# Nonlinear oscillations of a degenerate $He^3-He^4$ solution

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We derive an exact system of nonlinear equations that describe the dynamics of a degenerate $He^3-He^4$ solution, and compute the nonlinear correction to the propagation velocity of first sound (proportional to the amplitude) in the collisionless limit. Our nonlinear treatment requires the third functional derivative of the total energy with respect to the distribution function of $He^3$ quasiparticles in addition to the second functional derivative (the Landau function). We find the Fermi-liquid parameters in the limit of small $He^3$ concentrations from a microscopic calculation, and obtain an explicit expression for the correction. © *1996 American Institute of Physics.* [S1063-7761(96)01105-5]

## 1. INTRODUCTION

We will consider degenerate normal $He^3$ dissolved in $He^4$ to form a non-stratified solution with arbitrary concentration. The temperature is assumed to be so low that the contributions of phonons and rotons can be neglected. The dynamic properties of the system are described by a distribution function $n(\mathbf{p},\mathbf{r},t)$ for the interacting $He^3$ quasiparticles, and the density $\rho_4(\mathbf{r},t)$ and superfluid velocity $\mathbf{v}_s(\mathbf{r},t)$ of the $He^4$. A previous paper by Khalatnikov[1] solved the problem of finding a complete system of dynamical equations for these variables. Although the equations derived in this paper are correct in linear approximation, some of them differ significantly from those derived in the present paper. We will discuss this question at the end of the third section of the article after we have derived our nonlinear equations.

Even in pure $He^4$, nonlinear effects lead to a dependence of the velocity of sound on wave amplitude, and hence to changes in the profile of a sound wave. In a solution, the $He^3$ quasiparticles participate in the acoustic vibrations, which leads to a dependence of the nonlinear correction on the Fermi-liquid parameters. Because of the nonlinear character of Fermi-liquid oscillations, the very possibility of describing them using kinetic equations is often questioned. This problem was analyzed in Ref. 2, where it was shown that if

$$\frac{\delta\rho}{\rho}\ll\frac{\hbar\omega}{E_F},$$

where $E_F$ is the Fermi energy, $\rho$ is the density of the solution, and $\omega$ and $\delta\rho$ are the frequency and amplitude of the density-wave oscillations, then $He^3$ can be described to accuracy up to quadratic terms by the Boltzmann equation without a collision integral. The correction we find in this paper to the velocity of sound depends not only on the Landau function for the quasiparticle interactions, but also on the third functional derivative of the Fermi liquid energy of the $He^3$. Because the expressions that appear here are quite cumbersome, we will often put them in appendices to the basic text.

In the next part of the paper we find the parameters of the Fermi liquid required to compute the corrections, calculate them from a microscopic theory for the case of low-concentration solutions, and use them in a general explicit expression. We compare our expression for the correction to the velocity with the nonlinear correction to the velocity of sound in pure $He^4$ found previously by Khalatnikov *et al.*[1]

## 2. DYNAMIC EQUATIONS

Let $E_c$, $\mathbf{P}_c$ be the energy and momentum (per unit volume) of the liquid in a reference frame moving with the velocity of superfluid motion $\mathbf{v}_s$. Then the energy and momentum of the liquid in the rest (laboratory) frame will equal

$$E=E_c+\mathbf{P}_c\mathbf{v}_s+\frac{\rho\mathbf{v}_s^2}{2}, \tag{1}$$

$$\mathbf{J}=\mathbf{P}_c+\rho\mathbf{v}_s, \tag{2}$$

where $\rho_4$ is the density of $He^4$, $\rho=\rho_4+m\langle n_c\rangle$ is the density of the $He^3-He^4$ solution, $m$ is the mass of a $He^3$ atom, $n_c$ is the distribution function of $He^3$ quasiparticles in the comoving frame, and $\langle\ldots\rangle=\int\ldots d\tau$ $(d\tau=2d^3\mathbf{p}/(2\pi\hbar)^3)$. In the laboratory and co-moving frames the $He^3$ quasiparticles are characterized by real momentum, and the value of the total Hamiltonian of the system equals its energy. In the co-moving system

$$E_c=E_c(\{n_c\},\rho_4)=\tilde{E}_c(\{n_c\},\rho), \tag{3}$$

$$\mathbf{P}_c=\int\mathbf{p}n_c(\mathbf{p},\mathbf{r},t)d\tau=\langle\mathbf{p}n_c\rangle, \tag{4}$$

where $\{n_c\}$ denotes a functional dependence on $n_c$. The energy of a single quasiparticle in the corresponding system is the functional derivative $\delta E/\delta n$ for $\rho_4$ = const:

$$\varepsilon_c=\left(\frac{\delta E_c}{\delta n_c}\right)_{\rho_4}. \tag{5}$$

From (2) we have

$$\rho_4\mathbf{v}_s+\langle\mathbf{p}n\rangle=\langle\mathbf{p}n_c\rangle+m\mathbf{v}_s\langle n_c\rangle+\rho_4\mathbf{v}_s,$$

from which we obtain

$$\langle\mathbf{p}n\rangle=\langle(\mathbf{p}+m\mathbf{v}_s)n_c\rangle$$

  

and consequently

$$n(\mathbf{p})=n_c(\mathbf{p}-m\mathbf{v}_s). \tag{6}$$

Now, having derived (1)–(5), we can compute the Hamiltonian of the quasiparticles in the laboratory frame:

$$H=\left(\frac{\delta E}{\delta n}\right)_{\rho_4,v_s}=\varepsilon(\mathbf{p}-m\mathbf{v}_s)+\mathbf{p}\mathbf{v}_s-\frac{m\mathbf{v}_s^2}{2}. \tag{7}$$

The total momentum of the solution from (2) is

$$\mathbf{J}=\mathbf{P}_c+\rho\mathbf{v}_s=\langle \mathbf{p}n_c\rangle+\rho\mathbf{v}_s=\langle \mathbf{p}n\rangle+\rho_4\mathbf{v}_s. \tag{8}$$

Because the solution is a closed system, we must satisfy the conservation law

$$\frac{\partial J_i}{\partial t}+\frac{\partial \Pi_{ik}}{\partial r_k}=0. \tag{9}$$

Let us substitute (8) into (9),

$$\rho_4\frac{\partial v_{si}}{\partial t}+v_{si}\frac{\partial \rho_4}{\partial t}+\frac{\partial\langle p_i n\rangle}{\partial t}+\frac{\partial \Pi_{ik}}{\partial r_k}=0; \tag{10}$$

we will determine $\partial n/\partial t$ from the kinetic equation

$$\frac{\partial n}{\partial t}+\frac{\partial n}{\partial \mathbf{r}}\frac{\partial H}{\partial \mathbf{p}}-\frac{\partial n}{\partial \mathbf{p}}\frac{\partial H}{\partial \mathbf{r}}=\mathrm{St}\, n. \tag{11}$$

The law of conservation of momentum gives $\langle \mathbf{p}\ \mathrm{St}\, n\rangle=0$. Therefore, multiplying (11) by $p_i$ and averaging over $\mathbf{p}$, we obtain

$$\left\langle p_i\frac{\partial n}{\partial t}\right\rangle=-\left\langle p_i\frac{\partial n}{\partial r_k}\frac{\partial H}{\partial p_k}\right\rangle+\left\langle p_i\frac{\partial n}{\partial p_k}\frac{\partial H}{\partial r_k}\right\rangle$$
$$=-\frac{\partial}{\partial r_k}\left\langle p_i\frac{\partial H}{\partial p_k}n\right\rangle-\left\langle\frac{\partial H}{\partial r_i}n\right\rangle$$
$$=-\frac{\partial}{\partial r_k}\left(\left\langle p_i\frac{\partial H}{\partial p_k}n\right\rangle+\delta_{ik}\langle Hn\rangle\right)+\left\langle H\frac{\partial n}{\partial r_i}\right\rangle.$$

Using the fact that

$$\frac{\partial n(\mathbf{p})}{\partial r_i}=\left.\frac{\partial n_c(\mathbf{p})}{\partial r_i}\right|_{\mathbf{p}-m\mathbf{v}_s}-m\frac{\partial \mathbf{v}_s}{\partial r_i}\frac{\partial n(\mathbf{p})}{\partial \mathbf{p}}, \tag{12}$$

we can verify the equation

$$\left\langle H\frac{\partial n}{\partial r_i}\right\rangle=\frac{\partial E_c}{\partial r_i}+m\left\langle\frac{\partial \varepsilon}{\partial p_k}n_c\right\rangle\frac{\partial v_{sk}}{\partial r_i}-\frac{\partial E_c}{\partial \rho_4}\frac{\partial \rho_4}{\partial r_i}$$
$$+\left\langle p_k\frac{\partial n}{\partial r_i}\right\rangle v_{sk}-\frac{m\mathbf{v}_s^2}{2}\left\langle\frac{\partial n}{\partial r_i}\right\rangle.$$

Substituting it into the previous equation, we obtain after some computations

$$\left\langle p_i\frac{\partial n}{\partial t}\right\rangle=-\frac{\partial}{\partial r_k}\left(\left\langle p_i\frac{\partial \varepsilon(p-mv_s)}{\partial p_k}n\right\rangle+\langle p_i n\rangle v_{sk}\right.$$
$$\left.+\delta_{ik}[\langle \varepsilon n_c\rangle+\sigma\rho_4-E_c]\right)+m\left\langle\frac{\partial \varepsilon}{\partial p_k}n_c\right\rangle\frac{\partial v_{sk}}{\partial r_i}$$
$$-\langle p_k n\rangle\frac{\partial v_{sk}}{\partial r_i}+m\langle n\rangle v_{sk}\frac{\partial v_{sk}}{\partial r_i}+\rho_4\frac{\partial \sigma}{\partial r_i}, \tag{13}$$

where $\sigma=\partial E_c/\partial\rho_4$. In order to determine $\partial\rho_4/\partial t$, we write

$$\frac{\partial \rho_4}{\partial t}=\frac{\partial \rho}{\partial t}-m\left\langle\frac{\partial n}{\partial t}\right\rangle.$$

For the total density we must have

$$\frac{\partial \rho}{\partial t}+\mathrm{div}\,\mathbf{J}=0, \tag{14}$$

while we find $(\partial n_4/\partial t)$ from the kinetic equation

$$\left\langle\frac{\partial n}{\partial t}\right\rangle=-\left\langle\frac{\partial H}{\partial \mathbf{p}}\frac{\partial n}{\partial \mathbf{r}}\right\rangle+\left\langle\frac{\partial H}{\partial \mathbf{r}}\frac{\partial n}{\partial \mathbf{p}}\right\rangle=-\mathrm{div}\left\langle n\frac{\partial H}{\partial \mathbf{p}}\right\rangle.$$

To summarize,

$$\frac{\partial \rho_4}{\partial t}=-\mathrm{div}\left(\mathbf{J}-m\left\langle\frac{\partial H}{\partial \mathbf{p}}n\right\rangle\right). \tag{15}$$

Substituting (13) and (15) into (10), we obtain

$$\frac{\partial J_i}{\partial t}=\rho_4\left(\frac{\partial v_{si}}{\partial t}+\frac{\partial}{\partial r_i}\left(\sigma+\frac{\mathbf{v}_s^2}{2}\right)\right)$$
$$-\frac{\partial}{\partial r_k}\left(\left\langle p_i\frac{\partial \varepsilon(p-mv_s)}{\partial p_k}n\right\rangle+\langle p_i n\rangle v_{sk}\right.$$
$$+\langle p_k n\rangle v_{si}+\delta_{ik}(\langle \varepsilon n_c\rangle+\sigma\rho_4-E_c)+\rho_4 v_{si}v_{sk}$$
$$\left.-m\left\langle\frac{\partial H}{\partial p_k}n\right\rangle v_{si}\right)=-\frac{\partial \Pi_{ik}}{\partial r_k}.$$

From this we find an equation for $\mathbf{v}_s$ and the form of $\Pi_{ik}$:

$$\frac{\partial v_{si}}{\partial t}+\frac{\partial}{\partial r_i}\left(\sigma+\frac{\mathbf{v}_s^2}{2}\right)=0, \tag{16}$$

$$\Pi_{ik}=\left\langle p_i\frac{\partial \varepsilon(\mathbf{p}-m\mathbf{v}_s)}{\partial p_k}n\right\rangle+\langle p_i n\rangle v_{sk}+\langle p_k n\rangle v_{si}$$
$$+\rho_4 v_{sk}v_{si}-m\left\langle\frac{\partial H}{\partial p_k}n\right\rangle v_{si}+\delta_{ik}(\langle \varepsilon n_c\rangle$$
$$+\sigma\rho_4-E_c). \tag{17}$$

Now we also find the energy flux $\mathbf{Q}$ defined by the equation

$$\frac{\partial E}{\partial t}+\mathrm{div}\,\mathbf{Q}=0. \tag{18}$$

We have

$$\frac{\partial E}{\partial t}=\frac{\partial}{\partial t}\left(E_c+\mathbf{P}_c\mathbf{v}_s+\frac{\rho\mathbf{v}_s^2}{2}\right)=\left\langle\varepsilon\frac{\partial n_c}{\partial t}\right\rangle+\frac{\partial E_c}{\partial \rho_4}\frac{\partial \rho_4}{\partial t}$$
$$+\left\langle\mathbf{p}\frac{\partial n_c}{\partial t}\right\rangle\mathbf{v}_s+\mathbf{P}_c\frac{\partial \mathbf{v}_s}{\partial t}+\rho\mathbf{v}_s\frac{\partial \mathbf{v}_s}{\partial t}+\frac{\mathbf{v}_s^2}{2}\frac{\partial \rho}{\partial t}$$
$$=\left\langle H(\mathbf{p}+m\mathbf{v}_s)\frac{\partial n_c}{\partial t}\right\rangle+\left(\frac{\partial E_c}{\partial \rho_4}+\frac{\mathbf{v}_s^2}{2}\right)\frac{\partial \rho_4}{\partial t}+\mathbf{J}\frac{\partial \mathbf{v}_s}{\partial t}$$
$$=\left\langle H\frac{\partial n}{\partial t}\right\rangle-\left\langle\frac{\partial H}{\partial \mathbf{p}}n\right\rangle m\frac{\partial \mathbf{v}_s}{\partial t}+\left(\sigma+\frac{\mathbf{v}_s^2}{2}\right)\frac{\partial \rho_4}{\partial t}+\mathbf{J}\frac{\partial \mathbf{v}_s}{\partial t}.$$

Making use of (11), (15), and (16) we obtain

$$\frac{\partial E}{\partial t}+\operatorname{div}\left\{\left\langle H\frac{\partial H}{\partial \mathbf{p}}n\right\rangle+\left(\mathbf{J}-m\left\langle\frac{\partial H}{\partial \mathbf{p}}n\right\rangle\right)\left(\sigma+\frac{\mathbf{v}_s^2}{2}\right)\right\}=0,$$

$$\mathbf{Q}=\left\langle H\frac{\partial H}{\partial \mathbf{p}}n\right\rangle+\left(\mathbf{J}-m\left\langle\frac{\partial H}{\partial \mathbf{p}}n\right\rangle\right)\left(\sigma+\frac{\mathbf{v}_s^2}{2}\right). \tag{19}$$

The Boltzmann equation (11), the continuity equation (15), and the equation for the superfluid velocity (16) form a complete system of nonlinear hydrodynamic equations for the degenerate $He^3-He^4$ solution. As always, these equations are exact in the limit where $\rho_4$, $\mathbf{v}_s$, and the distribution function $n(\mathbf{p},\mathbf{r},t)$ vary slowly in space and time.

We now turn to a comparison with the results of Khalatnikov. In Ref. 1, the quasiparticle energy is given by the functional derivative $\delta E/\delta n$ for $\rho=$const. Taking (5) into account, we obtain

$$\tilde{\varepsilon}=\left(\frac{\delta\tilde{E}}{\delta n_c}\right)_p=\varepsilon-m\frac{\partial E_c}{\partial\rho_4}.$$

But the Hamiltonian in the laboratory system is

$$\tilde{H}=\left(\frac{\delta\tilde{E}}{\delta n}\right)_{\rho,v_s}=\tilde{\varepsilon}(\mathbf{p}-m\mathbf{v}_s)+\mathbf{p}\mathbf{v}_s-mv_s^2.$$

It is clear that, unlike $\hat{H}$, $H$ as defined by Eq. (7) is obtained from $\varepsilon$ via a Galilean transformation. Because the quasiparticle energy for a dilute gas of quasiparticles should always transform this way under a Galilean transformation, we must conclude that Eq. (7) is correct. When the $\Pi_{ik}$ and $\mathbf{Q}$ obtained from (17), (19) are expressed in terms of $n_c$ and $\tilde{\varepsilon}$, it is found that they are exactly equal to the fluxes obtained in Ref. 1. How are we able to obtain the same energy and momentum flux as Ref. 1 while using a different Hamiltonian function? The fact is that the kinetic equation introduced in Ref. 1 was derived using the Hamiltonian $\hat{H}$, for which the unknown was the distribution function in the comoving system $n(\mathbf{p}+m\mathbf{v})=n_c(\mathbf{p})$. By definition, however, both the Hamiltonian and the distribution function should enter into the kinetic equation in the laboratory frame. Thus, the authors of Ref. 1 derived the correct form of the fluxes because of a double error.

Since $\varepsilon$ and $\tilde{\varepsilon}$ are phenomenological quantities, the only significant difference between $H$ and $\hat{H}$ is the quadratic term $mv_s^2/2$. Therefore, the linear expressions from (1) remain correct once we relabel the phenomenological constants appropriately.

## 3. NONLINEAR COLLISIONLESS OSCILLATIONS

In specific calculations we will use the distribution function of quasiparticles $n_c$ in the co-moving system as our variable. At zero temperature, a small deviation from the equilibrium state (in which $\mathbf{P}_c=0$) can be characterized by a displacement of the Fermi surface that depends on the direction of the momentum. In this case, the perturbed function is a step function as before. This is related to the fact that in solving the Boltzmann equation by the method of successive approximations we represent $n$ in the form of a series in derivatives of the delta function:

$$n=n_0(p_F-p)+a_0(\mathbf{n})\delta(p_F-p)+a_1(\mathbf{n})\delta'(p_F-p)+a_2(\mathbf{n})\delta''(p_F-p)+\ldots,$$

where $\mathbf{n}=\mathbf{p}/p$. Accordingly, we can either work with this series or write the nonequilibrium function in the form $n_c(\mathbf{p})=\theta[p_1(\mathbf{n})-p]$, and its increment in the form

$$\delta n_c(\mathbf{p},\mathbf{r},t)=\theta[p_1(\mathbf{n})-p]-\theta[p_F-p],$$

where $p_1(\mathbf{n})-p_F$ is the displacement of the Fermi surface in the direction $\mathbf{n}$.

To quadratic accuracy in $\delta n_c$ (i.e., for sufficiently small deviations from equilibrium), the energy of Fermi excitations in the comoving system has the form

$$\varepsilon(\mathbf{p})=\varepsilon_0(p)+\int f(\mathbf{p},\mathbf{p}')\delta n_c(\mathbf{p}')d\tau'+\frac{1}{2}\int\phi(\mathbf{p},\mathbf{p}',\mathbf{p}'')\delta n_c(\mathbf{p}')\delta n_c(\mathbf{p}'')d\tau'd\tau'', \tag{20}$$

where $f$ is the ordinary Landau function and $\phi$ is the third variational derivative of the total energy of the system with respect to $\delta n_c$ at constant $\rho_4$, evaluated at $n_c(\mathbf{p})=n_0(p)$. The spin dependence will be omitted here and in what follows. Let us introduce the set of parameters given previously in Ref. 2:

$$v_F=\frac{p_F}{m^*}=\left.\frac{\partial\varepsilon_0}{\partial p}\right|_{p=p_F},\quad\left.\frac{\partial^2\varepsilon_0(p)}{\partial p^2}\right|_{p=p_F}=\frac{1}{M^*}, \tag{21}$$

$$\frac{m^*p_F}{\pi^2\hbar^3}f(p_F,p_F,\cos\alpha)=F(\alpha)=\sum_{l=0}^{\infty}F_lP_l(\cos\alpha), \tag{22}$$

$$\frac{\partial f}{\partial p}(p_F,p_F,\cos\alpha)=\frac{\partial f}{\partial p'}(p_F,p_F,\cos\alpha)=\frac{\pi^2\hbar^3}{m^*p_F^2}F^1(\alpha),$$

$$F^1(\alpha)=\sum_{l=0}^{\infty}F_l^1P_l(\cos\alpha), \tag{23}$$

$$\frac{m^*p_FN_3}{\pi^2\hbar^3}\phi(\alpha,\beta,\gamma)=\Phi(\alpha,\beta,\gamma)=\sum_{l_1l_2l_3}\Phi_{l_1l_2l_3}\sum_{m_1m_2m_3}\begin{pmatrix}l_1&l_2&l_3\\m_1&m_2&m_3\end{pmatrix}Y_{l_1m_1}(\theta,\varphi)\times Y_{l_2m_2}(\theta',\varphi')Y_{l_3m_3}(\theta'',\varphi''). \tag{24}$$

Here

$$\begin{pmatrix}l_1&l_2&l_3\\m_1&m_2&m_3\end{pmatrix}$$

are $3j$ symbols, $\alpha$, $\beta$, $\gamma$ are the angles between $\mathbf{p}$ and $\mathbf{p}'$, $\mathbf{p}$ and $\mathbf{p}''$, and $\mathbf{p}'$ and $\mathbf{p}''$ respectively, $\theta$, $\varphi$ are angles that determine the orientation of the vector $\mathbf{p}$, and $\theta'$, $\theta''$, $\varphi'$, $\varphi''$ are the analogous angles for $\mathbf{p}'\mathbf{p}''$. Then

$$\cos\alpha=\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\varphi-\varphi')$$

with analogous expressions for $\beta, \gamma$. $P_l(\cos\alpha)$ is a Legendre polynomial, $Y_{lm}(\theta,\varphi)$ is a spherical harmonic, and $N_3=p_F^3/3\pi^2\hbar^3$ is the number of $He^3$ atoms per unit volume of the equilibrium liquid. The parameters $\Phi_{l_1l_2l_3}$ defined by Eq. (24) are dimensionless. Because of the symmetry properties of the $3j$ symbols, when $\phi$ is a symmetric function these parameters equal zero whenever the sum $l_1+l_2+l_3$ is odd. The first two terms of the series (24) reduce to the form

$$\Phi(\alpha,\beta,\gamma)\approx\Phi_{000}+\Phi_{011}\sqrt{3}(\cos\alpha+\cos\beta+\cos\gamma). \tag{25}$$

The Hamiltonian in the laboratory frame is determined by Eq. (6). Because $\delta n_c$ appears in the majority of the integrals we must deal with, we write the kinetic equation (11) in the following form for convenience:

$$\frac{\partial n_c}{\partial t}+\frac{\partial n_c}{\partial \mathbf{r}}\frac{\partial H(\mathbf{p}+m\mathbf{v}_s)}{\partial \mathbf{p}}-\frac{\partial n_c}{\partial \mathbf{p}}\frac{\partial H(\mathbf{p}+m\mathbf{v}_s)}{\partial \mathbf{r}}-m\frac{\partial \mathbf{v}_s}{\partial t}\frac{\partial n_c}{\partial \mathbf{p}}=\mathrm{St}\, n|_{\mathbf{p}+m\mathbf{v}_s}. \tag{26}$$

We will use the continuity equation in the form (14). We want to investigate axially symmetric uniform oscillations of the liquid solution. In this case, using (7) and (20), and also introducing the parameters (21)–(24), we can write Eqs. (14), (16) and (26) to quadratic accuracy in the small deviation from equilibrium as follows:

$$\tilde{\rho}(x,t)=\frac{\delta\rho_4}{\rho_4},\quad \tilde{v}_s(x,t)=\frac{v_{sx}}{v_F},\quad \nu(\theta,x,t)=\frac{(p_1-p_F)}{p_F} \tag{27}$$

in the collisionless regime. In this case the collision integral, as shown in Ref. 2, can be omitted if the following condition is satisfied:

$$\frac{\delta\rho_4}{\rho_4}\sim\frac{v_s}{v_F}\sim\frac{\delta p}{p_F}\ll\frac{\hbar\omega}{E_F},$$

where $\delta p_F=p_1-p_F$, $\omega$ is the acoustic frequency, and $E_F$ is the Fermi energy. In order to treat approximations higher than quadratic, we must take collisions into account (see Ref. 2).

After some computations we obtain

$$\frac{\partial\nu}{\partial t}+\frac{\partial\nu}{\partial x}v_F\cos\theta+v_F\cos\theta\left(\alpha\frac{mc^2}{p_Fv_F}\frac{\partial\tilde{\rho}}{\partial x}+\cos\theta\frac{\partial\tilde{v}_s}{\partial x}+\int F(\alpha)\frac{\partial\nu(\theta')}{\partial x}d\Omega'+\frac{m}{p_F}\frac{\partial\tilde{v}_s}{\partial t}\right)+v_FB(\theta,\nu,\tilde{\rho},\tilde{v}_s)=0, \tag{28}$$

$$\frac{\partial\tilde{\rho}}{\partial t}+3\frac{\rho_3}{\rho_4}\int\frac{\partial\nu}{\partial t}d\Omega+v_F\frac{\partial\tilde{v}_s}{\partial x}\left(1+\frac{\rho_3}{\rho_4}\right)+3\frac{p_F}{m}\frac{\rho_3}{\rho_4}\int\frac{\partial\nu}{\partial x}\cos\theta d\Omega+v_FA(\nu,\tilde{\rho},\tilde{v}_s)=0, \tag{29}$$

$$\frac{\partial\tilde{v}_s}{\partial t}+\frac{c^2}{v_F}\frac{\partial\tilde{\rho}}{\partial x}+3\alpha\frac{c^2}{v_F}\frac{\rho_3}{\rho_4}\int\frac{\partial\nu}{\partial x}d\Omega+v_FD(\nu,\tilde{\rho},\tilde{v}_s)=0, \tag{30}$$

where $\rho_3$ ($\rho_4$) are the equilibrium densities of $He^3$ ($He^4$). $B$, $A$, $D$ are expressions that are quadratically nonlinear in the deviations (27), and are given in Appendix 1.

Let us consider one-dimensional collisionless small oscillations of a degenerate solution described by the system of Eqs. (28)–(30). As in Ref. 2, we can obtain corrections to the velocity of sound in the collisionless regime once we have solved the problem with given initial conditions. The importance of a specific formulation of the problem for a collisionless system was discussed in Ref. 3. Let us prescribe the initial conditions

$$\tilde{\rho}|_{t=0}=\rho_1(x),\quad \tilde{v}_s|_{t=0}=v_1(x),\quad \nu(\theta,x,t)|_{t=0}=g(\theta,x) \tag{31}$$

and solve the system (28)–(30) using the standard method of successive approximations in the amplitude ($\tilde{\rho}=\tilde{\rho}_1+\tilde{\rho}_2+\ldots, \tilde{v}_s=\tilde{v}_1+\tilde{v}_2+\ldots, \nu=\nu_1+\nu_2+\ldots$) using initial conditions (31) for the first approximation and zero for the subsequent approximations. We will find only the first and second approximations, because the system (28)–(30) itself is only quadratically accurate.

Let us apply a Laplace transform $\mathscr{L}$ with respect to $t$ and a Fourier transform $\mathscr{F}$ with respect to $x$ to the system:

$$\mathscr{F}\mathscr{L}[\nu]=\nu_{k,\omega}^{(+)}(\theta)=\int_{-\infty}^{+\infty}\int_0^{+\infty}\nu(\theta,x,t)\times\exp(-ikx+i\omega t)dx\,dt$$

and approximate the Landau function by its first two harmonics:

$$F=F_0+F_1\cos\alpha.$$

Then we obtain the following system for the transforms $\nu_1^{(+)}$, $\rho_1^{(+)}$, and $v_1^{(+)}$ of the first approximations $\nu_1,\tilde{\rho}_1,\tilde{v}_1$:

$$\nu_1^{(+)}[s-\cos\theta]=-\frac{g_k}{ikv_F}-\frac{m}{m^*}\frac{\cos\theta}{ikv_F}v_{1k}+\cos\theta\left(\alpha\frac{mc^2}{p_Fv_F}\rho_1^{(+)}-\frac{m}{m^*}sv_1^{(+)}+F_0C_1+\cos\theta[F_1C_2+v_1^{(+)}]\right), \tag{32}$$

$$C_1=\int\nu_1^{(+)}(\theta)d\Omega;\quad C_2=\int\nu_1^{(+)}(\theta)\cos\theta d\Omega,$$

$$-s\rho_1^{(+)}-3s\frac{\rho_3}{\rho_4}C_1-\frac{\rho_{1k}}{ikv_F}-3\frac{\rho_3}{\rho_4}\int\frac{g_k}{ikv_F}d\Omega+3\frac{m^*}{m}\frac{\rho_3}{\rho_4}C_2+v_1^{(+)}\left(1+\frac{\rho_3}{\rho_4}\right)=0, \tag{33}$$

$$-sv_1^{(+)}-\frac{v_{1k}}{ikv_F}+\frac{c^2}{v_F^2}\rho_1^{(+)}+3\alpha\frac{c^2}{v_F^2}\frac{\rho_3}{\rho_4}C_1=0. \tag{34}$$

where we have introduced the variable

$$s=\frac{\omega}{kv_F},$$

and $\rho_{1k}$, $v_{1k}$, and $g_k(\theta)$ are Fourier transforms of the initial conditions (31). Because we have limited the number of harmonics in the Landau function to two, Eq. (32) for the distribution function contains only two constants. The solution to this equation and the inverse transforms of the solutions as functions of the two variables $(x, t)$ are given in Appendix 2. As a result, the first-approximation transforms have the form

$$\rho_1^{(+)}=\frac{iX_1}{kv_F y(s)}, \tag{35}$$

where $X_1$ is a linear function of the initial conditions $v_{1k}$, $\rho_{1k}$, $g_k$. The basis of this method is that the transforms have poles as a function of complex $\omega$ that correspond to propagating waves. Equation (35) as it stands will have poles when $y(s_0)=0$. This condition implies two poles $s_0=u_0/v_F$, $-u_0/v_F$ on the real axis, corresponding to zero-temperature first sound (collisionless regime) propagating to the right and to the left. The authors of Refs. 1, 4 argued that there are no poles corresponding to zero-sound modes for the Fermi liquid parameters of a $He^3-He^4$ solution. In addition, as we show in Appendix 2, the transforms are not defined on the segment of the real axis from $-kv_F$ to $kv_F$, and can have cuts there corresponding to single-particle excitations. When we return to the variables $(x, t)$, we obtain a sum of contributions from various singularities of the transforms. In the reconstructed original, a separate term will correspond to each of these singularities. Retaining only the contribution from the acoustic pole, we obtain

$$\tilde{\rho}^{(1)}=Y(x-u_0t),\quad \tilde{v}^{(1)}=\mu Y(x-u_0t),$$
$$\nu^{(1)}=\mathscr{A}(\theta)Y(x-u_0t), \tag{36}$$

where

$$Y(x-u_0t)=Y(s_0,\rho_1(x-u_0t),v_1(x-u_0t),g(\theta,x-u_0t)),$$

i.e., it depends on $x-u_0t$ through the initial conditions of the problem. The constants and functions entering into this expression are defined in Appendix 2. In first approximation, Eq. (36) corresponds to a sound wave and represents a traveling wave of constant shape.

In second approximation, the equations for the transforms do not change; the only difference is that the system now has zero initial conditions, and contains the nonlinear terms $A$, $B$, $D$. Therefore, we can obtain a system of equations for the transforms $\nu_2^{(+)}$, $\rho_2^{(+)}$, $v_2^{(+)}$ in second approximation by making the following substitutions in (32)–(34):

$$\rho_1^{(+)}\to\rho_2^{(+)},\quad \nu_1^{(+)}\to\nu_2^{(+)},\quad v_1^{(+)}\to v_2^{(+)},$$

$$R=\rho_{1k}+3\frac{\rho_3}{\rho_4}\int g_k(\theta)d\Omega\to -v_F\mathscr{LF}[A(x,t)];$$

$$v_{1k}\to -v_F\mathscr{LF}[D(x,t)],\quad -g_k(\theta)-\frac{m}{m^*}\cos\theta v_{1k}$$
$$\to v_F\mathscr{LF}[B(\theta,x,t)], \tag{37}$$

where we have substituted the exact first-approximation quantities $\tilde{\rho}_1$, $\tilde{v}_1$, $\nu_1$ into $A$, $B$, $D$ (the nonlinear parts of Eqs. (28)–(30)). Consequently, in second approximation the transforms have the form (35), where $X_1$ is now a linear combination of $\mathscr{LF}A(x,t)$, $\mathscr{LF}B(\theta,x,t)$, $\mathscr{LF}D(x,t)$, which still have poles at $\omega=u_0k$ (if we substitute (36) into $A$, $B$, $D$). Thus, second-order poles appear in the second-approximation transforms, which give secular terms $\sim t$ when we transform back to the original variables. We solve the second-approximation equations and evaluate the inverse transforms in Appendix 3. We show there that in order to find the coefficient for the secular term that corresponds to the pole $\omega=u_0k$, in substituting the first-approximation functions into $A$, $B$, $D$ it is sufficient to save only terms that correspond to the same pole, i.e., use the functions (36). As a result, we obtain the following expression for the secular term corresponding to the pole $\omega=u_0k$ in the second-approximation original

$$\tilde{\rho}^{(2)}=-u_1tY\frac{\partial Y}{\partial x};\quad \tilde{v}^{(2)}=\mu\rho^{(2)};\quad \nu^{(2)}=\mathscr{A}(\theta)\rho^{(2)}, \tag{38}$$

while for $u_1$ we have the following expression:

$$u_1=-\frac{v_Fb}{y'(s_0)}\Bigg\{3\mu^2+\frac{\partial^3E_c}{\partial\rho_4^3}\frac{\rho_4^2}{v_F^2}$$
$$+3\frac{\rho_3}{\rho_4}\frac{m^*}{m}\Bigg[\int\Bigg(\mathscr{A}^2\Bigg(\mathscr{A}\frac{m^*}{M^*}+4\mu\cos\theta+\frac{\mu}{\cos\theta}$$
$$+3y\Bigg)-\mu\frac{\partial\mathscr{A}(\theta)}{\partial\cos\theta}\sin^2\theta+\mathscr{A}\frac{m}{m^*}\left(\alpha\frac{c^2}{v_F^2}-\mu\frac{u_0}{v_F}\right)$$
$$\times\left(2\mathscr{A}-\frac{\partial\mathscr{A}(\theta)}{\partial\cos\theta}\frac{\sin^2\theta}{\cos\theta}\right)+3\frac{m}{m^*}\left(z\frac{c^2}{v_F^2}+\mu^2\right)\mathscr{A}\Bigg)d\Omega$$
$$+3\int\Phi(\alpha,\beta,\gamma)\mathscr{A}\mathscr{A}'\mathscr{A}''d\Omega d\Omega' d\Omega''$$
$$+\int\mathscr{A}\mathscr{A}'\Bigg(\mathscr{A}\Bigg(3F^1(\alpha)+2F(\alpha)+\frac{\partial F(\alpha)}{\partial\cos\theta}\frac{\sin^2\theta}{\cos\theta}\Bigg)$$
$$-F(\alpha)\frac{\partial\mathscr{A}(\theta)}{\partial\cos\theta}\frac{\sin^2\theta}{\cos\theta}+3\frac{\partial F(\alpha)}{\partial\rho_4}\rho_4\Bigg)d\Omega d\Omega'\Bigg]\Bigg\}, \tag{39}$$

where $\mathscr{A}=\mathscr{A}(\theta)$, $\mathscr{A}'=\mathscr{A}(\theta')$, $\mathscr{A}''=\mathscr{A}(\theta'')$.

We know that the secular term should be cancelled by the term arising from the correction to the velocity of sound in the first-approximation functions:

$$\tilde{\rho}^{(1)}=Y[x-(u_0+\delta u)t].$$

Expanding this expression for small $\delta u\, t$ and setting the result equal to the term linear in $t$ in (38), we find that the correction to the velocity $u_0$ of the first approximation (36) is $\delta u=u_1\tilde{\rho}^{(1)}$. Accordingly, the first-approximation expressions for sound waves are:

$$\tilde{\rho}^{(1)}=Y[x-(u_0+u_1\tilde{\rho}^{(1)})t],\quad \tilde{v}^{(1)}=\mu\tilde{\rho}^{(1)},$$
$$\nu^{(1)}=\mathscr{A}(\theta)\tilde{\rho}^{(1)}. \tag{40}$$

We can confirm this by verifying that (40) is a solution to the system (28)–(30) to accuracy up to quadratic terms in the amplitude. As is clear from (40), the correction to the velocity deforms the profile of a simple traveling acoustic wave (36), analogous to the Riemann solutions of hydrodynamics and the propagation of large-amplitude sound in $He^4$ (see Ref. 1).

## 4. GAS APPROXIMATION

At low concentrations, bare $He^3$ quasiparticles dissolved in a superfluid background form a dilute degenerate Fermi gas of slow-moving particles with a short interaction radius. In this description, we can describe the particles of the solution using only a single parameter—the $s$-wave scattering length $a$ for a collision of two bare quasiparticles—and classify the states of a single $He^3$ atom located in superfluid $He^4$ at rest using the continuous energy spectrum

$$\varepsilon=E_0+\frac{p^2}{2M},\quad E_0\approx-2.8\text{ K},\quad M\approx2.33m, \tag{41}$$

where $m$ is the mass of a $He^3$ atom and $M$ is the bare mass.

In $He^4$ at rest, the energy of the system can be written to second order in perturbation theory in the form

$$E=E_4^0+\sum_{\mathbf{p}\sigma}\left(E_0+\frac{p^2}{2M}\right)n_\sigma(\mathbf{p})+\frac{8\pi a\hbar^2}{M}\sum_{ik}n_in_kQ_{ik}-\frac{128\pi^2a^2\hbar^4}{M}\sum_{iklm}\frac{n_in_kn_lQ_{ik}}{p_i^2+p_k^2-p_l^2-p_m^2}, \tag{42}$$

where $E_4^0$ is the energy of pure $He^4$, $Q_{ik}=1/4-\sigma_i\sigma_k$, $\sigma$ is a spin variable, and the labels $i$, $k$, $l$, $m$ enumerate the spin states. Varying the energy (42) twice and three times with respect to $n_\sigma(\mathbf{p})$, we obtain respectively (see Ref. 4)

$$f_{\sigma\sigma'}(\alpha)=\frac{2\pi a\hbar}{M}\left[1+2\lambda\left(2+\frac{\cos\alpha}{2\sin(\alpha/2)}\times\ln\frac{1+\sin(\alpha/2)}{1-\sin(\alpha/2)}\right)\right]-\frac{8\pi a\hbar^2}{M}\sigma\sigma'\times\left[1+2\lambda\left(1-\frac{1}{2}\sin\frac{\alpha}{2}\ln\frac{1+\sin(\alpha/2)}{1-\sin(\alpha/2)}\right)\right], \tag{43}$$

$$\phi_{\sigma\sigma'\sigma''}(\alpha,\beta,\gamma)=-\frac{128\pi^2a^2\hbar^4}{p_F^2M}\times\left[\frac{Q_{\sigma\sigma'}}{\cos\gamma+\cos\beta\cos\alpha-1}+\frac{Q_{\sigma\sigma''}}{\cos\alpha+\cos\gamma\cos\beta-1}+\frac{Q_{\sigma'\sigma''}}{\cos\alpha+\cos\beta-\cos\gamma-1}\right], \tag{44}$$

where $\lambda=p_Fa/\pi\hbar$ is a small parameter.

Using (42)–(44) from Appendix 4, we can calculate all the parameters needed in Eq. (39) for the correction to the velocity. Approximating the functions $F(\alpha)$, $F^1(\alpha)$, $\Phi(\alpha,\beta\gamma)$ by the first two terms of Eqs. (22)–(24) and substituting the computed parameters into the expressions for $u_0$, $u_1$, we obtain to accuracy up to terms $\sim p_F^5$ (this accuracy comes from writing the energy in the form (42))

$$u_0=c_0\left\{1+\epsilon_1\frac{\rho_3}{\rho_4}+\epsilon_2\frac{\rho_3}{\rho_4}\frac{p_F^2}{c_0^2M^2}\right\}, \tag{45}$$

$$u_1=c_0\left\{1+\gamma+\frac{\rho_3}{\rho_4}[T+(2-\gamma)\epsilon_1]+\frac{\rho_3p_F^2}{\rho_4c_0^2M^2}\times[W+(2-\gamma)\epsilon_2]\right\}. \tag{46}$$

The constants $\epsilon_1$, $\epsilon_2$, $T$, $W$, and $\gamma$ are defined in the same Appendix. The analogous quantity for pure $He^4$ at zero temperature (see Ref. 1) is $u_1=(1+\gamma)c_0$.

We are deeply grateful to I. A. Fomin for discussing a number of questions.

## APPENDIX A

$$B=\frac{\partial\nu}{\partial x}\left(\cos\theta[\nu m^*/M^*+y\tilde{\rho}]+\tilde{v}_s+\sin^2\theta\int\frac{\partial F(\alpha)}{\partial\cos\theta}\nu'd\Omega'\right)+\cos\theta\left(3\int\Phi\frac{\partial\nu'}{\partial x}\nu''d\Omega'd\Omega''+\int\left[\frac{\partial F(\alpha)}{\partial\rho_4}\rho_4\frac{\partial}{\partial x}(\tilde{\rho}\nu')+[2F(\alpha)+F^1(\alpha)]\frac{\partial\nu'}{\partial x}\nu'+F^1(\alpha)\frac{\partial}{\partial x}(\nu\nu')\right]d\Omega'+\frac{\partial\tilde{v}_s}{\partial x}\left[\nu\cos\theta+\frac{m}{m^*}\tilde{v}_s\right]+\frac{\partial\tilde{\rho}}{\partial x}\left[y\nu+z\frac{mc^2}{p_Fv_F}\tilde{\rho}\right]\right)-\frac{\partial\nu(\theta)}{\partial\cos\theta}\sin^2\theta\left(\alpha\frac{mc^2}{p_Fv_F}\frac{\partial\tilde{\rho}}{\partial x}+\cos\theta\frac{\partial\tilde{v}_s}{\partial x}+\frac{m}{P_F}\frac{\partial\tilde{v}_s}{\partial t}+\int F(\alpha)\frac{\partial\nu'}{\partial x}d\Omega'\right), \tag{47}$$

$$A=3\frac{\rho_3}{\rho_4}\int\left[\frac{2}{v_F}\frac{\partial\nu}{\partial t}\nu+3\frac{m^*}{m}\frac{\partial\nu}{\partial x}\nu\cos\theta+\frac{\partial}{\partial x}(\tilde{v}_s\nu)\right]d\Omega+\frac{\partial}{\partial x}(\tilde{\rho}\tilde{v}_s), \tag{48}$$

$$D=\frac{\partial^3E_c}{\partial\rho_4^3}\frac{\rho_4^2}{v_F^2}\frac{\partial\tilde{\rho}}{\partial x}\tilde{\rho}+3\frac{\rho_3}{\rho_4}\left(2\alpha\frac{c^2}{v_F^2}+y\frac{m^*}{m}\right)\int\frac{\partial\nu}{\partial x}\nu d\Omega+3z\frac{c^2}{v_F^2}\frac{\rho_3}{\rho_4}\frac{\partial}{\partial x}\left(\tilde{\rho}\int\nu d\Omega\right)+\tilde{v}_s\frac{\partial\tilde{v}_s}{\partial x}$$

$$+3\frac{\rho_3}{\rho_4}\frac{m^*}{m}\int\frac{\partial F(\alpha)}{\partial\rho_4}\rho_4\frac{\partial\nu'}{\partial x}\nu d\Omega d\Omega', \tag{49}$$

$$\frac{c^2}{\rho_4}=\frac{\partial^2 E_c}{\partial\rho_4^2},\quad \frac{\partial F}{\partial\rho_4}=\frac{m^* p_F}{\pi^2\hbar^3}\frac{\partial f}{\partial\rho_4},\quad \alpha=\frac{\partial\varepsilon_0}{\partial\rho_4}\frac{\rho_4}{mc^2},$$

$$z=\frac{\partial^2\varepsilon_0}{\partial\rho_4^2}\frac{\rho_4^2}{mc^2},\quad y=\frac{\partial^2\varepsilon_0}{\partial p\partial\rho_4}\frac{\rho_4}{v_F},$$

$$\nu=\nu(\theta),\quad \nu'=\nu(\theta'),\quad \nu''=\nu(\theta''),$$

$$d\Omega=\sin\theta d\theta/2.$$

## APPENDIX B

The system (32)–(34) is solved as follows. Using Eq. (32), we express $\nu_1^+$ in the following way:

$$\nu_1^{(+)}=\left\{-\frac{g_k}{ikv_F}-\frac{m}{m^*}\frac{\cos\theta}{ikv_F}v_{1k}+\cos\theta\left(\alpha\frac{mc^2}{p_Fv_F}\rho_1^{(+)}-\frac{m}{m^*}sv_1^{(+)}+F_0C_1+\cos\theta[F_1C_2+v_1^{(+)}]\right)\right\}\Big/[s-\cos\theta], \tag{50}$$

after which we substitute into the definitions

$$C_1=\int\nu_1^{(+)}(\theta)d\Omega;\quad C_2=\int\nu_1^{(+)}(\theta)\cos\theta d\Omega,$$

which gives a linear inhomogeneous system consisting of the two equations for $C_1$, $C_2$. Solving it we obtain

$$C_1\Delta=\left[1-F_1\left(s^2w-\frac{1}{3}\right)\right]\int\frac{M(\theta)}{ikv_F}d\Omega+F_1sw\int M(\theta)\frac{\cos\theta}{ikv_F}d\Omega+\alpha\frac{mc^2}{p_Fv_F}w(1+F_1/3)\rho_1^{(+)}+sw\left(\frac{\delta m}{m^*}-\frac{m}{m^*}\frac{F_1}{3}\right)v_1^{(+)}, \tag{51}$$

$$C_2\Delta=(1-F_0w)\int M(\theta)\frac{\cos\theta}{ikv_F}d\Omega+F_0sw\int\frac{M(\theta)}{ikv_F}d\Omega+\alpha\frac{mc^2}{p_Fv_F}sw\rho_1^{(+)}+v_1^{(+)}\times\left(s^2w\frac{\delta dm}{m^*}-\frac{1}{3}(1-F_0w)\right), \tag{52}$$

where

$$\Delta=1-F_0w-F_1s^2w+\frac{F_1}{3}-\frac{F_0F_1w}{3},$$

$$M(\theta)=-\frac{g_k(\theta)+(m/m^*)v_{1k}\cos\theta}{s-\cos\theta},$$

$$w=\int\frac{\cos\theta}{s-\cos\theta}d\Omega=-1+\frac{s}{2}\ln\frac{s+1}{s-1}.$$

Now, $\omega$, and therefore $s$, are complex quantities; hence, in order to give meaning to the last integral and integrals like it (i.e., in which the expression under the integral sign contains $s-\cos\theta$ in the denominator) we must make a cut in the complex plane $\omega$ along the real axis from $-kv_F$ to $kv_F$. This cut, and the condition of reality when $s<-1$ and $s>-1$, specifies one branch of the complex multivalued function $\ln[(s+1)/(s-1)]$. Substituting (51) and (52) into (33), (34), we obtain a linear inhomogeneous system consisting of the two equations for $\rho_1^+$, $v_1^+$, from which we find

$$\rho_1^{(+)}=\frac{i}{kv_Fy(s)}([R+R_1]sr-[v_{1k}+R_2]b),$$

$$v_1^{(+)}=\frac{i}{kv_Fy(s)}([v_{1k}+R_2]sr-[R+R_1]a), \tag{53}$$

where

$$y(s)=-s^2r^2+ab,\quad b=1+\frac{\rho_3}{\rho_4}+\frac{3}{\Delta}\frac{\rho_3}{\rho_4}\left[s^2w\left(\frac{\delta m}{m}-\frac{\delta m}{m^*}+\frac{m}{m^*}\frac{F_1}{3}\right)-\frac{m^*}{3m}(1-F_0w)\right],$$

$$r=-1-3\alpha\frac{mc^2}{p_Fv_F}\frac{\rho_3}{\rho_4}\frac{w}{\Delta}\left(\frac{F_1}{3}-\frac{\delta m}{m}\right),$$

$$a=\frac{c^2}{v_F^2}\left(1+3\alpha^2\frac{mc^2}{p_Fv_F}\frac{\rho_3}{\rho_4}\frac{w}{\Delta}\left(1+\frac{F_1}{3}\right)\right),$$

$$R_1=3\frac{\rho_3}{\rho_4}\frac{s}{\Delta}\left(\int Md\Omega\left[1-F_1\left(s^2w-\frac{1}{3}\right)-\frac{m^*}{m}F_0w\right]+\int M\cos\theta d\Omega\left[F_1sw-\frac{m^*}{sm}(1-F_0w)\right]\right),$$

$$R=\rho_{1k}+3\frac{\rho_3}{\rho_4}\int g_k(\theta)d\Omega,$$

$$R_2=-\frac{3\alpha}{\Delta}\frac{c^2}{v_F^2}\frac{\rho_3}{\rho_4}\left(\int Md\Omega\left[1-F_1\left(s^2w-\frac{1}{3}\right)\right]+F_1sw\int M\cos\theta d\Omega\right).$$

We recover the original functions we are looking for using the inverse Laplace transform $\mathscr{L}^{-1}$ and inverse Fourier transform $\mathscr{F}^{-1}$:

$$\mathscr{F}^{-1}\mathscr{L}^{-1}[\nu_{k\omega}^{(+)}]=\nu(\theta,x,t)=\int_{-\infty}^{+\infty}\int_{-\infty+i\alpha}^{+\infty+i\alpha}\nu_{k\omega}^{(+)}\times(\theta)e^{ikx-i\omega t}\frac{dk}{2\pi}\frac{d\omega}{2\pi},\quad \alpha>0,$$

$$\tilde{\rho}_1=\mathscr{F}^{-1}\mathscr{L}^{-1}[\rho_1^{(+)}],\quad \tilde{v}_1=\mathscr{F}^{-1}\mathscr{L}^{-1}[v_1^{(+)}],$$

$$\nu_1=\mathscr{F}^{-1}\mathscr{L}^{-1}[\nu_1^{(+)}]. \tag{54}$$

The unattenuated singularities of the functions $\rho_1^+$, $v_1^+$, $\nu_1^+$ in the complex $\omega$ plane will be: a cut from $-kv_F$ to $kv_F$ and a pole at $\omega=kv_F\cos\theta$ (only for $v_1^+$), which correspond to single-particle excitations, in particular the unaccelerated motion of particles from the initial state (a particle in unaccelerated motion with constant velocity $kv_F\cos\theta$ along the $x$ axis), and the two poles $\omega=u_0k$, $-u_0k$—if $y(s_0)=0$, $s_0=u_0/v_F$, $-u_0/v_F$—corresponding to first sound at zero temperature (collisionless regime) propagating to the right and to the left. There are no real zero-sound solutions to the equation $y(s)=0$ (see Refs. 1, 4). Retaining only the contribution from the pole $\omega=u_0k$ ($s_0=u_0/v_F$), we find

$$\tilde{\rho}^{(1)}=\int_{-\infty}^{\infty}\int_{-\infty+i\alpha}^{\infty+i\alpha}\frac{iY_k}{(\omega-u_0k)}\exp(-i\omega t+ikx)\frac{dk}{2\pi}\frac{d\omega}{2\pi}$$

$$=Y(x-u_0t)\ , \tag{55}$$

where $Y_k=([R+R_1]sr-[v_{1k}+R_2]b)/y'(s_0)=dy/ds(s_0)$, and consequently

$$Y(x-u_0t)=-\frac{b}{y'(s_0)}\left(R\mu+v_1(x-u_0t)-3\frac{\rho_3}{\rho_4}\frac{m^*}{m}\int M(\theta)\frac{s_0-\cos\theta}{\cos\theta}A(\theta)d\Omega\right),$$

$$\mu=-\frac{r}{b}s_0. \tag{56}$$

In $R$, $M(\theta)$ we replace $\rho_{1k}$, $v_{1k}$, $g_k(\theta)$ by $\rho_1(x-u_0t)$, $v_1(x-u_0t)$, $g(\theta,x-u_0t)$ respectively, and in $w$, $r$, $b$ we substitute $s=s_0$. Analogously we find for $v_s^{(1)}$, $\nu^{(1)}$:

$$\tilde{v}_s^{(1)}=\mu Y(x-u_0t),\quad \nu^{(1)}=\mathscr{A}(\theta)Y(x-u_0t), \tag{57}$$

where

$$\mathscr{A}(\theta)=\frac{\cos\theta}{s_0-\cos\theta}\frac{q+d\cos\theta}{\Delta},$$

$$q=(1+F_1/3-F_1s_0^2w)\left(\alpha\frac{mc^2}{p_Fv_F}-\frac{m}{m^*}s_0\mu\right)+F_0s_0w\mu,$$

$$d=\alpha\frac{mc^2}{p_Fv_F}F_1s_0w+\mu\left(1-F_0w-F_1s_0^2w\frac{m}{m^*}\right).$$

## APPENDIX C

As we said in the main text, in order to obtain a system of equations for the second approximation it is sufficient to make the replacement (37) in Eqs. (32)–(34); consequently, the transforms of the required functions have the following form:

$$\rho_2^{(+)}=\frac{i\mathscr{L}\mathscr{F}[Z(A,B,D)]}{kv_Fy(s)}, \tag{58}$$

where $Z(A,B,D)$ is a linear function of $A$, $B$, $D$, i.e., the nonlinear expressions (47)–(49) (in which we substitute the exact first approximation), which is easy to write down making the replacement (37) in (53). The transforms of $\nu_1$, $\tilde{v}_1$ are obtained analogously. The problem is to identify the secular terms (proportional to $t$) that give a correction to the velocity of a sound wave (36). The secular term in the original arises from the second-order poles in the transforms. Equation (58) already has a pole at $\omega=u_0k$ if $y(s_0)=0$ (see Appendix 2). The numerator of Eq. (58) has a pole at $\omega=u_0k$ only when the function $Z(A,B,D)=Z(x-u_0t)$; consequently, in order to obtain the secular term corresponding to the pole $\omega=u_0k$ in $Z(A,B,D)$ we must substitute the part of the first approximation corresponding to the sound wave (36). Then

$$A(x,t)=A_1Y\frac{\partial Y}{\partial x},\quad D(x,t)=D_1Y\frac{\partial Y}{\partial x},$$

$$B(\theta,x,t)=B_1(\theta)Y\frac{\partial Y}{\partial x}$$

and consequently $Z(x-u_0t)\sim Y\partial Y/\partial x$. Taking into account only the second-order pole $\omega=u_0k$ in (58), we obtain

$$\rho^{(2)}(x,t)=u_1\mathscr{L}^{-1}\mathscr{F}^{-1}\left[\frac{iLF\left[Y\frac{\partial Y}{\partial x}\right]}{\omega-u_0k}\right]=u_1Y\frac{\partial Y}{\partial x}*\delta(x-u_0t)=u_1\int_{-\infty}^{\infty}\int_0^tY(y-u_0\tau)\frac{\partial Y}{\partial y}\delta(x-y-u_0(t-\tau))dyd\tau=u_1tY(x-u_0t)\frac{\partial Y}{\partial x}, \tag{59}$$

where * denotes convolution. It is easy to see that we obtain Eq. (56) for $u_1/v_F$ with the replacement of $R$, $v_1$, $-M(\theta)\times(s_0-\cos\theta)$ by $A_1$, $D_1$, $B_1$, respectively. After some simple but tedious transformations, we finally obtain Eq. (39) for $u_1$, and (38) for the secular part (which gives a correction to (36)) in second approximation. Analysis shows that the other singularities of Eq. (58) give no correction to the first-approximation velocity (36).

## APPENDIX D

Since we know the explicit forms (43), (44) of the functions $f(\alpha)$ and $\phi(\alpha,\beta,\gamma)$, we can compute their harmonics in the expansions (22), (24). Neglecting some simple but tedious computations, we present the values of the first two harmonics:

$$F_0=2\lambda\left[1+\frac{4}{3}\lambda(2+\ln 2)\right],\quad F_1=\frac{8}{5}\lambda^2(7\ln 2-1), \tag{60}$$

$$\Phi_{000}=32\lambda^2\ln 2,\quad \Phi_{011}=\frac{32}{3\sqrt{3}}\lambda^2(\ln 2-1). \tag{61}$$

The change in the Landau function $\delta f(\alpha)$ with the change in the number of particles in equilibrium $\delta N_3=p_F^2\delta p_F/\pi^2\hbar^3$ is written as follows:

$$\delta f(\alpha)=2\frac{\partial f}{\partial p}(p_F)\delta p_F+3N_3\int\phi(\mathbf{p},\mathbf{p}'\cdot\mathbf{p}'')|p_Fd\Omega''\frac{\delta p_F}{p_F}. \tag{62}$$

Because we have already approximated the Landau function by its first two harmonics in these calculations, it is sensible to retain only the first two harmonics in the expansions (23), (24) of the functions $F^1(\alpha)$, $\Phi(\alpha,\beta,\gamma)$ as well. Equation (62) gives the expressions for these harmonics:

$$2F_0^1+3\Phi_{000}=\frac{\partial F_0}{\partial p_F}p_F-F_0-\frac{\partial F_1}{\partial p_F}p_F\frac{F_0}{3}\frac{m}{m^*},$$

$$2F_1^1+3\sqrt{3}\Phi_{011}=\frac{\partial F_1}{\partial p_F}\frac{3p_F}{3+F_1}-F_1.$$

Taking (60), (61) into account, we find

$$F_0^1=\frac{4}{3}\lambda^2(2-35\ln 2),\quad F_1^1=\frac{4}{5}\lambda^2(19-13\ln 2). \quad (63)$$

The free energy of the solution obtained from (42) by integration equals

$$F=F_4^0+N_3E_0+\frac{3}{10}\frac{p_F^2}{M}N_3\left[1+\frac{10}{9}\lambda+\frac{4}{21}\lambda^2(11-2\ln 2)\right], \quad (64)$$

where $N_3$ ($N_4$) is the concentration of $He^3$ ($He^4$). Differentiating (64) with respect to $N_3$, we obtain

$$\mu_3=E_0+\frac{p_F^2}{2M}\left[1+\frac{4}{3}\lambda+\frac{4}{15}\lambda^2(11-2\ln 2)\right]. \quad (65)$$

The change in the chemical potential $\mu_3$ as the number of particles changes by $\delta N_3$ equals

$$\delta\mu_3=\frac{p_F}{m^*}\delta p_F+\frac{\pi^2\hbar^3}{p_F m^*}F_0\delta N_3. \quad (66)$$

Taking (65) into account, we obtain from this

$$\frac{m^*}{M}=1+\frac{8}{15}\lambda^2(7\ln 2-1). \quad (67)$$

There still remains the undetermined parameter $1/M^*=\partial^2\varepsilon/\partial p^2(p_F)$. To determine it we write the change in $\delta v_F$, i.e., $\delta\{\partial\varepsilon/\partial p\}(p_F)$ for the variation $\delta N_3$, as follows:

$$\delta v_F=(F_0^1+m^*/M^*)\delta p_F/m^*, \quad (68)$$

from which, taking (63), (67) into account, we obtain

$$\frac{m^*}{M^*}=1+\frac{4}{5}\lambda^2(49\ln 2-2).$$

All the variations (62), (66), (68) are easily obtained from Eq. (20), in which we understand the variation $\delta n$ to be the difference between infinitesimally separated equilibrium states $\theta[p_F'-p]-\theta[p_F-p]$ that differ by $p_F'-p_F=\delta p_F\sim\delta N_3$.

In order to find the unknown parameters that take into account the interaction of the Fermi quasiparticles with the superfluid background, let us find $\mu_4$, i.e., the chemical potential of $He^4$, which is easy to obtain by differentiating (64) with respect to $N_4$:

$$\mu_4=\mu_4^0+N_3\frac{\partial E_0}{\partial N_4}+\frac{3}{10}p_F^2N_3\frac{\partial}{\partial N_4}\left(\frac{1}{M}\right). \quad (69)$$

Using (65), (67), (69), we compute the following parameters to accuracy determined by writing the energy (42) to second approximation:

$$c^2=c_0^2\left[1+\frac{\rho_3}{\rho_4}z_0+\frac{3}{10}\frac{p_F^2}{mMc_0^2}(2y_0^2-e)\right],\quad \frac{c^2}{\rho_4}=\frac{\partial\mu_4}{\partial\rho_4},$$

$$\frac{\partial(c^2/\rho_4)}{\partial\rho_4}\frac{\rho_4^2}{c_0^2}=-1+2\gamma+\frac{\rho_3}{\rho_4}\alpha_0+\frac{3}{10}\frac{\rho_3}{\rho_4}\frac{p_F^2\beta}{Mmc_0^2},$$

$$\frac{\partial\mu_3}{\partial\rho_4}\frac{\rho_4}{mc_0^2}=x-\frac{p_F^2y_0}{2Mmc_0^2},$$

$$\frac{\partial^2\mu_3}{\partial\rho_4^2}\frac{\rho_4^2}{mc_0^2}=z_0+\frac{p_F^2(y_0^2-e/2)}{Mmc_0^2},$$

$$\frac{\partial v_F}{\partial\rho_4}=-y_0-\frac{8}{15}\lambda^2(7\ln 2-1)(\xi-y_0),$$

where

$$x=\frac{\rho_4}{m_3c_0^2}\frac{\partial E_0}{\partial\rho_4},\quad y_0=\frac{\rho_4}{M}\frac{\partial M}{\partial\rho_4},\quad z_0=\frac{\rho_4^2}{mc_0^2}\frac{\partial^2E_0}{\partial\rho_4^2},$$

$$e=\frac{\rho_4^2}{M}\frac{\partial^2M}{\partial\rho_4^2},\quad \delta m=M-m,\quad \alpha_0=\frac{\rho_4}{mc_0^2}\frac{\partial^3E_0}{\partial\rho_4^3},$$

$$\beta=\rho_4^3M\frac{\partial^3(1/M)}{\partial\rho_4^3},\quad \gamma=\frac{\rho_4}{c_0}\frac{\partial c_0}{\partial\rho_4},\quad \xi=\frac{\rho_4}{\lambda^2}\frac{\partial\lambda^2}{\partial\rho_4},$$

where $c_0$ is the velocity of sound in pure $He^4$ at $T=0$. Substituting the computed parameters in the expressions for $u_0$, $u_1$ given in the previous paragraph, and approximating the functions $F(\alpha)$, $F^1(\alpha)$, $\Phi(\alpha,\beta\gamma)$ by the first two terms of expansions (22)–(24), we obtain Eqs. (45), (46) to accuracy up to terms $\sim p_F^5$, where

$$\epsilon_1=\frac{z_0}{2}-\frac{\delta m}{2M}+\frac{x}{2}\left(x\frac{m}{M}+2\frac{\delta m}{M}\right),$$

$$\epsilon_2=-\frac{3}{10}\frac{m}{M}[1-x]\left[x-1+2\frac{M}{m}\right]+\frac{3}{10}\frac{M}{m}\left[1+y_0^2-\frac{e}{2}\right]-\frac{y_0}{2}\left[x+\frac{\delta m}{m}\right],$$

$$T=3\frac{m}{M}(x-1)\left(1+(x-1)\left(3-y_0+\frac{m}{M}[x-1]\right)\right)+\frac{1}{2}\alpha_0+\frac{3}{2}\frac{\delta m}{M}(z_0+1)\left(x+\frac{\delta m}{m}\right)-\frac{m}{M}(1+\gamma)\left(x^2+\frac{\delta m}{m}\right),$$

$$W=\frac{9}{2}(x-1)\left(\frac{2}{3}y_0^2-\frac{9}{5}(y_0-1)+\frac{m}{M}(x-1)\left(2-\frac{11}{10}y_0+\frac{2}{3}\frac{m}{M}(x-1)\right)\right)+\frac{3}{2}\frac{\delta m}{M}y_0+\frac{9}{10}\frac{m}{M}(z_0+1)\left(x+\frac{\delta m}{m}-\frac{5}{6}y_0\frac{M}{m}\right)+\frac{M}{m}\left(\frac{3}{20}\beta+\frac{7}{5}+y_0^2-\frac{8}{5}y_0\right)+\frac{3}{5}x(1+\gamma)\times\left(\frac{5}{3}y_0-2\frac{\delta m}{M}-2x\frac{m}{M}\right)-\left(x+\frac{\delta m}{m}\right)\left(\frac{3}{4}e+\frac{9}{5}\frac{\delta m}{M}-\frac{8}{75}\frac{m^2}{M^2}(x-1)^2\lambda^2s^2(43+44\ln 2)\right).$$

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