Model-independent description of the $dt$ and $d^3He$ systems near low-energy resonances

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An expansion of the effective radius is employed in a model-independent description of the $dt$ and $d^3He$ systems in the vicinity of the low-energy $^5He^*(3/2^+)$ and $^3Li^*(3/2^+)$ resonances. The Coulomb-nuclear scattering lengths ($a_{cu}$) and effective radii ($r_{cu}$) for states with orbital angular momentum $l=0$, as well as the astrophysical function $s(E)$, are extracted from existing experimental data for the cross sections of the nuclear fusion reactions $dt\rightarrow n\alpha$ and $d^3He\rightarrow p\alpha$ in the vicinity of the resonances (data on elastic $dt$ and $n\alpha$ scattering are also employed in the case of the $dt$ system). Extensive use is made of the generalization of the Schwinger-Smorodinskii equation for the effective radius to the case of potentials with a Coulomb barrier. A bound is established on the value of the Coulomb-nuclear effective radius $r_{cu}'$ with an arbitrary value of the orbital angular momentum $l$, which does not depend on the specific form of the strong potential $V_s(r)$. Numerical calculations of the form factors $P$ and $Q$ appearing in the expansion of the effective radius are performed for different models of $V_s(r)$, and the problem of the stability of the results obtained toward possible variations of the form of the strong interaction is discussed. The analytic structure of the scattering amplitude near the elastic threshold is investigated in the presence of absorption, i.e., open reaction channels, in the system. Two series of Coulomb poles in the complex $k$ plane, which converge near the elastic threshold ($k=0$), are found for the $dt$ scattering amplitude.

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1. INTRODUCTION

The resonant nuclear reactions $dt\rightarrow n\alpha+17.59$ MeV and $d^3He\rightarrow p\alpha+18.35$ MeV, which have large energy releases, play an important role in nuclear physics (including thermonuclear fusion and $\alpha$-catalysis), astrophysics, etc. The cross sections of these reactions in the vicinity of the $s$-wave $^5He^*(3/2^+)$ and $^3Li^*(3/2^+)$ resonances have been measured with extremely high accuracy, making it possible to perform a detailed analysis of the $dt$ and $d^3He$ systems using an expansion of the effective radius. In these systems the resonance wave ($l=0$, $J^P=3/2^+$) dominates both in elastic scattering and in the fusion reaction and largely determines the reaction cross section $\sigma(E)$. As was shown in Refs. 5 and 6, the elastic scattering amplitude for slow charged particles (as opposed to neutral particles) can be reconstructed, in principle, from experimental data for the reaction cross section in the region of a low-energy resonance. This makes it possible to find the Coulomb-nuclear parameters $a_{cu}$ and $r_{cu}$ and to perform a reliable extrapolation of the cross section $\sigma(E)$ to the region of small energy values $E\lesssim 10$ keV, which plays a significant role in physical applications. Direct measurement of the reaction cross section at small energies is extremely difficult because of the exponentially small permeability of the Coulomb barrier.

We use a model-independent approach, based on an expansion of the effective radius previously developed in the theory of systems with a Coulomb attraction potential distorted at short distances by the strong interaction (particularly in the theory of $\bar{p}p$ and $K^-\alpha$ atoms).

As a rule, we shall henceforth use the Coulomb units $\hbar=m=a_B=1$, where $m$ is the reduced mass and $a_B$ is the Bohr radius of the system.

2. LOW-ENERGY COULOMB-NUCLEAR PARAMETERS

The expansion of the effective radius for particles of like charge has the form

$$\sigma(k^2) = a_{cu} + a_{cu}k^2 + a_{cu}k^4 + \ldots,$$

$$s(E) = \frac{1}{a_{cu}} + \frac{1}{2} a_{cu} k^2 + \frac{1}{4} a_{cu} k^4 + \frac{1}{6} a_{cu} k^6 + \ldots,$$

where $l=0$, $\beta(k^2) > 0$ hold [from the unitarity condition, see Eq. (4)-(6) below],

$$\alpha(k^2) = a_{cu} + a_{cu} k^2 + a_{cu} k^4 + \ldots,$$

$$\beta(k^2) = \beta_0 + \beta_2 k^2 + \ldots,$$

$$a_{cu} = \frac{a_{cu}^0 + i \beta_0}{\sqrt{a_{cu}^0 + \beta_0^2}},$$

$$\Delta(k^2) = \frac{2(\alpha_{cu} - i \beta_{cu})}{8(\alpha_{cu} - i \beta_{cu})},$$

$$\Delta_{cu}(k^2), a_{cu},$$ and $r_{cu}$ are the Coulomb-nuclear phase of the scattering amplitude, scattering length, and effective radius, respectively.
$P$ and $Q$ are the form factors, $k = \sqrt{2E}$, $E$ is the energy in the center-of-mass system, $\eta = \ln k a_\rho$ is the Sommerfeld parameter, 
\[ D_\zeta(\eta) = (e^{2\eta} - 1)^{-1}, \quad \ln \eta = \text{Re} \psi(\eta) - \frac{1}{2} \ln \eta^2, \] 
(3')
$\psi(z) = \Gamma(z)/\Gamma(1)$, $\Gamma(z)$ is the gamma function, and $D_\zeta(\eta)$ is the permeability of the Coulomb barrier.

Since the thresholds of the nearest open channels ($\Delta = 17.59$ and $18.35$ MeV for the fusion reactions under consideration) and closed channels ($\Delta = -2.22$ and $-1.46$ MeV for the $(d,pn)$ and $^3\text{He}(d,2p)$ reactions) significantly exceed the energies of the $^5\text{He}^*$ and $^3\text{He}^*$ resonances, the expansion of the effective radius is applicable in the resonance energy range ($E < 50$ and $200$ keV, respectively, for $dt$ and $d^3\text{He}$ scattering). Due to the presence of the open channels, i.e., absorption in the system, the low-energy parameters (3) are complex.

The resonance $s$ wave plays the dominant role in both elastic scattering and the fusion reaction. Confining ourselves to its contribution, from the unitarity condition for the fusion reaction cross section we obtain
\[ a_s(E) = \frac{4\pi}{3E^2}(1 - |S_{0s}|^2) = \frac{8\pi^2}{3E^2} D_\zeta(\eta)s(E), \] 
(4)
where
\[ S_{0s} = e^{2i\rho_0} \frac{a(k) - ib(k)}{a(k) - ib(k)}, \] 
(5)
\[ \rho_0 = \text{arg}^+(1 + i\eta) \] is the Coulomb phase of the $s$ scattering amplitude,
\[ a(k) = a(k^2) - 2b(\eta), \quad b_\zeta(k) = \beta(k^2) \pm 2\pi D_\zeta(\eta), \] and $s(E)$ is the astrophysical function:
\[ s(E) = \beta(k^2) \] 
(we note that our definition of the astrophysical function differs from that used in Refs. 1 and 2; see Appendix A). Describing the experimental data in Refs. 1 and 2 by this expression, we obtain a series of sets of low-energy parameters for the $dt$ system (see Table I; the quality of the fit is illustrated by Fig. 1). When the $\alpha_i$ and $\beta_i$ are calculated, the results of Ref. 16 on elastic $dt$ scattering are also included in the treatment. This, however, does not significantly alter the low-energy parameters (compare the set of parameters corresponding to $\alpha_0 = 0.238$ in Table I with the last row, which were obtained without consideration of the results in Ref. 16; the difference between the values of $\chi^2$ is associated with the low accuracy of the elastic scattering data). It is noteworthy that we set the form factors $P$ and $Q$ equal to zero in (1), since their consideration is beyond the range of accuracy of the approximation used (for further details, see Appendix A). We also stress that it becomes possible to extract the low-energy parameters from experimental data on the fusion reaction cross section owing to the interference between the nuclear and Coulomb interactions.

As is seen from Table I, the consistent variation of all four parameters $\alpha_i$ and $\beta_i$ ($i = 0, 1$) scarcely alters the value of $\chi^2$ ($\chi^2 < 1$ for $0.22 < \alpha_0 < 0.32$). Therefore, additional criteria are needed to select the low-energy parameters. For this purpose, we examine some properties of the effective radius $r_\zeta$ for charged systems.

### 3. BOUNDS ON $r_\zeta$ IN THE RESONANCE CASE

The effective radius $r_\zeta$ can be expressed in terms of the real wave function $\phi_\zeta(r) = rR_\zeta(r, k = 0)$ with zero energy:
\[ r_\zeta = \pm \frac{2}{3}\frac{a_\zeta}{\beta_\zeta} \sqrt{2(1 - 2)} \langle \frac{n}{3} \rangle = \frac{8}{3(1)^{3/2}} \int_0^\infty \frac{\rho^2}{|a|^2} \frac{d\rho}{\rho}, \] 
(7)
where
\[ \tilde{\xi}(r) = \frac{2^{2/3} + 1}{2^{1/3} + 1} E_{21}(\rho), \]
\[ \eta(\rho) = 2^{2/3} + 1 \rho^{2/3} E_{21}(\rho). \]
(8)

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*As shown in Fig. 1, the astrophysical function $s(E)$ for the fusion reaction $dt\rightarrow nu$. The solid curve was calculated from Eq. (6) using the parameters in Table II; the experimental points were taken from Ref. 1 (filled circles) and from Ref. 2 (+).*
with \( p = \sqrt{8r_0a_B} \) and \( K_J(p) \) and \( J_J(p) \) are Bessel functions of imaginary argument. The function \( \psi_0(r) \) is normalized by the asymptotic condition

\[
\psi_0(r) = -\frac{\xi_0(r)}{\sqrt{4Q}} e^{i \frac{\sqrt{8Q}}{a_B} r}, \quad r \gg r_0.
\]

so that the integral in (7) converges (here and in the following \( r_0 \) denotes the characteristic radius of action of the nuclear forces). At the instant when an \( l \) level appears we have

\[
\psi_{l(l)}^{(1)} = \psi_l(r), \quad (10)
\]

where

\[
\chi(r) = \frac{e^{i \sqrt{8Q} r/a_B}}{r} \exp \left( -\frac{\sqrt{8Q}}{a_B} r \right), \quad r \to \infty,
\]

(the exponential decay of the wave function at infinity is due to the Coulomb barrier). In this case Eq. (7) is simplified significantly:

\[
F_{l(l)}^{(1)} = 2(2l+1)! \int_0^\infty dr \left[ \frac{\alpha}{a_B^2} \xi_0(r)^2 - \chi_l^2(r) \right],
\]

(11)

where

\[
\alpha = \frac{1}{2l+1} \left( \frac{4l+3}{3(l+1)!} \right) (c_0 = 1, c_1 = 0.509, \text{etc.});
\]

and in the following quantities referring to the instant when the level appears are marked with a tilde.

We make several remarks regarding Eq. (11).

1) When the Coulomb interaction is "turned off" \((a_B \to \infty)\), we have \( p \to 0 \) and \( \xi(0) = \eta(0) = 1 \), and (11) takes the form:

\[
F_{l(l)}^{(1)} = 2(2l+1)! \int_0^\infty dr \chi_l^2(r), \quad l > 1,
\]

(12)

where, according to (9), the function \( \chi_l(r) \) is normalized by the condition \( \chi_l(0) = 1 \) when \( r \to \infty \).

2) As for \( s \) states, in the limit \( a_B \to \infty \) the radial function \( \chi_s(r) \) delocalizes, and the expression (11) transforms into the familiar formula of Schwinger19 and Smorodinski:20

\[
\bar{r}_s = \bar{r}_s^{(1)} = 2 \int_0^\infty [1 - \chi_s(r)] dr, \quad \lim_{r \to \infty} \chi_s(r) = 1.
\]

(13)

3) In the case of Coulomb repulsion, the wave function of the zero-energy \( s \) state is localized at distances \( r \ll a_B \). The expression (11) then takes the form

\[
\bar{r}_0 = \bar{r}_0^{(1)} = 2 \int_0^\infty [\frac{\xi_0^2(r)}{a_B^2} - \chi_0^2(r)] dr
\]

\[
- \frac{1}{3} a_B - 2 \int_0^\infty \chi_0^2(r) dr,
\]

(14)

which is a generalization of Eq. (13) to the case of potentials with Coulomb repulsion at large distances.

Some useful bounds on the value of \( F_{l(l)}^{(1)} \) follow from (11) and (14). Let \( R_c \) be the shortest distance at which the strong interaction begins to be negligibly small compared with the Coulomb interaction (in the R-matrix approach9 the role of \( R_c \) is played by the charged channel radius). Since \( \chi_l(r) = -\frac{\xi_l(r)}{\sqrt{8Q}} \) when \( r > R_c \) and \( k_{l(l)}^{(1)} = \infty \), discarding the positive term \( 2 \int_0^\infty \chi_l^2(r) dr \) in (11), we arrive at the inequality

\[
F_{l(l)}^{(1)} < \frac{a_B}{2 \sqrt{3}} \frac{1}{3 (l+1)!} H_l(\bar{r}_s), \quad \bar{r}_s = \frac{\bar{r}_s}{a_B},
\]

(15)

where

\[
H_l(s) = 1 - \frac{3}{2} \int_0^\infty dx K_{2l+1}(x) y^l dy.
\]

In particular, for an \( s \) wave

\[
H_l(s) = 1 - \frac{1}{4} \rho^2 [K_{2l+1}(\rho) - K_{2l}^{(1)}(\rho)]
\]

\[
= \begin{cases} 
\frac{3}{4} \rho^2 + \frac{3}{2} \rho^4 \ln \rho + O(\rho^6), & \rho \to 0, \\
\frac{1}{8} \pi e^{-2 \rho^2} [\rho^2 + 7 + \ldots] + \ldots, & \rho \to \infty,
\end{cases}
\]

(16)

where \( \rho = \sqrt{3} x \) (Fig. 2). We stress that the bound (15) is valid for any short-range potential and for arbitrary \( l \).

Let us demonstrate the effectiveness of the bound on \( F_{l(l)}^{(1)} \). Table I contains several sets of low-energy parameters. They are all acceptable according to the criterion of a minimum for \( x_2 \), the set with \( a_B = 0.238 \) corresponding to the absolute minimum of \( x_2 \), which is equal to 0.62. However, it corresponds to the physically unacceptable value \( \bar{R}_s \approx 8 \) fm, which significantly exceeds the sum of the charge radii of \( d \) and \( s \) and should, therefore, be eliminated. Assuming that \( R_c = 5 \) fm for the \( ds \) system, we can eliminate all the sets with \( a_B < 0.26 \), for which Re \( r_{l(l)} = 2 a_B a_0 > 5.1 \) fm.

In the case of exact resonance (i.e., for \( 1/a_B = 0 \)) the points corresponding to \( ds \) and \( d^3s \) should lie on the solid line.
plays a very important role. 852 of the particles increase and the charged channel radius $d_{3\text{e}}$ calculations of this system within the R-matrix present case, which is realized, for example, in the incomplete binding. We note that the value of the same curve in Fig. 2, which corresponds to the equality sign in (15). Using (7), we generalize the inequality (15) to the case of finite values of $a_3$, (see Appendix B):

$$r_{2e} \leq \frac{1}{3} R(1/x_2) + a_2 f_2(x_2) + a_3 f_3(x_2),$$  

(17)

where

$$f_1(x) = \frac{p_1^2}{2} [f_1(p) K(p) + f_2(p) K_2(p)],$$  

(18)

$$f_2(x) = \frac{p_1^4}{192} [f_2(p) - f_2(p)].$$  

$x = \frac{1}{a_3}$, and $\mu = \sqrt{a_3}$. For example, setting $a_3 = 0.27$ and $R_{\text{d}} = 5$ fm for the $dt$ system, we obtain $p_1 = 1.29$, $R_{\text{d}} = 0.577$, $f_1 = 5.83(-2)$, and $f_2 = 0.80(-3)$, and (17) gives $r_{2e} \leq (0.192 + 0.106 + 0.001)a_3 = 5.01$ fm. Thus, the correction for the "incomplete binding" of the $dt$ system\textsuperscript{9} increases the value of $r_{2e}$ by approximately 10%. This, in turn, corresponds to a decrease in $R_{\text{d}}$ (see Table II) by the same 10%, which was taken into account in Table II. See also Fig. 2, in which the displacement of the points for the $dt$ and $d^2H$ systems from the solid curve is due to consideration of the incomplete binding. We note that the value $R_{\text{d}} = 4.8$ fm for the $dt$ system in Table II is consistent with the choice of the charged channel radius $R_{\text{d}} = 5$ fm used in most recent calculations of this system within the R-matrix approach\textsuperscript{1,2}. It also follows from (15) and (16) that when the charges of the particles increase and $R_{\text{d}} = 3a_3$, the effective radius $r_{2e}$ is small (compared to $r_{3e}$) and is exponentially close to its limiting value, which is equal to $a_2/3$. This can be illustrated in the examples of the Breit model (see Appendix B) and the $\delta$ potential, which allow an exact solution. In these two cases, for $p_1 = \sqrt{R_{\text{d}}}/a_3 \ll 1$ we obtain

$$r_{2e} = \frac{1}{2} a_3 d_1 e^{-2x_2} p_1^2 \left[1 + \frac{d_2}{p_1} + \ldots\right],$$  

(19)

where $d_1 = 3/8$ and $d_2 = 7/4$ for the Breit model and $d_1 = 3/4$ and $d_2 = 3/4$ for the $\delta$ potential. Therefore, in the present case, which is realized, for example, in the $\alpha$ system,\textsuperscript{11} Coulomb renormalization of the effective radius plays a very important role.

To conclude this section, we make two remarks.

1) In addition to the stringent upper bounds (15) and (17), we can also obtain a useful, though fairly rough, lower bound on $r_{3e}$ [see Eq. (B5) in Appendix B].

2) The introduction of terms proportional to $r_{1e}$, $r_{2e}$, etc., into the expansion (1) enables us to take into account phenomenologically the effective radius of the system (its "non-pointlike character") under the condition $kr_{3e} \ll 1$. This distinguishes the low-energy reactions with very light nuclei considered here, for example, from the collisions of heavy nuclei, whose description requires the introduction of model conceptions regarding the structure of the nuclei and the reaction mechanisms.

4. $\alpha \alpha$ SCATTERING AND LOW-ENERGY PARAMETERS OF THE $dt$ SYSTEM

The lower bound on $r_{3e}$ can be obtained from an analysis of the elastic $\alpha\alpha$ scattering near the threshold of the $dt$ channel.

The $^3\text{He}^*(3/2^+)$ resonance plays a significant role both in the $s$ wave for low-energy $dt$ scattering and in the $d$ wave of the $\alpha\alpha$ system near the threshold of the $\alpha\alpha\rightarrow dt$ reaction (i.e., for the neutron energies $E_n = E_d = 22.065$ MeV). Utilizing the analyticity and unitarity conditions, as well as the $T$ invariance of the nuclear interaction, we can express the elements of the $S$-matrix of such a two-channel system in terms of the low-energy parameters of the $dt$ system:

$$S_{\alpha\alpha} e^{i\delta_{\alpha\alpha} - iv_{\alpha\alpha}} = \frac{\Gamma}{S_0} e^{i\nu_{\alpha\alpha} - i\delta_{\alpha\alpha}}$$  

(20)

$$S_{\alpha\alpha} = S_{\alpha\alpha} + S_{\alpha\alpha} e^{i\delta_{\alpha\alpha} - iv_{\alpha\alpha}}$$  

(20')

$$S_{\alpha\alpha} = S_{\alpha\alpha} e^{i\delta_{\alpha\alpha} - iv_{\alpha\alpha}}$$  

(20'')

(compare, for example, Refs. 22 and 23). Here $\delta_{\alpha\alpha}(k)$ is the Coulomb phase, and $\delta_{\alpha\alpha}(k)$ is the Coulomb-nuclear phase of the $dt$ scattering amplitude, $V = |S_{\alpha\alpha}|$ is the inelasticity parameter (0 $\leq$ $r$ $\leq$ 1, $r$ = 1 being below the threshold of the $dt$ channel), $S_{\alpha\alpha} = \sqrt{1 - r^2}$, $e^{i2\beta(k)} = \exp(2i\delta(k))$, $E_d > E_{\alpha\alpha}$,

$$A(k) = \frac{a(k) - i\beta(k)}{a(k) - i\beta(k)}$$  

(21)

Finally, $\psi$ is the phase of the potential $\alpha\alpha$ scattering. This is the only additional parameter (beside $\alpha_1$ and $\beta_1$) that determines the elastic $\alpha\alpha$ scattering in the resonance $D_{23}$ wave. In the energy range under consideration the phase $\psi$ can be assumed constant, its numerical value being small.\textsuperscript{11}

It follows from (20') that

$$\mu - \phi = \frac{1}{2} \arctan \frac{b_2}{a}$$  

(22)
FIG. 3. Energy dependence of the phase $\mu - \varphi$ for the $D_{33}^1$ wave of elastic $na$ scattering. Curves 1, 2, and 3 correspond to the sets with $a_0 = 0.238, 0.28,$ and $0.32$. The experimental points were taken from Ref. 24.

We note that the functions $a(k)$ and $b_+(k)$ appearing therein have zeros at resonance:

$$a(k_0) = 0, \quad 22.148 \text{ MeV} \lessgtr E \lessgtr 22.149 \text{ MeV},$$

$$b_+(k_0) = 0, \quad 22.195 \text{ MeV} \lessgtr E \lessgtr 22.196 \text{ MeV},$$

the values of the energies $E_1$ and $E_2$ being close to one another. Therefore, the energy dependence of the phase shift $\mu - \varphi$ turns out to be very sensitive to the magnitude of $a_0$ (see Fig. 3, as well as Ref. 21). This allows us to obtain the bounded interval $0.26 \lessgtr a_0 \lessgtr 0.28$, which gives

$$\text{Re} a_{1e} = -76 \pm 4 \text{ fm}, \quad \text{Im} a_{1e} = -310 \pm 0.2 \text{ fm},$$

$$\text{Re} r_{1e} = 4.9 \pm 0.3 \text{ fm}, \quad \text{Im} r_{1e} = -0.34 \pm 0.04 \text{ fm},$$

$$s(0) = 1.29 \pm 0.01.$$  

(23)

The values presented are determined directly from existing experimental data and have small errors, especially $s(0)$. Thus, taken together, the bound (15) and the phase-shift analysis data for $na$ scattering at $22.16 < E_n < 22.21$ MeV significantly reduce the uncertainty in extracting the low-energy parameters of the $dt$ system.

Using the set of values for the low-energy parameters $a_0$ and $b_0$ presented above, we can calculate the astrophysical function $s(E)$ and the fusion reaction cross section at the minimal energies $E \lessgtr 5$ keV. The results of the calculation are shown in Fig. 4, where the solid curve corresponds to the set of data from Table I with $a_0 = 0.27$, while the two dashed curves correspond to $a_0 = 0.26$ and 0.28.

As is seen from Fig. 4, the experimental points in this energy range have irregular positions and are located outside the error corridor following from the bounds on $s(E)$ established above. Our calculated values of $s(E)$ have a small error ($\approx 1\%$), because they were obtained on the basis of the experimental data for $\sigma_s(E)$ at resonance, which have 1% accuracy. Since the data from Ref. 24 on elastic $na$ scattering near the $^3\text{He}^*$ resonance were also used in the calculations, the further increase in their accuracy at $E_n \approx E_0 \approx 22$ MeV makes it possible to obtain a more reliable upper bound on $a_0$ and seems very desirable.

A similar analysis was performed for the $d^3\text{He}$ system (see Appendix B). The corresponding low-energy parameters are presented in Table II. In this case the term with the form factor $P$ should also be taken into account in the expansion (1) (see Appendix A). The increase in the number of fitting parameters along with the insufficient accuracy of the experimental data at $E - E_0$ (the error $\approx 4\%$ for the cross section of the $d^3\text{He} \rightarrow p\alpha$ reaction) preclude determining the values of $a_0$ and $b_0$ with as high an accuracy as for the $dt$ system. Therefore, the corresponding parameters in Table II should not be considered final.

Let us now compare the results obtained with the results of other investigators. The effective-radius approximation was previously used to describe the $dt \rightarrow na$ and $d^3\text{He} \rightarrow p\alpha$ reactions by Barit and Sergeev, who treated the
experimental data known at that time by varying three inde-
pendent parameters, viz., $\Re a_\beta$, $\Im a_\beta$, and $\Re r_{\lambda\nu}$, and assuming $\Im r_{\lambda\nu} = 0$. A "continuous ambiguity" was dis-
covered in extracting the low-energy parameters (see Table I in Ref. 5), which is consistent with the data presented in Table I of this paper (selection criteria in addition to the
criterion of a minimum for $\chi^2$ were not considered in Ref. 5). Despite the considerable uncertainty in the values of the scattering lengths and the effective radius ($\sim 15\%$ for $\Re a_\beta$ and $30\%$ for $\Re r_{\lambda\nu}$), Barut and Sergeev demonstrated the advan-
tages of an approach based on expansion of the
effective radius over the $R$-matrix approach in the one-
level approximation, in which the charged channel radius can vary from 3 to 7 fm. In the more recent studies, this model parameter was set equal to 5 fm. However, the ambiguity
associated with the choice of the number of $R$-matrix poles
still remains. For example, the following values of the au-
thority function at zero have been given in the literature for
the $dt \to n\alpha$ reaction: $\Re a_\beta = 1.272 \pm 0.007$, 1.2S7, and 1.28S for one-level, two-level, and multilevel fits, respectively.
The $d\bar{t}$ system was also considered within the resonance
coupled channel model in Ref. 25. Conversion of the param-
eters obtained therein to the scattering length and the effec-
tive radius give values of $a_\beta$ and $r_\beta$ that are close to the set
with $m_\alpha = 0.30$ from Table I and $\Re r_{\lambda\nu} = 1.299$.
The values of $r(0)$ and especially of $e_\beta$ and $r_\beta$ have a considerable spread. The additional selection criteria that we
considered make it possible to reduce this uncertainty.

5. ANALYTIC STRUCTURE OF THE SCATTERING
AMPLITUDE NEAR THE ELASTIC THRESHOLD

The positions of the $S$-matrix poles on the complex $k$
plane (which correspond to both bound and virtual or quan-
titative levels) are determined from the equation
$\delta(0,k) = i\nu$, which takes the following form in the effective-
radial approximation:6

$$\lambda + [2 \phi(1 + \lambda^{-1}) + \ln \lambda] \frac{1}{\alpha_{\lambda\nu}} + \frac{1}{\alpha_{\lambda\nu}^2} = \nu_\lambda^2 k^2, \quad (24)$$

where $\lambda = -i\alpha_{\lambda\nu}$. This equation does not depend on the model of the strong (short-range) potential $V_\lambda(r)$.

It is known27 that in the absence of absorption, i.e., when we have $\Im \alpha_{\lambda\nu} = \Im r_{\lambda\nu} = 0$, the $S$-matrix poles for poten-
tials with a barrier are arranged symmetrically with respect to the imaginary axis in the $k$ plane, and $\Im r_{\lambda\nu} = 0$. When there are open channels, the nuclear interaction breaks this symmetry and displaces the Coulomb poles away from the negative imaginary semiaxis ($k_{\lambda\nu} = -i\alpha_{\lambda\nu} a_{\lambda\nu}$) for a "purely Coulomb" repulsive potential, where $n = 1, 2, \ldots$. In case of resonance, in which $|\alpha_{\lambda\nu}| > |r_{\lambda\nu}|$, there are two poles, $R$ and $R'$ with $\Im k_{\lambda\nu} = 0$ close to zero. A numerical solution of Eq. (24) with the parameters presented in Ref. 25 makes it possible to obtain two additional series of Coulomb poles, $k_{\lambda\nu} = -i\alpha_{\lambda\nu}$, which crowd together near the elastic threshold
$k_{\lambda\nu} = 0$, along with them. In the limit $n > 1$, for the poles in the right-hand half-plane ($\Re k_{\lambda\nu} > 0$) we have

$$\nu_{\lambda\nu} = n + \nu_0 - \frac{c_1}{(n + \nu_0)^2} + \frac{c_2}{(n + \nu_0)^3} + \ldots, \quad (25)$$

where $-1/2 < \Re \nu_0 < 1/2$.

$$\nu_{\lambda\nu} = \frac{1}{2\pi i} \ln \left(1 - 2ia_\beta \right), \quad c_1 = a_\beta^2 \left(1 - b \right), \ldots, \quad (25')$$

$\nu_{\lambda\nu} = 2\pi a_\beta^2 / b$, and $b = 3r_\beta / a_\beta$. It can be shown that the asymptotic expansion (25) has good accuracy even for $n > 1$. The series of Coulomb poles in the left-hand $k$ half-
plane is described by a similar expression.6

The positions of the poles in the $k$ plane that correspond to the low-energy parameters of the $dt$ system from Table II are given in Table III, which also presents the corresponding values of the variable $\nu_{\lambda\nu} - n$. It is seen from the table that, as $n$ increases, these values rapidly reach their limiting value $\nu_0$, which is specified by the ratio $a_\beta / a_\beta - (a_\beta - (bE))^1$. The presence of absorption in the $dt$ system, which breaks the symmetry between the left- and right-hand poles noted above, is especially pronounced for the imaginary parts of the resonance poles $R$ and $R'$. As $n$ increases, the Coulomb poles $k_{\lambda\nu}$ converge near the elastic threshold, which is essentially a special point of the scattering matrix, their residues decreasing rapidly.36

$$S(k) = \frac{\gamma_0}{E - E_\nu}, \quad E = E_{\nu}k^2 / 2, \quad (26)$$

$$\gamma_0 = (1 - \exp(-i\pi e_\nu)) \left(n\pi n_{\nu}+1\right)^2, \quad \Re \nu > 0. \quad (27)$$

Therefore, besides the resonance poles $R$ and $R'$, only the first few Coulomb poles make an appreciable contribution to the scattering amplitude. We note that the residues $\gamma_0 = 0$ when $a_\beta / a_\beta = 0$. Thus, the poles considered are directly related to the Coulomb interaction and vanish when it is turned off.

6. CONCLUSIONS

A model-independent approach to the investigation of the resonance low-energy scattering of charged particles has been developed using an expansion of the effective radius. It permits extrapolation of the fusion reaction cross section $\sigma(E)$ and the astrophysical function $a(E)$ from the reso-
nance region to the energies $E \to 0$. The low-energy param-
eters $a_{\lambda\nu}$, $r_{\lambda\nu}$, and $s(0)$ have been calculated for the $dt$ and $d^3$He systems. As is seen from Table II, the effective radius $r_{\lambda\nu}$, unlike $a_{\lambda\nu}$, is almost real. This is easily understood within the optical-potential model, which makes it possible to obtain the estimate $\Im r_{\lambda\nu}/(\Re r_{\lambda\nu} - W/T)$ (see Appendix

---

**Table III. Positions of the poles of the $dt$ scattering amplitude near the elastic threshold.**

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\nu_{\lambda\nu}$</th>
<th>$\nu_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.334 $- 0.868i$</td>
<td>1.659 $- 0.966i$</td>
</tr>
<tr>
<td>1</td>
<td>0.503 $+ 0.502i$</td>
<td>0.323 $+ 0.155i$</td>
</tr>
<tr>
<td>2</td>
<td>0.874 $- 0.426i$</td>
<td>0.303 $- 0.954i$</td>
</tr>
<tr>
<td>3</td>
<td>0.258 $- 0.304i$</td>
<td>0.306 $- 0.956i$</td>
</tr>
</tbody>
</table>

$^1$ The value $n = 0$ corresponds to the poles $R$ (right-hand half-plane, $\Re k > 0)$ and $R'$ (left-hand half-plane, $\Re k < 0$), and the values $n > 1$ correspond to two Coulomb series of poles.
The linear approximation error reaches interaction: $\Delta E_{Jv}$. E is measured in kiloelectron-volts. The linear approximation of the expansion (A3) at $E \approx 2$ keV for the $dt$ system we have accuracy better than 0.1%; however, when $E$ increases, the accuracy decreases. In the case of the $dt$ system for the energies $E \approx 70$ keV, which was justified because of the small penetrability of the Coulomb barrier. For example, $D_{Jv} \approx 5 \times 10^{-5}$ holds at $E = 8$ keV (see the values of $E_{0}$ and $E_{1}$ below). Hence we have $J = 0$ and $v = 0$ or 1 are the total angular momentum and the vibrational quantum number of the $d\mu$ mesomolecule, $R_{0}$ is the value of the wave function of the nuclear subsystem at zero (i.e., when the $d$ and $t$ nuclei “fuse”), and

$$E_{0} = \frac{m}{2} e_{0}.$$  

(A7)

where $e_{0}$ is the binding energy of the $d\mu$ mesomolecule, $\mu = m_{d}(m_{d} + m_{t})/(m_{d} + m_{t} + m_{u})$, and $m = m_{d} m_{t}/(m_{d} + m_{t})$. The values of $R_{0}$ and $e_{0}$ were calculated to high accuracy in the studies of Ponomarev et al. Using their numerical values, we have $E_{0} = 7.97$ keV, $E_{1} = 8.26$ keV, and $R_{0} = 7.27(-4)$. Hence for the set of parameters $\alpha_{0} = 0.27$ (see Table I) we obtain

$$\Gamma_{0} = (9.18 \pm 0.09) \times 10^{-3} \text{ eV},$$

$$\Gamma_{1} = (7.63 \pm 0.07) \times 10^{-3} \text{ eV},$$

$$\Delta E_{0} = - (10.3 \pm 1.0) \times 10^{-3} \text{ eV},$$

$$\Delta E_{1} = - (8.6 \pm 0.9) \times 10^{-3} \text{ eV}.$$  

These values are consistent with the values known in the literature, but they have a higher accuracy. 

d. In performing the numerical fit for the $dt$ system, whose results are presented in Tables I and II, we adopted the form factors $P = Q = 0$ in (1). We discuss the features of this approximation. 

It was shown in Ref. 8 in the case of the Yukawa, Hulthen, exponential, and Gaussian potentials that the form factors $P$ and $Q$ are numerically small. We carried out a more detailed investigation of this question. Using the equations presented in Ref. 8 [see Eqs. (9.14) and (9.15) in that paper], we performed a calculation of the low-energy parameters, including $P$ and $Q$, for various model potentials

$$V_{s}(r) = \frac{g}{2r} \left[ \frac{t}{x_{s}} \right]_{t} \cdot h = m = 1.$$  

(A9)

(1) in which $x_{s}$ was obtained for the resonance case, i.e., at the instant when the respective level appears. It is seen from Table IV that we have $-0.05 < C < 0.01$. As for the next term (in $x_{s}$) in the expansion (1), we can neglect it, since 1) $Q$ is generally several times smaller than $P$ (with the exception of the cases in which $P$ is anomalously small) and 2) the corresponding term in the expansion of the effective radius has an additional small parameter of order $(k_{F})^{2} - 0.1$ (in the case of the $dt$ system for the energies $E \approx 70$ keV, which are included in the treatment). The variation of $P$ from $-0.05$ to 0.05 leads to uncertainties in the low-energy parameters that exceed the range of accuracy of the present approach, i.e., $\alpha_{0} = 0.270 \pm 0.006$, $\beta_{0} = 0.110 \pm 0.005$, $s(0) = 1.292 \pm 0.001$, which justifies the choice of $P = Q = 0$ for the $dt$ system.
make a correction for the incomplete binding of the system comparison with the Coulomb interaction as \( r > R \). For this reason we should be included in the treatment in this case.

**APPENDIX B**

We give the necessary explanations for the derivation of Eqs. (15) and (17). For \( s \) states Eq. (7) takes the form

\[
\frac{d^2\psi}{dr^2} + \frac{1}{r} \frac{d\psi}{dr} + \left[ V_s(r) + \frac{1}{2} k^2 r^2 \right] \psi = 0,
\]

where \( r \gg R_s, \psi = \psi_0(r), \psi_0(r) \) is the wave function with a zero energy in the potential \( V_s(r)+\frac{1}{2}k^2 r^2 \), and \( u_s(r) \) is the so-called comparison function (Ref. 7):

\[
u_s(r) = \frac{\psi_0(r)}{\rho_0(r)}, \quad 0 < r < \infty \]  

i.e., the solution of the Schrödinger equation in the absence of the strong potential \( V_s(r) \), which has the same asymptotic form as \( \psi_0(r) \).

If we discard the last term in (B1) and take the minimum distance at which the strong interaction can be neglected in comparison with the Coulomb interaction as \( r = R_s \), we arrive at the inequality (17). At exact resonance (when \( u_s \to \infty \) or \( u_0 = 0 \)) it transforms into the bound (15) on \( R_s \), so that in the last two terms on the right hand side of (17) make a correction for the incomplete binding of the system or the resonance mismatch.

This correction is especially important when \( R_s \) is calculated in the case of the \( d^3 \)He system. It is seen from Fig. 5, in which the solid curve was constructed using Eq. (17) and the dashed curve was constructed using Eq. (15) and corresponds to \( \tilde{R}_s \), that consideration of the incomplete binding of the system for \( a_2 \leq 5 \times 10^{-3} \) significantly reduces the value of the radius \( R_s \). The range \( a_2 = 1 \times 10^{-3} - 3 \times 10^{-3} \) corresponds to the physically reasonable values \( R_s = 5 - 4.5 \) fm. We note that for the mirror systems under consideration \( R_s(d^3 \text{He}) < R_s(d^3 \text{He} \rangle = 5 \) fm holds, since the Coulomb interaction is two times weaker in the \( d^3 \)He system than in the \( d^3 \)He system. The point in Fig. 5 marks the value \( a_2 = 2 \times 10^{-3} \) (see Table II). In addition, the limiting value \( a_2 = 1/60 \) is shown.

We now obtain the lower bound for \( r_{cs} \). According to the meaning of \( R_s \), the strong interaction vanishes at \( r > R_s \), and the Coulomb interaction can be neglected in the range \( 0 < r < R_s \) we make the situation somewhat rougher in the narrow region where \( V_s(r) < V_c(r) \). Therefore, we now obtain the lower bound for \( r_{cs} \).

\[
\frac{1}{2} \tilde{R}_c = \int_0^R \left[ \frac{1}{2} \tilde{\chi}^2 + \tilde{\chi} \right] d\tilde{r}, \quad \tilde{\chi} = \tilde{\chi}(R_s), \quad \tilde{R}_c = 2 \int_0^{\tilde{R}_c} \left[ \frac{1}{2} \tilde{\chi}^2 + \tilde{\chi} \right] d\tilde{r},
\]

where \( \tilde{\chi}(r) \) is the wave function with \( E = 0 \) in the strong potential at the instant when an \( s \) level appears, which satisfies the usual boundary condition \( \tilde{\chi}(r = R_s) = 1 \) and differs from \( \chi_0(r) \) only by a scaling factor. Since

\[
\int_0^{\tilde{R}_c} \left[ \frac{1}{2} \tilde{\chi}^2 + \tilde{\chi} \right] d\tilde{r} = \frac{1}{6} \int_0^{\tilde{R}_c} H_{\tilde{D}}(\tilde{r}) d\tilde{r}, \quad \tilde{R}_c = 2 \int_0^{\tilde{R}_c} \left[ \frac{1}{2} \tilde{\chi}^2 + \tilde{\chi} \right] d\tilde{r},
\]

we obtain

---

**TABLE IV. Some parameters for short-range attraction potentials.**

<table>
<thead>
<tr>
<th>No.</th>
<th>( \psi(r) )</th>
<th>( E_0 )</th>
<th>( \Gamma_R )</th>
<th>( \Gamma_\rho )</th>
<th>( \bar{Q} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \exp(-\alpha x^2) )</td>
<td>1.67081</td>
<td>2.120</td>
<td>6.610(2)</td>
<td>2.800(2)</td>
</tr>
<tr>
<td>2</td>
<td>( \exp(-\alpha x^n) )</td>
<td>1.75002</td>
<td>1.070</td>
<td>-6.624(2)</td>
<td>-2.064(3)</td>
</tr>
<tr>
<td>3</td>
<td>( \exp(-\alpha x^n) )</td>
<td>1.64009</td>
<td>8.730(1)</td>
<td>-1.048(2)</td>
<td>-1.440(3)</td>
</tr>
<tr>
<td>4</td>
<td>( \theta (1-x) x )</td>
<td>1.44580</td>
<td>8.720(1)</td>
<td>-2.664(2)</td>
<td>1.450(4)</td>
</tr>
<tr>
<td>5</td>
<td>( (a^2-x)^{-1} )</td>
<td>1</td>
<td>5</td>
<td>3.012(2)</td>
<td>7.810(5)</td>
</tr>
<tr>
<td>6</td>
<td>( \sinh x )</td>
<td>6.31400(1)</td>
<td>3.330</td>
<td>2.030(2)</td>
<td>2.960(3)</td>
</tr>
<tr>
<td>7</td>
<td>( (x^2-1) )</td>
<td>3.04617</td>
<td>6.260(2)</td>
<td>-1.500(2)</td>
<td>-2.420(3)</td>
</tr>
<tr>
<td>8</td>
<td>( \exp(x) )</td>
<td>1.44580</td>
<td>3.540</td>
<td>1.246(1)</td>
<td>1.565(3)</td>
</tr>
<tr>
<td>9</td>
<td>( \exp(-x) )</td>
<td>2.66400</td>
<td>1.450</td>
<td>-1.812(2)</td>
<td>-9.480(4)</td>
</tr>
<tr>
<td>10</td>
<td>( \exp(-x) )</td>
<td>2.89527</td>
<td>1.060</td>
<td>-2.790(2)</td>
<td>5.710(4)</td>
</tr>
<tr>
<td>11</td>
<td>( \delta(x) )</td>
<td>2.5674(1)</td>
<td>1</td>
<td>-3.265(2)</td>
<td>1.77(3)</td>
</tr>
<tr>
<td>12</td>
<td>( (1-x)(1-x) )</td>
<td>5.8764(1)</td>
<td>4.647</td>
<td>4.94(2)</td>
<td>-7.30(4)</td>
</tr>
<tr>
<td>13</td>
<td>( (1-x)(1-x) )</td>
<td>3.25250(1)</td>
<td>4.856</td>
<td>-3.79(3)</td>
<td>-7.27(4)</td>
</tr>
<tr>
<td>14</td>
<td>( (1-x)(1-x) )</td>
<td>7.3335</td>
<td>7.870(1)</td>
<td>-2.785(2)</td>
<td>-5.24(4)</td>
</tr>
<tr>
<td>15</td>
<td>( (x+1)^{-1} )</td>
<td>1.72057</td>
<td>3.793</td>
<td>-5.33(3)</td>
<td>4.98(4)</td>
</tr>
<tr>
<td>16</td>
<td>( (x+1)^{-1} )</td>
<td>4.6746(1)</td>
<td>4.499</td>
<td>-4.99(4)</td>
<td>-5.99(4)</td>
</tr>
<tr>
<td>17</td>
<td>( (x+1)^{-1} )</td>
<td>3.66478</td>
<td>2.695</td>
<td>2.740(2)</td>
<td>-7.17(3)</td>
</tr>
<tr>
<td>18</td>
<td>( (1-x)(1-x) )</td>
<td>5.13770</td>
<td>8.240(1)</td>
<td>-2.942(2)</td>
<td>-9.99(4)</td>
</tr>
<tr>
<td>19</td>
<td>( \text{comb} )</td>
<td>7.71056(1)</td>
<td>3.052</td>
<td>-8.31(2)</td>
<td>3.08(3)</td>
</tr>
<tr>
<td>20</td>
<td>( \text{comb} )</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>21</td>
<td>( \text{comb} )</td>
<td>1</td>
<td>1</td>
<td>-3.750(2)</td>
<td>3.010(3)</td>
</tr>
<tr>
<td>22</td>
<td>( \text{comb} )</td>
<td>0</td>
<td>2</td>
<td>-4.167(2)</td>
<td>4.17(3)</td>
</tr>
</tbody>
</table>

---

On the other hand, for \( d^3 \)He we have \( (kr_s)^2 \leq 0.5 \) for \( E \leq 400 \) keV; therefore, the form factor [or \( a_2 \), see (2)] should be included in the treatment in this case.
Neglecting here the difference between $\bar{r}_i$ and $\bar{r}_s$, but taking into account the correction that is linear with respect to $a_0$ in (17), we arrive at the lower bound

$$r_i \approx \bar{r}_i + a_0 f_i(x_i) = \bar{r}_i + a_0 \frac{4}{3} H(x_i) \eta(x_i), \tag{B5}$$

which gives $r_i > 0.16a_B$ in the case of the $dt$ system. As a result, $r_i = 0$ holds for $0 < r < r_i$ and the integral discarded in (B1) equals zero. Also, $R_i = 2R_s$, and (B4) transforms into (15).

We present expansions of $\xi_0(r)$ and other functions at small and large distances. In the limit $r \to 0$ we have

$$\xi_0(r) = 1 + 2r \ln r + C_1 + O(r^2 \ln r), \tag{B7}$$

$$f_1(r) = \frac{8}{3} r^2 + \ldots, \quad f_2(r) = \frac{2}{3} r^3 + \ldots,$$

where $C_1 = \ln 2 - 1 + C_2$.

At large distances ($r \to \infty$), it is convenient to express these functions in terms of $\rho = \sqrt{2} r$:

$$\xi_0(r) = c_0 \rho^{1/2} e^{-\rho/3} \left( 1 + \frac{3}{8} \rho + \ldots \right), \tag{B8}$$

$$f_1(r) = \frac{1}{2} \rho^2 + \ldots, \quad f_2(r) = \frac{1}{128} \rho^3 + \ldots,$$

where $c_0 = \sqrt{2}$. Since the functions $f_1$ and $f_2$ increase rapidly as $r \to 1$, the inequality (17) becomes meaningless, if $R_s > a_B$ [however, in the case of the $dt$ and $d^4$He systems, the bound (17) is fully effective].

**APPENDIX C**

We illustrate the dependence of the low-energy scattering parameters ($a_0$, $r_i$, etc.) on the depth of the strong potential in the case of a rectangular well:

$$V_s(r) = -\frac{8}{2R_s^3} (R - r).$$
the ns level appears). Expanding (C3) and (C4) near a pole, we obtain

$$a_r = \frac{1}{\kappa_s (\kappa_s' - \kappa_s)} + \frac{1}{\kappa_s'} \left[ \frac{1}{\kappa_s} \left( \frac{1}{\kappa_s} \kappa_s' - \cdots \right) \right] R,$$

$$r_s = 1 - \frac{u + i w}{\kappa_s'} + \cdots, \quad R,$$

$$P = \frac{-1}{12} \left[ \frac{1}{2 \kappa_s'^2} \left( \frac{3}{2} \kappa_s'^2 (u + i w) + \cdots \right) \right] R,$$

Here \(w = (U - U_j)/S\), \(w = W_1 U\), \(U_j = e_{2R}^2\), and \(U_1 = A_1^2 / \kappa_s'^2\) are the potential (well). We also present the expansion for the dimensionless parameter \(\xi:\)

$$\xi = \frac{1}{2w} \left[ \frac{d^2}{d R^2} \left( \frac{w}{\kappa_s^2} \right) \right] R$$

$$\text{Im} \ r_s = -\frac{w}{\kappa_s'^2} + \cdots - \frac{W}{\kappa_s'^2}$$

with consideration of the fact that \(W < U\) holds at low energies.

\(^{1}\)Besides the criterion of a minimum for \(\xi^2\).

\(^{2}\)In this section we consider the case of arbitrary \(l\), since the transition to \(l = 0\) does not introduce any appreciable simplifications. We also neglect the absorption in the system.

\(^{3}\)Here and in the following the order of magnitude of the number is indicated in parentheses, i.e., \(a_{(\cdot)} = a_{(\cdot)} \cdot 10^{(\cdot)}\).

\(^{4}\)The conditions \(R \approx 0\) and \(|Re \, a_1| > r_s\) mean that even slight deepening of the strong potential \(V(r)\) is sufficient for the appearance of a bound state in it.

\(^{5}\)For example, \(a = 3\), if \(a = 0.28\) (for the values of the other parameters, see Table I).

\(^{6}\)In \(l_z = 0\), holds, then we have \(n = 0\), and the unitary limit is achieved for the \(d\rightarrow n\) reaction: \(|S_{d, n}| = 1\). The set with \(n = 0.29\) in Table I corresponds to this case.

\(^{7}\)Direct measurements of the reaction cross sections at such low energies are hardly possible. At the same time, exact values of \(a_{(\cdot)}(E)\) are of interest for theoretical research, especially for calculating fusion reactions in astrophysics. We thank Yu. V. Petrov for drawing our attention to the problem of calculating the astrophysical function into this energy range.

\(^{8}\)Here we assume \(Z_i Z_j > 0\). The case of Coulomb attraction, in which the system has a spectrum of atomic levels that is distorted by the strong interaction at short distances \((r < a_{(\cdot)}^2)\), has been considered (on the basis of a similar equation) both for \(Z\) levels in and for states with a nonzero angular moment \(l\) (Ref. 26).

Translated by P. Shelest

Popov et al. 858