Dynamics of soliton-like wave signals propagating in smoothly inhomogeneous and weakly nonstationary nonlinear media

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By employing the nonlinear Klein–Gordon equations with small nonlocal corrections we develop a self-consistent description of one-dimensional soliton-like wave signals in smoothly inhomogeneous nonstationary nonlinear media. For the quasisoliton velocity and the field’s integral characteristics we develop a closed system of ordinary differential equations written in a form common in relativistic mechanics. We study the interaction, accompanied by frequency conversion, of quasisolitons with the waves of the medium parameters; in particular, the laws that govern the penetration of these waves into the bulk of an advancing dense-plasma barrier and their reflection from the barrier. Finally, we demonstrate the possibility of the signal frequency growing considerably without temporal-spectrum broadening by employing the example of relativistic quasisolitons propagating in an initially homogeneous and stationary plasma with small additional ionization in the medium. © 1996 American Institute of Physics. [S1063-7761(96)028609-0]
Equation (2) shows that in the limit $p=0$, which corresponds to a homogeneous stationary medium with purely local nonlinearity, Eq. (1) becomes the nonlocal Klein–Gordon equation of the type

$$\frac{\partial^2 \psi}{\partial t^2} - c^2 \frac{\partial^2 \psi}{\partial x^2} + V(\lvert \psi \rvert^2) \psi = 0.$$  

(3)

Suppose that Eq. (3) has solutions that are localized and at rest in space:

$$\psi = \Psi_0(x) \exp(i(\omega_0 t + \phi_0)),$$

(4)

where $\Psi_0$ is a real function that satisfies the ordinary differential equation

$$\frac{d^2 \Psi_0}{dx^2} + \frac{\omega_0^2 - V(\lvert \Psi_0 \rvert^2)}{c^2} \Psi_0 = 0.$$  

(5)

Then, in view of its Lorentz invariance, Eq. (3) (along with Eq. (4)) also defines a complete set of traveling solitons, where $\omega_0$ is a real function that satisfies the ordinary differential equation

$$\frac{d^2 \omega_0}{dx^2} + \frac{\omega_0^2 - V(\lvert \Psi_0 \rvert^2)}{c^2} \omega_0 = 0,$$

(6)

whose velocity $v$ is lower than $c$ ($v<c$). In the solution (6), $\omega_0$ is the frequency of field oscillations in the soliton’s “proper” reference frame ($t' = (t - vx/c)/\sqrt{1 - v^2/c^2}$ and $x' = (x - vt)/\sqrt{1 - v^2/c^2}$ are the soliton’s “proper” time and position); $\omega_v = \omega_0/\sqrt{1 - v^2/c^2}$, $k = \omega/c$, and $q = c/v$ are the frequency, wave number, and phase velocity of the soliton in the laboratory reference frame; and $\phi_0$ is an arbitrary constant phase shift.

By way of an example we take a medium with cubic nonlinearity

$$V(\lvert \psi \rvert^2) = \omega_v^2 (1 - \alpha \lvert \psi \rvert^2),$$

(7)

characterized by two parameters, $\omega_v^2$ and $\alpha$ (for a plasma $\omega_v$ is the square of the plasma frequency, and $\alpha$ is determined by the type of nonlinearity). Here $\Psi_0(x,t)$ has the simple analytic form

$$\Psi_0(x,t) = \frac{\omega_v^2 \sqrt{1 - \alpha \lvert \psi \rvert^2}}{\cos \left( \frac{x}{\sqrt{1 - \alpha \lvert \psi \rvert^2}} - \frac{\omega_v^2}{c} \right)},$$

(8)

where the frequency of oscillations in the “proper” reference frame, $\omega_v$, can vary between 0 and $\omega_v$ ($0<\omega_v^2<\omega_v^2$).

We now return to the original equation (1) with the operator $V$ defined by (2).

If the soliton solutions (6) exist with the potential $V = V(\lvert \psi \rvert^2)$, it is natural to assume that in a smoothly inhomogeneous medium with slowly varying parameters ($\mu=0$), certain pulsed signals that are close to (6) in structure can propagate, at least along finite propagation paths. Reasoning from this assumption, we solve Eqs. (1) and (2) in the form that is asymptotic in $\mu$:

$$A = A_\mu(\eta, \mu_1) + \mu A_1(\eta, \mu_2) + \cdots \exp(i\phi).$$

(9)

Here the field’s phase $\phi$, which allows for linear corrections of the phase wavefront of the signal, is described by the following expression:

$$\phi = \phi_0 + \int_0^t \left( \omega(\mu_1 t') - k(\mu_1 t') \right) dt' - k(\mu_1) \eta$$

$$+ \mu \int_0^t \left( \omega(\mu_1 t') - k(\mu_1 t') \right) dt' - k(\mu_1) \eta,$$

where $\phi_0$ is const; $\eta = x - xo(t)$ is measured from the signal’s center $xo(t)$, whose propagation velocity is $v = dx_d/dt; \omega(\mu_1)$ and $k(\mu_1)$ are the frequency and wave number ($d\omega/dx_d\mu = k$); and $\mu_0$, $\mu_1$, and $\mu_k$ are small corrections to the frequency and wave number.

We plug (9) into (1) and (2). Then in zeroth-order perturbation theory in the small parameter $\mu$, we have

$$\begin{align*}
V = V(\mu xo(t), \mu xo(t)), \\
\mu xo(\mu xo(t)), A_0 = 0, \\
2i(\omega - k c^2) \frac{\partial A_0}{\partial \eta} = 0,
\end{align*}$$

(10)

which implies

$$A_0 = \Psi_0 \left( \frac{\eta}{\sqrt{1 - \alpha (\mu xo(t)/c)^2}} \right),$$

(11)

$$\omega = \omega_0 \sqrt{1 - \alpha (\mu xo(t)/c)^2},$$

where $\Psi_0$ is the solution of Eq. (5) with

$$V = V(\mu xo(t), \mu xo(t)).$$

(12)

Actually, the expressions (11) reflect the expected quasi-soliton structure of the pulsed signals that at time $t$ are at the point with coordinate $xo(t)$. Zeroth-order perturbation theory provides no insight into propagation of the signal (11) and the variation of the “proper” frequency $\omega_0(\mu_1)$. Hence we write the inhomogeneous equation for $A_\mu(\eta, \mu_1)$ obtained via first-order perturbation theory in $\mu$ after plugging (9) into (1) and (2):

$$\begin{align*}
\left( \omega^2 - c^2 - k^2 c^2 - V(\mu xo(t) + \mu_1 xo(t)) \right) A_1 + \\
2 \left( \omega \mu xo(t) + \mu xo(t) \right) \delta A}{\partial \eta} &= F_1 - iF_2, \\
F_1 &= \left( \omega + \frac{\delta A}{\delta \eta} \right) A_2 + 2i \mu xo(t) \frac{\partial A_1}{\partial \eta} + 2 \mu xo(t) \frac{\partial A_0}{\partial \eta} - \\
&- \Im \left[ \epsilon e^{i(\delta A + \mu xo(t))} \right] \mu xo(t) + \mu xo(t) \frac{\partial A_0}{\partial \eta}, \\
F_2 &= \frac{\partial A_0}{\partial \eta} + 2 \mu xo(t) \frac{\partial A_1}{\partial \eta} + 2 \mu xo(t) \frac{\partial A_0}{\partial \eta} - \\
&- \Re \left[ \epsilon e^{i(\delta A + \mu xo(t))} \right] \mu xo(t) + \mu xo(t) \frac{\partial A_0}{\partial \eta}.
\end{align*}$$

The right-hand side of Eq. (12) is determined by the derivatives of the soliton parameters (11) with respect to the “slow” time $t = \mu_1 t$ and the “slow” coordinate $x = \mu_1 x$; and
the left-hand side is the result of linearizing (10). The homogeneous equation that follows from (12) if we set \( F_1 \) and \( F_2 \) to zero has two solutions localized in \( \eta \):

\[
A^{(1)} - iA_d(\eta, \mu_t), \quad A^{(2)} = \frac{\partial A_d(\eta, \mu_t)}{\partial \eta}.
\]

For the inhomogeneous equation (12) to have a solution localized in \( \eta \), the functions \( F_1(\eta, \mu_t) \) and \( F_2(\eta, \mu_t) \), according to the Fredholm alternative theorem, must be "orthogonal" to \( A^{(1)}(\eta, \mu_t) \) and \( A^{(2)}(\eta, \mu_t) \), respectively:

\[
\int_{-\infty}^{\infty} F_1(\eta, \mu_t) A_d(\eta, \mu_t) d\eta = 0,
\]

\[
\int_{-\infty}^{\infty} F_2(\eta, \mu_t) \frac{\partial A_d(\eta, \mu_t)}{\partial \eta} d\eta = 0.
\]

If these conditions are not met, the additional term \( A_d(\eta, \mu_t) \) builds up in a resonant manner in the process of the quasisoliton's motion, which rather rapidly leads to a breakdown of the representation (9) with \( F_1 \) and \( F_2 \).

After integrating by parts several times with allowance for the orthogonality conditions (14):

\[
\int_{-\infty}^{\infty} F(\eta, \mu_t) A_d(\eta, \mu_t) d\eta = 0,
\]

we arrive at the following expressions for the orthogonality conditions (14):

\[
dN = X, \quad d = \frac{m_0 \omega}{c^2 \sqrt{1 - \omega^2 c^2}} \frac{dN}{d\mu_t} - \frac{\omega_0 \omega}{c^2 \sqrt{1 - \omega^2 c^2}} dN = -N, \quad d\Sigma, \quad (15, 16).
\]

where we have introduced new notation for the integral characteristics of the field and medium:

\[
N = \omega \int_{-\infty}^{\infty} A_d^2 \psi(\xi, \tau) d\xi, \quad m_0 = \int \left( \frac{d\psi}{d\xi} \right)^2 + \frac{\omega_0}{c^2} \psi^2 d\xi, \quad (17, 18)
\]

\[
W_d(\xi, \tau) = \int_0^{\infty} \left( \psi(\xi, \tau) \right)^2 d\xi, \quad (19)
\]

\[
\chi = \int_{\tau_0}^{\infty} \left( \psi(\xi, \tau) \right) d\xi, \quad (20)
\]

\[
\Sigma = \Re \int_{\tau_0}^{\infty} \left( e^{-i \varphi} \psi d\xi \right) d\xi, \quad (21)
\]

In what follows we call \( N \) the "number of quanta," \( m_0 \) the "rest mass," and \( W_d(\xi, \tau) \) the "effective potential energy" of the soliton-like signal (9). The quasisoliton’s integral parameters \( N \) and \( m_0 \) and the "proper" frequency are expressed in terms of one another at any fixed values of the arguments \( x = x_d(i) \) and \( \tau \) of the potential \( V(\xi, \tau, \psi, \psi^*) \).

Both the number of quanta \( N \) and the rest mass \( m_0 \) are independent of the quasisoliton velocity, and they are completely determined by the proper frequency \( \omega_0 \) and the properties of the medium at the field location.\(^{15}\)

The variation of \( N \) is related to the operator \( \delta \), which takes weak dissipative and nonlocal nonlinear processes into account. However, if this operator acts on the soliton \( \Psi \) and does not change the phase \( \phi \) of the field, \( \chi \) in Eq. (15) vanishes (\( \chi = 0 \)) and \( N \) is conserved in the course of the quasisoliton’s motion \((N = \text{const})\).

The left-hand side of Eq. (16) is written in a form common in relativistic dynamics \((m_0 \neq m)\), the rest mass, \( m = m_0 / \sqrt{1 - \omega^2 c^2} \) is the mass in the laboratory reference frame, and \( p = m v \) is the momentum. Extending the analogy between the quasisoliton (9) and a particle, we can call the right-hand side of Eq. (16) a force. The first term in the force is expressed in terms of the \( x \)-derivative of \( W_d(\xi, \tau) \) at point \( x = \mu_x(t) \), which suggests interpreting \( W_d \) as the effective potential energy of the signal. Equation (19), which at \( \tau = t \) defines \( W_d(\xi, \tau) \), is obtained as a result of averaging the soliton’s "potential energy density"

\[
W = \int_0^{\infty} \left( \psi(\xi, \tau, \psi, \psi^*) d\psi_d^2 \right).
\]
The right-hand side of Eq. (22) contains the partial derivative of $W_{\text{eff}}(\vec{x},\vec{i},\vec{T})$ with respect to $\vec{r}$, which vanishes in stationary media ($v(\vec{u},\vec{i},\vec{T})=v(\vec{u},\vec{T})$). The second term gives the variation of the quasisoliton mass due to creation or annihilation of quanta with energy $\omega c^2$. The last term, which affects the mass of the signal, is proportional to the work done by the force $\Sigma$ per unit time, or $v\Sigma$.

For a medium with the cubic nonlinearity (7) and $\alpha=\text{const}$, $\omega_0=\omega_0(\vec{x},\vec{T})$, the number of quanta $N$, the rest mass $m_0$, and the effective potential energy of a quasisoliton are

$$\begin{align*}
N &= N_0 - \frac{\omega_0^2 - \omega_r^2}{\alpha} \omega_0, \\
m_0 &= m_0 + \frac{2\omega_0^2 + \omega_r^2}{\alpha^2}, \\
W_{\text{eff}} &= N_0 - \frac{\omega_0^2 - \omega_r^2}{\alpha} \omega_0, \\
\text{where } \omega_r^2 &= \omega_0^2 \left(1 + \frac{2\alpha^2 - \omega_0^2}{\omega_r^2}\right),
\end{align*}$$

Here the number of quanta $N$ cannot exceed the maximum permissible value $N_{\text{max}}=2\alpha/\alpha$ ($0<\alpha=N_{\text{max}}$), which implies equality of the proper frequencies $\omega_0^{(0)}$ and of the rest masses $m_0^{(0)}$:

$$\begin{align*}
\omega_0^{(0)}(N=N_{\text{max}}) &= \frac{\omega_0}{\sqrt{2}}, \\
m_0^{(0)}(N=N_{\text{max}}) &= \frac{\sqrt{2}N_0\omega_0}{3\alpha^2}.
\end{align*}$$

When $N=N_{\text{max}}$, the two proper frequencies $\omega_0^{(0)}$ and the two rest masses $m_0^{(0)}$ differ considerably:

$$\begin{align*}
\omega_0^{(-)} &= \omega_0^{(0)}, \\
m_0^{(-)} &= \frac{N_{\text{max}}\omega_0}{2\alpha^2}, \\
m_0^{(+)}, m_0^{(+)} &= \frac{N_{\text{max}}\omega_0}{c^2}.
\end{align*}$$

The “light” quasisoliton with rest mass $m_0^{(-)}$ satisfies the quasimonochromatic condition $1/\omega_0^{(-)} < 2\pi$, while the “heavy” quasisoliton with rest mass $m_0^{(+)}$ does not.

3. INHOMOGENEOUS NONSTATIONARY MEDIUM WITH LOCAL NONLINEARITY

Let the operator $H$ (in 2) be identically zero ($H=0$) or, in other words, let the medium be locally nonlinear and nonabsorbing. This means that the number of quanta $N$ is conserved ($N=\text{const}$) and that Eqs. (16) and (22) describing the dynamics of a quasisoliton signal assume the following form:

$$\begin{align*}
\frac{d}{dt} \frac{m_0}{\sqrt{1-\nu^2}} &= -\frac{c}{2} \frac{\partial}{\partial T} W_{\text{eff}}(\vec{x},\vec{T}), \\
\frac{d}{dt} m_0 &= \frac{c}{2} \frac{\partial}{\partial T} W_{\text{eff}}(\vec{x},\vec{T}),
\end{align*}$$

Here we see that the two simplest limiting cases are an inhomogeneous stationary medium (in which case $\partial W_{\text{eff}}(\vec{x},\vec{T})/\partial T=0$) and a homogeneous nonstationary medium (in which case $\partial W_{\text{eff}}(\vec{x},\vec{T})/\partial T=0$), whereupon the quasisoliton’s mass $m=m_0/\sqrt{1-\nu^2}$ or momentum $p=um=m_0/\sqrt{1-\nu^2}$ is conserved, respectively.

In a stationary situation ($\nu=\text{const}$) the medium “does not perform work on the quasisoliton,” and the medium’s inhomogeneity only changes the velocity $\nu$, which for media with cubic nonlinearity is equivalent to preservation of the frequency $\omega$ ($\omega=\text{const}$) and alteration of the wave number $k$ ($\omega/c=\text{const}$). From (23) and (24) we can easily obtain

$$\begin{align*}
\omega_0^{(0)} &= A_{11} \omega_0, \\
m_0^{(0)} &= A_{22} \frac{2\alpha^2 \omega_0^2 + 1}{A_{11}},
\end{align*}$$

where

$$A_{11} = \frac{1}{2} \frac{1}{\sqrt{1-\alpha N^2}}.$$
Initiated by Nonlocal Nonlinear Processes

We analyze the self-action of quasisoliton signals in initially stationary homogeneous media,

$$\frac{\partial}{\partial t} \text{W}_{\text{eff}}(\vec{x},t_1,\vec{j}) = 0, \quad \frac{\partial}{\partial t} \text{W}_{\text{eff}}(\vec{x},t,\vec{j}) = 0,$$

which simultaneously exhibit both local and nonlocal nonlinear properties ($\delta \neq 0$). By way of example, we examine the propagation of relativistic quasisolitons in a plasma in the presence of weak additional ionization. It is convenient to express the electric field $E$, the magnetic field $H$, and the velocity $v_x$ of ordered electron motion in this problem in terms of the vector potential $A(x,t) = A_x(x,t)\hat{y} + A_y(x,t)\hat{z}$.
where $n_0$ and $e$ are the electron mass and charge.

We ignore bulk electron losses during the passage of the pulse and describe the relative perturbation of plasma concentration due to additional ionization, $\Delta n = \Delta n/m$, where $n$ is the plasma concentration, with the following model equation:

$$\frac{\partial n}{\partial t} = n_0 (E)^2 - \frac{e}{m_e c^2} \left( \frac{1}{2} \frac{e A}{m_e c^2} \right)^2 A + \omega_0^2,$$

where $n_0$ is a coefficient with dimensions of frequency, and $E_i$ is the characteristic field, which governs the intensity of additional ionization. By assuming that the relative perturbation of the electron mass caused by relativistic motion, $\delta m = mc^2/2c^2$, is much larger than $\delta m = mc^2/2c^2$, which means that nonlinear relativistic effects are much stronger than ionization effects, we can easily derive the following equation for $A(x,t)$:

$$\frac{\partial A}{\partial t} - \frac{c}{4\pi} \frac{\partial A}{\partial t} + \frac{1}{2} \left( \frac{1}{2} \frac{e A}{m_e c^2} \right)^2 + \frac{\mu_0}{c E_i^2} \frac{\partial A}{\partial t} = 0,$$

where

$$\frac{1}{4\pi} \frac{\partial A}{\partial t} \left( \frac{1}{2} \frac{e A}{m_e c^2} \right)^2,$$

is the current of the electrons produced with zero velocity $v_0$ and $\omega_0 = 4\pi c^2 N/m_0$ is the plasma frequency.

For a circularly polarized wave, in which $A_x = A e^{i\phi} = A e^{i\phi} = A e^{i\phi}$, instead of a vector equation for $A$ we can write an equivalent equation for the complex-valued amplitude $A = A_x + iA_y = A e^{i\phi}$.

$$\frac{\partial A}{\partial t} - \frac{c}{4\pi} \frac{\partial A}{\partial t} + \frac{1}{2} \left( \frac{1}{2} \frac{e A}{m_e c^2} \right)^2 + \frac{\mu_0}{c E_i^2} \frac{\partial A}{\partial t} = 0,$$

where

$$\frac{\partial A}{\partial t} \left( \frac{1}{2} \frac{e A}{m_e c^2} \right)^2.$$

According to Eqs. (37) and (15)–(21), when relativistic quasisolitons propagate in a plasma, additional weak polarization results in a simultaneous decrease in the number of quanta $N$ and conversion of the central frequency $\omega$ of the signal, since $\chi \neq 0$ and $\xi \neq 0$. If Eqs. (15), (16), and (22) are applied in the quasimonochromatic limit ($\omega > 1$, with $r$ the duration of the signal),\(^6\) we easily obtain the following equations for $N$ and $\omega$:

$$dN/dt = -2 \gamma N^3, \quad d\omega/dt = \gamma w u N^2,$$

where

$$\gamma = \frac{1}{24} \frac{\nu e^2 a_0^2}{E_i^2 m_e c^2}.$$

The solution of the system (38) with initial conditions $N(t=0) = N(0)$ and $\omega(t=0) = \omega(0)$ is

$$\omega = \omega^{0}(1 - 4\pi N^{0}/(1+4\pi N^{0}))^{1/4},$$

where

$$N = N^{0}(1 - 4\pi N^{0}/(1+4\pi N^{0}))^{1/4},$$

We see that additional ionization can considerably increase the frequency of a soliton-like signal whose field structure is preserved by relativistic nonlinearity. The signal remains quasimonochromatic, and as the central frequency of the signal increases, the signal’s temporal spectrum narrows and its duration increases accordingly. Significant conversion of the signal frequency without broadening the temporal spectrum sets the problem considered here apart from that studied in Ref. 17, where ionization nonlinearity dominates and no quasisolitons can exist.

5. CONCLUSION

The self-consistent description of one-dimensional soliton-like signals in smoothly homogeneous and nonstationary nonlinear media developed in this work reduces the wave problem to the solution of a closed system of ordinary differential equations for the quasisoliton velocity and the integral characteristics of the wave field. It has proved possible to write this system of equations in a familiar relativistic form, and to interpret the quasisoliton as a set of coupled quanta to which an integral parameter, acting as the effective mass, can be assigned. The number of quanta, the effective mass, and the quasisoliton velocity uniquely determine the field’s frequency. The theory makes it possible to examine specific problems, such as the possibility of conversion of the carrier frequency of pulsed wave signals without broadening their temporal spectra in media with combined local and nonlocal nonlinearities, and the interaction of quasisolitons with waves in the parameters of the medium.

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\(^6\)Here we are dealing with the “tradiional” quasioptics method, in which the carrier frequency and the field’s “central” wave vector are fixed. The ideas of generalizing the quasioptical approximation to arbitrary smoothly homogeneous stationary media proposed in Refs. 13 and 14 are close to the situation analyzed in the present work.

\(^7\)Multiplying the real and imaginary parts of Eq. (12) by $A_x(\eta,\mu)$ and $A_y(\eta,\mu)$ and integrating the result with respect to $\eta$ from $-\infty$ to $+\infty$, we arrive at relationships determining the small corrections to the frequency $\omega$, and the wave number $k$, in terms of the integral perturbation characteristics $e^{i\eta} A_{\eta}(\eta,\mu)$ and $\mu_0 A_{\eta}(\eta,\mu)$. If the conditions (15) and (16) are met, there is no buildup of these corrections in the course of quasisoliton’s motion.

\(^8\)The linear limit these aspects are examined in Refs. 1–3.
where $O(\omega^2)$ stands for terms that are second order in $\omega$.