is spanned by the vectors

$$(e_0, e_1, \ldots, \hat{e}_K, \ldots, \hat{e}_J, \ldots, e_d).$$

We then consider the BRST operator

$$D^*_{KJ} = Q^*_0 + Q^*_1 + \ldots + \hat{Q}^*{}_K + \ldots \ldots + \hat{Q}^*{}_J + \ldots + Q^*_d \quad (21)$$

acting on $\Phi_{C_{KJ}\oplus\Lambda^*}$. The space of sections $M_{C_{KJ}}$ of the chiral de Rham complex over $A_{C_{KJ}}$ is given by the cohomology of $\Phi_{C_{KJ}\oplus\Lambda^*}$ with respect to the operator D^*_{KJ}. It is a localization of M_{C_K} (M_{C_J}) with respect to the multiplicative system generated by

$$\prod_\mu (a_{K\mu}[0])^{m_\mu} (\prod_\mu (a_{J\mu}[0])^{m_\mu}),$$

with

$$\sum_\mu m_\mu w^*_{K\mu}(C_{KJ}) = 0 \quad \left(\sum_\mu m_\mu w^*_{J\mu}(C_{KJ}) = 0\right).$$

Analogously, the localization maps can be defined for the cones that are intersections of an arbitrary number of maximal-dimension cones [4].

The localization maps defined above allow calculating the cohomology of the chiral de Rham complex on E as the Čech cohomology of the covering by A_{C_I}, $I = 1, \ldots, d$, [4]:

$$0 \to \oplus_{C_I} M_{C_I} \to \oplus_{C_{KJ}} M_{C_{KJ}} \to \to \ldots M_{C_{12\ldots d}} \to 0. \quad (22)$$

This finishes the calculation of D_Δ-cohomology.

The next step is to restrict the chiral de Rham complex on E to a CY manifold $CY \subset \mathbb{P}^{d-1}$. We define the function W on E, linear on the fibers of E, such that in the coordinates $a_{I\mu}$ on A_{C_I},

$$W = a_{I0}\left(1 + \sum_{\mu\neq 0,I} (a_{I\mu})^d\right). \quad (23)$$

We next introduce the corresponding screening currents and screening charges

$$S_{I0}(z) = \alpha_{I0}(z)\left(1 + \sum_{\mu\neq 0,I} (a_{I\mu})^d(z)\right),$$
$$S_{I\mu}(z) = \alpha_{I\mu} a_{I0}(z)(a_{I\mu})^{d-1}(z), \quad \mu \neq 0, I, \quad (24)$$
$$Q_\mu = \oint dz\, S_{I\mu}(z), \quad \mu \neq I,$$

and the BRST operator

$$D_W = \sum_{\mu\neq I} Q_\mu. \quad (25)$$

(It is the D_{Δ^*} differential in the notation in [4].) The cohomology of M_{C_I} with respect to D_W gives the space of sections $M_{C_I}|_W$ of chiral de Rham complex on $A_{C_I} \cap CY$ determined by the system of equations

$$a_{I0} = 0,$$
$$1 + \sum_{\mu\neq 0,I} (a_{I\mu})^d = 0. \quad (26)$$

The cohomology of the chiral de Rham complex on CY is calculated by the Čech complex of the covering [4]:

$$0 \to \oplus_{C_I} M_{C_I}|_W \to \to \oplus_{C_{KJ}} M_{C_{KJ}}|_W \to \ldots M_{C_{12\ldots d}}|_W \to 0. \quad (27)$$

This yields the $D_\Delta + D_{\Delta^*}$ cohomology.

A more general function W and BRST operator (25) can be considered by adding the monomial that corresponds to the internal point from the dual polytope Δ^* [4, 7].

This completes the review of the chiral de Rham complex and its cohomology construction on a CY hypersurface in $\mathbb{P}^{d-1}$.

3. LINE BUNDLE TWISTED CHIRAL DE RHAM COMPLEX

In this section, the generalization of Borisov's construction producing the $O(N)$-twisted chiral de Rham complex on a CY hypersurface is proposed.

We twist the fermionic screening charges Q^*_i as

$$Q^*_i \to S^*_i[N_i] = \oint dz\, z^{N_i} S^*_i(z), \quad N_i \in \mathbb{Z}. \quad (28)$$

Then the old charges Q^*_i can be considered zero modes of the screening currents

$$S^*_i(z) = \sum_n S^*_i[n] z^{-n-1}$$

and BRST operator (11) can be considered a particular case of the more general one

$$D_I^* = S_0^*[N_0] + \ldots + \hat{S^*}_I[N_I] + \ldots + S_d^*[N_d]. \quad (29)$$

Now the question is: what is the cohomology of space (12) with respect to this new BRST operator?

It follows by direct calculation that the fields $a_{I\mu}(z)$, $\alpha_{I\mu}(z)$, $\alpha^*_{I\mu}(z)$ in (13) still commute with the new differential D_I^*, but instead of $a^*_{I\mu}(z)$ we have to take

$$\nabla_{I\mu}(z) = a^*_{I\mu}(z) + N_\mu z^{-1} a^{-1}_{I\mu}(z). \quad (30)$$

The last term in this expression can be regarded as coming from a $U(1)$ gauge potential on A_{C_I}. We see in what follows that this is indeed so and the modes of the fields $\nabla_{I\mu}(z)$ can be regarded as a string version of covariant derivatives.

In terms of this new $bc\beta\gamma$ fields, the $N=2$ Virasoro superalgebra currents are given by

$$\begin{aligned} G_I^- &= \sum_\mu \alpha_{I\mu}\nabla_{I\mu} = \sum_n G_I^-[n] z^{-n-\frac{3}{2}}, \\ G_I^+ &= -a_{I0}\partial\alpha^*_{I0} - \sum_{\mu\neq 0,I} \alpha^*_{I\mu}\partial a_{I\mu} = \\ &= \sum_n G_I^+[n] z^{-n-\frac{3}{2}}, \\ J_I &= a_{I0}\nabla_{I0} + \sum_{\mu\neq 0,I} \alpha^*_{I\mu}\alpha_{I\mu} = \sum_n J_i[n] z^{-n-1}, \\ T_I &= \frac{1}{2}(\nabla_{I0}\partial a_{I0} - \partial\nabla_{I0} a_{I0}) - \alpha_{I0}\partial\alpha^*_{I0} + \\ &+ \sum_{\mu\neq 0,i}\left(\nabla_{I\mu}\partial a_{I\mu} + \frac{1}{2}(\partial\alpha^*_{I\mu}\alpha_{I\mu} - \alpha^*_{I\mu}\partial\alpha_{I\mu})\right) = \\ &= \sum_n L_I[n] z^{-n-2}. \end{aligned} \quad (31)$$

To calculate the cohomology, we consider the vertex operator

$$V_{(0,p^*)}(z) = \exp[p^* X](z),$$

where $p^* \in \Lambda^*$. We find

$$\begin{aligned} S^*_\mu[N_\mu](z_1)V_{(0,p^*)}(z_2) &= z_{12}^{N_\mu} S^*_\mu(z_1) V_{(0,p^*)}(z_2) = \\ &= z_{12}^{N_\mu + p^*(e_\mu)} e_\mu \cdot \psi^* \times \\ &\times \exp[e_\mu \cdot X^* + p^* \cdot X](z_2) + \ldots, \quad \mu \neq I. \end{aligned} \quad (32)$$

Hence, the state $|(0,p^*)\rangle$ corresponding to the vertex $V_{(0,p^*)}(0)$ is in

$$\mathrm{Ker}(S^*_\mu[N_\mu]) \quad \text{if} \quad p^*(e_\mu) \geq -N_\mu.$$

The (Ramond-sector) state saturating the inequality is $\left|\left(0, -\sum_{\mu\neq I} N_\mu w^*_{I\mu}\right)\right\rangle$ and has the properties

$$\begin{aligned} \nabla_{I\mu}[k]\left|\left(0, -\sum_{\mu\neq I} N_\mu w^*_{I\mu}\right)\right\rangle &= 0, \quad k \geq 0, \\ a_{I\mu}[k]\left|\left(0, -\sum_{\mu\neq I} N_\mu w^*_{I\mu}\right)\right\rangle &= \\ = \alpha_{I\mu}[k]\left|\left(0, -\sum_{\mu\neq I} N_\mu w^*_{I\mu}\right)\right\rangle &= \\ = \alpha^*_{I\mu}[k-1]\left|\left(0, -\sum_{\mu\neq I} N_\mu w^*_{I\mu}\right)\right\rangle &= 0, \\ k > 0. \end{aligned} \quad (33)$$

Proposition. The cohomology $\mathbb{M}_{C_I}$ of $\Phi_{C_I \oplus \Lambda^*}$ with respect to differential (29) is generated from the vacuum state

$$|\Omega_I\rangle = \left|\left(0, -\sum_{\mu\neq I} N_\mu w^*_{I\mu}\right)\right\rangle \quad (34)$$

by the creation operators of fields (13) and (30).

The proof is similar to the proof of Proposition 6.5. in [4].

The vacuum $|\Omega_I\rangle$ defines the trivializing isomorphism of modules (over the chiral de Rham complex on A_{C_I})

$$g_I : \mathbb{M}_I \to M_I \quad (35)$$

by the rule

$$\begin{aligned} g_I|\Omega_I\rangle &= |0\rangle, \\ g_I(\nabla_{I\mu}[k])g_I^{-1} &= a^*_{I\mu}[k], \\ g_I(a_{I\mu}[k])g_I^{-1} &= a_{I\mu}[k], \\ g_I(\alpha_{I\mu}[k])g_I^{-1} &= \alpha_{I\mu}[k], \\ g_I(\alpha^*_{I\mu}[k])g_I^{-1} &= \alpha^*_{I\mu}[k]. \end{aligned} \quad (36)$$

We therefore call the vacuum $|\Omega_I\rangle$ the trivializing vacuum.

We consider the subspace $\mathbb{M}^0_{C_I}$ generated from $|\Omega_I\rangle$ by the operators $a_{I\mu}[0]$ and $\alpha_{I\mu}[0]$. The operator $G_I^-[0]$ acts on this subspace by

$$\delta_I = \sum_{\mu\neq I} \alpha_{I\mu}[0]\nabla_{I\mu}[0].$$

It is natural to think that $\mathbb{M}^0_I$ is holomorphic de Rham complex over A_{C_I} with coefficients in a holomorphic line bundle.

On the intersections $A_{C_I} \cap A_{C_J}$, the relations between the coordinates

$$a_{I0} = a_{J0}(a_{JI})^d, \quad a_{I\mu} = a_{J\mu}a_{JI}^{-1}, \quad \mu \neq I, J, \quad a_{IJ} = a_{JI}^{-1}, \tag{37}$$

can be used to find the relations between the trivializing vacua

$$\begin{gathered} g_I|\Omega_I\rangle = \prod_{\mu\neq I}(a_{I\mu}[0])^{N_\mu}|\Omega_I\rangle = |0\rangle, \\ g_I^{-1}g_J|\Omega_J\rangle \equiv g_{IJ}|\Omega_J\rangle = |\Omega_I\rangle, \\ g_{IJ} = (a_{(J)I}[0])^{N_1+N_2+\ldots+N_d-dN_0}, \end{gathered} \tag{38}$$

as well as between the sections

$$g_{IJ} : \mathbb{M}_J^0 \to \mathbb{M}_I^0. \tag{39}$$

The functions g_{IJ} in (38) are the transition functions of the line bundle on E that is induced from the $O(N)$-bundle on $\mathbb{P}^{d-1}$ by the canonical projection map $\pi : E \to \mathbb{P}^{d-1}$, where

$$N = N_1 + \ldots + N_d - dN_0. \tag{40}$$

Thus the set of modules $\mathbb{M}_{C_I}^0$ with the differentials δ_I and the transition functions (38) define the holomorphic de Rham complex on E with coefficients in the line bundle $\pi^*O(N)$.

One can extend this finite dimensional discussion to the infinite dimensional case. To do that we consider the relation between the currents $G_I^-(z)$ and $G_J^-(z)$ on the intersection $A_{C_I} \cap A_{C_J}$. Because of (37) and (17), we find

$$\begin{gathered} G_I^-(z) = G_J^-(z) + Nz^{-1}\alpha_{JI}(z)a_{JI}^{-1}(z) \Leftrightarrow \\ \Leftrightarrow G_I^-[k] = G_J^-[k] + N\sum_m \alpha_{JI}[m]a_{JI}^{-1}[k-m]. \end{gathered} \tag{41}$$

In the finite-dimensional case, the differentials δ_I are consistent on the intersections $A_{C_I} \cap A_{C_J}$: the difference $\delta_I - \delta_J$ coming from the different trivializations is canceled by gauge transformation of the gauge potential

$$A_{I\mu} = A_{J\mu} - g_{IJ}^{-1}\frac{\partial g_{IJ}}{\partial a_{J\mu}}.$$

A similar event should occur in the infinite-dimensional situation. Because the first Chern class on E is zero, the second term in expression (41) is due to different trivializations defined on $A_{C_I} \cap A_{C_J}$ and has to be canceled by a gauge transformation of the gauge potential:

$$\begin{gathered} G_I^-[k] = G_J^-[k] + N\sum_m \alpha_{JI}[m]a_{JI}^{-1}[k-m] - \\ - \sum_{\nu\neq J}\left(g_{IJ}^{-1}\alpha_{J\nu}\frac{\partial g_{IJ}}{\partial a_{J\nu}}\right)[k] = G_J^-[k]. \end{gathered} \tag{42}$$

Hence, the current $G^- \equiv G_I^-$ as well as the $N = 2$ Virasoro superalgebra are defined globally if we take transformations of the gauge potential into account and extend the map (39) to the map

$$g_{IJ}(z) = (a_{(J)I}(z))^N : \mathbb{M}_J \to \mathbb{M}_I. \tag{43}$$

If the cone C_{IJ} is a face of the cone C_I (C_J) and spanned by the vectors

$$(e_0, \ldots, \hat{e}_I, \ldots, \hat{e}_J, \ldots, e_d),$$

we can consider the BRST operator

$$\begin{gathered} D_{IJ}^* = S_0^*[N_0] + \ldots + \hat{S^*}_I[N_I] + \ldots \\ \ldots + \hat{S^*}_J[N_J] + \ldots + S_d^*[N_d] \end{gathered} \tag{44}$$

acting on $\Phi_{C_{IJ}\otimes\Lambda^*}$.

The cohomology $\mathbb{M}_{C_{IJ}}$ of $\Phi_{C_{IJ}\otimes\Lambda^*}$ with respect to differential (44) is the localization of $\mathbb{M}_{C_I}$ ($\mathbb{M}_{C_J}$) with respect to the multiplicative system generated by

$$\prod_\mu (a_{I\mu}[0])^{m_\mu} \quad \left(\prod_\mu (a_{J\mu}[0])^{m_\mu}\right),$$

with

$$\sum_\mu m_\mu w_{I\mu}^*(C_{IJ}) = 0 \quad \left(\sum_\mu m_\mu w_{J\mu}^*(C_{IJ}) = 0\right).$$

The module $\mathbb{M}_{C_{IJ}}$ is generated from the vacuum vector

$$|\Omega_{IJ}\rangle = \left|\left(0, -\sum_{\mu\neq I,J} N_\mu w_{I\mu}^*\right)\right\rangle \tag{45}$$

by the creation operators of the fields $a_{I\mu}(z)$, $\nabla_{I\mu}(z)$, $\mu \neq I, J$, $a_{IJ}(z)$, $a_{IJ}^{-1}(z)$, $a_{IJ}^*(z)$, $\alpha_{I\mu}(z)$, $\alpha_{I\mu}^*(z)$, $\mu \neq I$. $\mathbb{M}_{C_{IJ}}$ can also be generated from the vacuum

$$|\tilde{\Omega}_{IJ}\rangle = \left|\left(0, -\sum_{\mu\neq I,J} N_\mu w_{J\mu}^*\right)\right\rangle = (a_{IJ}[0])^{N-N_I-N_J}|\Omega_{IJ}\rangle \tag{46}$$

by the creation operators of the fields $a_{J\mu}(z)$, $\nabla_{J\mu}(z)$, $\mu \neq I, J$, $a_{JI}(z)$, $a_{JI}^{-1}(z)$, $a_{JI}^*(z)$, $\alpha_{J\mu}(z)$, $\alpha_{J\mu}^*(z)$, $\mu \neq J$.

Analogously, the modules $\mathbb{M}_{C_{IJ\ldots K}}$ and localization maps can be defined for the cones

$$C_{IJ\ldots K} = C_I \cap C_J \cap \ldots \cap C_K.$$

Relation (46) is a particular case of compatibility conditions to be satisfied for localization maps. They are as follows. For each maximal-dimension cone C_I, the trivializing vacuum $|\Omega_I\rangle$ defines a linear function $\omega_I^* \in \Lambda^*$ on this cone:

$$\begin{gathered}\prod_{\mu\neq I} a_{(I)\mu}^{-N_\mu}(0) = \exp[-\omega_I^* X](0),\\ \omega_I^* = dN_0 w_{I0}^* + \sum_{\mu\neq 0,I} N_\mu w_{I\mu}^*.\end{gathered} \tag{47}$$

It is easy to see that the collection of ω_I^* satisfies the obvious compatibility condition. Namely, on the cone $C_{IJ} = C_I \cap C_J$, the functions ω_I^* and ω_J^* coincide and are given by the function $\omega_{IJ}^* \in \Lambda^*$ of the trivializing vacuum $|\Omega_{IJ}\rangle$:

$$\begin{gathered}\prod_{\mu\neq I,J} a_{(I)\mu}^{-N_\mu}(0) = \exp[-\omega_{IJ}^* X](0),\\ \omega_{IJ}^* = dN_0 w_{I0}^* + \sum_{\mu\neq 0,I,J} N_\mu w_{I\mu}^*.\end{gathered} \tag{48}$$

It can be verified that similar compatibility conditions are also satisfied for the functions $\omega_{IJ\ldots K}^*$ on the cones

$$C_{IJ\ldots K} = C_I \cap C_J \cap \ldots \cap C_K.$$

Then the numbers $N_0, \ldots, N_d$ of screening current modes define the support function ω^* on Σ [9, 10] of the toric divisor of the bundle $\pi^* O(N)$ on E.

Hence, similarly to (22), we have the Čech complex of the covering by A_{C_I}, $I = 1, \ldots, d$,

$$0 \to \oplus_{C_I}\mathbb{M}_{C_I} \to \oplus_{C_{KJ}}\mathbb{M}_{C_{KJ}} \to \ldots \mathbb{M}_{C_{12\ldots d}} \to 0 \tag{49}$$

which gives the cohomology of the chiral de Rham complex on E twisted by $\pi^* O(N)$.

The restriction of the twisted chiral de Rham complex to a CY hypersurface is straightforward because BRST operator (25) commutes with operators (29) and acts within each of the modules $\mathbb{M}_{C_{IJ\ldots K}}$. Therefore, the complex

$$\begin{gathered}0 \to \oplus_{C_I}\mathbb{M}_{C_I}|_W \to \oplus_{C_{KJ}}\mathbb{M}_{C_{KJ}}|_W \to\\ \to \ldots \mathbb{M}_{C_{12\ldots d}}|_W \to 0\end{gathered} \tag{50}$$

gives the cohomology of the $O(N)$-twisted chiral de Rham complex on a CY hypersurface. This completes the construction.

4. ELLIPTIC GENUS CALCULATION

In this section, we calculate the elliptic genus of the twisted chiral de Rham complex, closely following [5]. The q^0 coefficient of the elliptic genus is related to the Hodge numbers of the sheaf $O(N)$ on a CY manifold. To justify the construction in Sec. 3, we calculate it for the torus $T^2 \subset \mathbb{P}^2$ and $K3 \subset \mathbb{P}^3$.

The discussion in Sec. 3 and the arguments in [5] allow extending Definition 6.1 in [5] to the case under discussion: the elliptic genus is given by the supertrace over the Čech cohomology space of the twisted chiral de Rham complex.

The calculation is greatly simplified using the torus $(\mathbb{C}^*)^d$ that acts on E [5]. We compute the function

$$\begin{gathered}\rho_N(CY, t_1, \ldots, t_d, y, q) = \sum_{k=1}^{d} (-1)^{k-1} \times\\ \times \sum_{C_{I_1}, \ldots, C_{I_k}} \text{super}\ \mathrm{Tr}_{\mathbb{M}_{C_{I_1\ldots I_k}}} \times\\ \times \left(\prod_{i=1}^{d} t_i^{K_i} y^{J[0]} q^{L[0]-\frac{c}{24}}\right),\end{gathered} \tag{51}$$

where t_i are the formal variables grading the torus action with the help of generators K_i (whose explicit form is obvious) and then take the limit $t_i \to 1$, $i = 1, \ldots, d$, to obtain the elliptic genus $\mathrm{Ell}_N(CY, y, q)$. It is guite helpful for the subseguent computations to write the $N = 2$ Virasoro superalgebra acting on $\mathbb{M}_{C_{IJ\ldots K}}$ in coordinates (6):

$$\begin{gathered}G_{IJ\ldots K}^- = -z^{-1}\omega_{IJ\ldots K}^* \cdot \psi - \deg^* \cdot \partial\psi +\\ + \psi \cdot \partial X^*,\\ G_{IJ\ldots K}^+ = -\deg^* \cdot \partial\psi^* + \psi^* \cdot \partial X,\\ J_{IJ\ldots K} = -z^{-1}\deg \cdot \omega_{IJ\ldots K}^* + \deg \cdot \partial X^* -\\ - \deg^* \cdot \partial X + \psi^* \cdot \psi,\\ T_{IJ\ldots K} = \frac{1}{2}(\partial\psi^* \cdot \psi - \psi^* \cdot \partial\psi) +\\ + \partial X \cdot (\partial X^* - z^{-1}\omega_{IJ\ldots K}^*) -\\ - \frac{\deg}{2} \cdot \partial(\partial X^* - z^{-1}\omega_{IJ\ldots K}^*) - \frac{\deg^*}{2} \cdot \partial^2 X,\end{gathered} \tag{52}$$

where

$$deg = e_0 = \frac{1}{d}(e_1 + \ldots + e_d). \tag{53}$$

The supertraces over the modules $\mathbb{M}_{C_{I_1\ldots I_k}}$ can be calculated as the supertraces over the spaces $\Phi_{C_{IJ\ldots K}\otimes\Lambda^*}$, which are the complexes with respect to the differentials

$$\begin{gathered}D_{IJ\ldots K}^* = S_0^*[N_0] + \ldots + \hat{S}^*{}_I[N_I] + \ldots\\ \ldots + \hat{S}^*{}_J[N_J] + \ldots + \hat{S}^*{}_K[N_K] + \ldots + S_d^*[N_d].\end{gathered}$$

Because of (52), we obtain

$$\rho_N(CY, t_1, \dots, t_d, y, q) = y^{-\frac{d-2}{2}+d-1} \sum_{w^*\in\Lambda^*} \prod_{i=1}^{d} t_i^{\langle w^*, e_i\rangle} \sum_{C\subset\Sigma} (-1)^{\operatorname{codim} C} \times \times \sum_{k\in C} y^{-\langle \deg^*, k\rangle + \langle w^*-\omega^*, \deg\rangle} \times \times q^{\langle w^*-\omega^*, k\rangle} G(y^{-1}, q)^d, \quad (54)$$

where

$$G(y,q) = \prod_{k\geq 1} \frac{(1-yq^{k-1})(1-y^{-1}q^k)}{(1-q^k)^2} \quad (55)$$

and the factor y^{d-1} is caused by the Ramond vacuum. This is a generalization of the elliptic genus expression obtained in [5]. It can be rewritten in terms of the theta functions. For that, we need to use the trick in [5] to eliminate the contribution of positive-codimension cones. Then we apply the "truly remarkable identity"

$$\prod_{k\geq 1} \frac{(1-tyq^{k-1})}{(1-tq^{k-1})} \frac{(1-t^{-1}y^{-1}q^k)}{(1-t{-}1q^k)} = \sum_{n\in\mathbb{Z}} t^n (1-yq^n)^{-1} G(y,q) \quad (56)$$

to write the maximal-dimension cones contribution as an infinite product. Thus, we obtain

$$\rho_N(CY, t_1, \dots, t_d, y, q) = y^{-\frac{d-2}{2}+d-1} \times \times \sum_{I=1}^{d} \left(\prod_{i=1}^{d} t_i^{-\langle \omega_I^*, e_i\rangle}\right) \frac{\Theta_{1,1}(t_I^d, q)}{\Theta_{1,1}(t_I^d y, q)} \times \times \prod_{J\neq I} \frac{\Theta_{1,1}(t_I^{-1} t_J y^{-1}, q)}{\Theta_{1,1}(t_I^{-1} t_J, q)}, \quad (57)$$

where

$$\Theta_{1,1}(u,q) = = q^{1/8} \prod_{n=0} (1-u^{-1}q^{n+1})(1-uq^n)(1-q^{n+1}) = = q^{1/8} \sum_{n\in Z} (-1)^n q^{(n^2-n)/2} u^{-n}. \quad (58)$$

In the limit as $q \to 0$ and $t_i \to 1$, the elliptic genus is related to the Hodge numbers of the sheaf $O(N)$ on the CY hypersurface

$$\mathrm{Ell}_N(CY, y) = y^{-\frac{d-2}{2}+d-1} \times \times \sum_{p,q} (-1)^{p+q} h^{p,q}(CY, O(N)) y^q. \quad (59)$$

We use this fact as a check of the construction in two simplest examples, the torus T^2 in $\mathbb{P}^2$ and $K3$ in $\mathbb{P}^3$.

Taking the $t_i \to 1$ limit by l'Hopital's rule, we find

$$\mathrm{Ell}_N(T^2, y) = 3N(1-y)y^{-\frac{1}{2}}, \quad \mathrm{Ell}_N(K3, y) = 2((N^2+1) + (10-2N^2)y + + (N^2+1)y^2)y^{-1}. \quad (60)$$

We see that these expressions correctly reproduce the corresponding Hodge numbers.

5. CONCLUDING REMARKS

In this note, we presented a generalization of Borisov's construction of the chiral de Rham complex on toric CY manifolds to include the CY hypersurfaces with line bundles. It is shown that including the nonzero modes of the screening currents associated to the points of the polytope Δ of $\mathbb{P}^{d-1}$ into Borisov's differential, we obtain $O(N)$-twisted chiral de Rham complex on the CY hypersurface. Moreover, we established a relation between the number of screening current modes and the toric divisor support function for the line bundle $O(N)$ on $\mathbb{P}^{d-1}$.

We hope that the construction discussed above can be applied for the quantization of monad bundles in heterotic string models. Another possible application appears if we consider the twisting line bundle as a Chan–Paton bundle of a bound state of $(2d-4, 2d-6)$ D-branes on a CY manifold. In this context, it would be interesting to generalize the construction to also include Chan–Paton sheafs describing more general bound states of the D-branes on CY manifolds [3].

There are two more questions to be mentioned. The first question, which is obvious, is to extend the discussion to CY hypersurfaces in general toric manifolds. The second one is a possible mirror-symmetry generalization. In the construction of Borisov, the differentials associated to the pair of reflexive polytopes Δ and Δ^* come into play on equal footing, which makes the mirror symmetry explicit [4, 5]. For the generalization considered in this paper, this democracy seems to be broken. Indeed, if we first take the cohomology with respect to the differential D_{Δ^*}, which is unchanged, we obtain the usual (untwisted) chiral de Rham complex on the toric manifold $\mathbb{P}_{\Delta^*}$. It is difficult to believe that taking then the cohomology with respect to the generalized differential D_Δ as a second step, we restrict the chiral de Rham complex to a mirror CY hypersurface in $\mathbb{P}_{\Delta^*}$. Therefore, the question is how to extend the mirror symmetry to this case. The more general setup is the simultaneous generalization of differentials D_Δ and D_{Δ^*} by nonzero screening current modes.

Recently, paper [11] has appeared where a different construction of the chiral de Rham complex twisted by a vector bundle is presented. It would be interesting to understand the relation with our approach.

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