Quasioptics of smoothly inhomogeneous isotropic media

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A generalization of the quasioptical description of vector wave fields to arbitrary smoothly inhomogeneous media is proposed. A study is made of the effect of curvature and torsion of the propagation tracks on the focusing and defocusing properties of the equivalent quasioptical line and also on the polarization of the waves (pseudogyrotropy). The treatment is restricted to the so-called aberrationless approximation of quasioptics, taking into account field phase corrections of at most second order. © 1996 American Institute of Physics. [S1063-7761(96)00203-7]

1. INTRODUCTION

The term quasioptics was adopted by physicists at the beginning of the sixties in connection with the development of microwave and laser technology. Initially it had a predominantly "instrumental" direction and referred to devices similar to optical devices but requiring allowance for diffraction effects in the description of the wave processes in them (open laser resonators, mirror and lens transmission lines for microwave radiation, fiber optics, etc.). However, the generality of the methods of calculating fields in quasioptical systems based on the use of a truncated parabolic wave equation gradually led to an extended interpretation of the term—quasioptics came to describe the branch of physics concerned with waves of arbitrary nature (both linear and nonlinear) in processes with relatively narrow frequency and angular spectra.1

There exists a deep and constructive analogy between the propagation of wave beams and packets in smoothly inhomogeneous media and in quasistransmission lines (of lens or mirror type). By virtue of the transverse localization of the fields around "propagation paths," the inhomogeneous medium can be regarded as a collection of distributed phase correctors: linear (of prism type and responsible for bending of the track), quadratic (of lens type and determining focusing or defocusing), cubic, quartic, etc. (leading to extension of the quasioptics of two-dimensional smoothly inhomogeneous media to the case of strong aberrations; these studies also obtained criteria for the applicability of the so-called aberrationless approximation, which takes into account only the quadratic distributed phase correctors. It was also shown in Refs. 7 and 8 that even when the aberrationless approximation cannot be regarded as an approximation to the true wave field it still remains informative and makes it possible to recover the correct solution by means of an asymptotic procedure analogous to the one developed in the diffraction theory of aberration.2

2. THE EQUATION OF QUASIOPTICS IN A SMOOTHLY INHOMOGENEOUS MEDIUM

For definiteness, we shall consider the electrodynamic problem of the propagation of a beam of monochromatic electromagnetic waves in a smoothly inhomogeneous stationary medium with permittivity \( \varepsilon(\omega, \mathbf{r}) \). The complex amplitudes of the electric, \( \mathbf{E}(\mathbf{r}) \), and magnetic, \( \mathbf{B}(\mathbf{r}) \), fields are described by Maxwell's equations:

\[
\text{curl}\ \mathbf{E} = i k_0 \mathbf{B}, \quad \text{curl}\ \mathbf{B} = -i k_0 \epsilon(\mathbf{r}) \mathbf{E},
\]  

(2.1)

where \( k_0 = \omega/c \). Equations (2.1) can be significantly simplified if the transverse dimension \( \Lambda \) of the beam everywhere along the propagation path is, on the one hand, small on the scale \( L_w = \epsilon/|\nabla \varepsilon| \) of the inhomogeneities of the medium but, on the other hand, large on the scale of the wavelength \( \lambda \). In
this case, the problem contains two small parameters: the 
width \( r = \lambda / \Delta \) of the angular spectrum of the beam and the 
ratio \( \mu = \lambda / l_0 \). The aim of the paper is to construct asympt- 
Iotic solutions of the system (2.1) with respect to the param- 
eters \( v \) and \( \mu \).

It is intuitively clear (and this will be shown below) that 
for \( v, \mu \ll 1 \) the wave beam is localized in a certain neigh- 
borhood of the geometric-optics ray (we shall call it the refer- 
ence ray\(^5\)), the canonical equations of which have the form

\[
\begin{align*}
\frac{d\theta_0}{d\tau} &= p, & \frac{dp}{d\tau} &= -\frac{1}{2} \nabla e.
\end{align*}
\]  

(2.2)

Here \( r_0(\tau) \) is the radius vector of the points on the reference 
ray; \( \theta(\tau) \), which is normalized by \( k_0 \), is the instantaneous 
wave vector \( (p = k_0 r_0(\tau)) \), in which \( k_0 \) is the unit vector tangen-
t to the ray), and the variable \( \tau \) is related to the path 
length \( s \) of the ray by \( d\tau = ds/\sqrt{e(r_0)} \).

The reference ray is a space curve with its principal nor-
mal \( n = (dv/ds) \) (for some remarks concerning the proper-
ties of the comoving orthogonal coordinate system, see Ap-
endix 1.)

The local structure of the electromagnetic field in a wide 
(on the scale of \( \lambda \)) wave beam is always close to that of a 
plane wave, i.e., the fields \( E \) and \( B \) are almost perpendicular 
to the direction of propagation and to each other. Represent-
ing the field in the form

\[
E(r) = E_0(r) + E_1(r) + f(r) = E_0(r) + E_1(r) + O(r^2, \mu^2),
\]  

and substituting in Eqs. (2.1) as expressed in the above cur-
vilinear coordinate system (in which the unit vectors \( e_1 \) and 
\( e_2 \) are to be assumed to be independent of the variable \( \tau \)), we 
obtain

\[
\begin{align*}
1 &= \frac{\partial}{\partial \tau} + \frac{1}{h} \frac{\partial E_1}{\partial r} + \frac{1}{h} \frac{\partial E_1}{\partial \theta} + \frac{1}{k_0} \frac{\partial E_1}{\partial \phi} + \frac{1}{k_0} \frac{\partial E_1}{\partial \phi}, \\
p_0 &= \frac{\partial}{\partial \tau} + \frac{1}{h} \frac{\partial E_1}{\partial r} + \frac{1}{h} \frac{\partial E_1}{\partial \theta},
\end{align*}
\]  

(2.6)

Here and in what follows, summation over repeated dummy 
indices is understood.

A wave beam with arbitrary polarization (including the 
case of inhomogeneity over the cross sections) can be repres-
ented in the form of a superposition of two beams with 
mutually orthogonal homogeneous polarizations (linear, cir-
cular, or elliptic).\(^6\) For each of the “partial” beams, it is 
possible to introduce the scalar field amplitude \( E_1 \) 
\( (E_1 = E_1 + E_1, \text{ where } e_1 \text{ is a unit complex polarization vector}) \).

In \( E_1 \), we isolate a phase factor with characteristic lon-
gitudinal scale \( \lambda \):

\[
E_1 = \frac{1}{k_0} W(r) e^{i \theta(r)} W(\theta(r)) d\tau.
\]  

(2.7)

The field amplitude \( W(r, \xi_1, \xi_2) \) of the beam, varying 
smoothly in space, has a characteristic transverse scale \( \lambda \ll \lambda 
(\lambda / \lambda - r < \lambda) \) and characteristic longitudinal scale \( L_1 \ll \lambda \).

In inhomogeneous media possessing focusing or defocusing 
properties, the longitudinal scale of variation of \( W \) can be 
determined both by diffraction effects (and then \( \lambda / L_1 - \nu \)) 
as well as by refraction effects (in this case \( \lambda / L_1 - \nu \)). 

Sub-
stituting (2.7) in (2.6), we obtain up to terms of second order 
in \( v \) and \( \mu \) the equation

\[
\frac{3}{4} \frac{\partial W}{\partial \tau} + \frac{1}{4} \frac{\partial^2 W}{\partial \xi_1^2} + \frac{1}{4} \frac{\partial^2 W}{\partial \xi_2^2} + \frac{1}{4} \frac{\partial^2 W}{\partial \xi_1 \xi_2} - \mu \frac{\partial W}{\partial \phi} = 0,
\]  

(2.8)

The sign in front of the quadratic form in (2.8) is chosen 
such that positive values of the coefficients \( \alpha_{mn} \) 
correspond to focusing properties of the medium. More pre-
cisely, if the signature \( \Sigma \) of the quadratic form is 2, the 
medium possesses focusing properties with respect to all direc-

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tions; if $\Sigma=0$, then in one direction the medium focuses and in the other it defocuses; if $\Sigma=-2$, the medium defocuses in all directions.

In actual calculations, it is more convenient to represent the coefficients $a_{ij}$ in (2.8) in terms of derivatives in the direction of the principal normal $n$ and binormal $m$ "attached" to the gradient of the permittivity in the medium:

$$a_{ij} = R_{ij}(\tau) R_{j\ell}(\tau) \dot{R}_{\ell\ell} ,$$

$$\beta_{11} = \frac{3}{2} \frac{\partial^2 \varepsilon}{\partial \tau^2} , \quad \beta_{22} = -\frac{1}{2} \frac{\partial^2 \varepsilon}{\partial \tau^2} ,$$

$$\beta_{12} = -\frac{1}{2} \frac{\partial^2 \varepsilon}{\partial \tau \partial \theta} , \quad \dot{R}(\theta) = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} .$$

The angle $\theta(\tau)$ in the rotation operator $\dot{R}(\theta)$ is determined by Rytov's law (2.4) (the derivatives in the above expressions are taken at points on the reference beam).

It can be seen from (2.9) that curvature of the reference beam is always a focusing factor (in the rectifying plane). In addition, the focusing and defocusing properties of the medium are also determined by the second derivatives of the permittivity along the transverse directions.

Equation (2.8) is written down in the orthogonal curvilinear coordinate system $(r, \phi, \chi)$, but the form of the differential operators retained in it after truncation makes it possible to establish a direct analogy with the problem of wave propagation in a lens-like medium elongated along the $T$ axis. This analogy can also be taken further to discrete optical lines--chains of quadratic phase correctors (thin lenses). The absence of axial symmetry ($a_{12} \neq a_{21}, a_{13} \neq a_{31}$) leads to astigmatism--each individual element of the line maps a point into two crossed segments. As few as two lenses for which the planes of the principal normal sections do not coincide (torison effect) blur the image. In geometrical optics, astigmatism is usually included among the aberrations, but in quasioptics it can be taken into account in an approximation that (with a certain degree of license) we shall call aberrationless.

Pseudogyrotropy

Sometimes it is advantageous, for one reason or another, to represent the field of a wave beam in a skew coordinate system $(r, \eta, \phi)$ associated with the natural trihedral. For this, it is necessary to apply a rotation transformation to the coordinates $(r_1, \eta_2)$ and field $[E_{\phi \ell} = R_{\phi \ell}(\theta) \eta_\ell, E_\phi, E_\eta, E_\chi]$. Bearing in mind that the derivative of the rotation operator is $R_{\phi \ell}(\theta) = R_{\phi \ell}(\theta + \pi/2)$, we obtain after truncating (2.6) the system of equations

$$2i \frac{\partial}{\partial \tau} \dot{R}_{\phi \ell} \dot{W}_\ell + \frac{1}{2} \frac{\partial^2 \varepsilon}{\partial \tau^2} \dot{W}_\ell - R_{\phi \ell}(\theta) \tau_\eta \tau_\phi \tau_\ell \dot{W}_\ell = \frac{2i}{k_0} \sqrt{\varepsilon_0} \Phi \dot{W}_\ell ,$$

$$2i \frac{\partial}{\partial \tau} \dot{R}_{\phi \ell} \dot{W}_\ell + \frac{1}{2} \frac{\partial^2 \varepsilon}{\partial \tau^2} \dot{W}_\ell - R_{\phi \ell}(\theta) \tau_\eta \tau_\phi \tau_\ell \dot{W}_\ell = -\frac{2i}{k_0} \sqrt{\varepsilon_0} \Phi \dot{W}_\ell ,$$

$$\dot{B}_{\phi \ell} = \frac{\partial \Phi}{\partial \tau} + T(\tau) \tau_\eta \tau_\phi \tau_\ell \dot{W}_\ell = \frac{\partial \Phi}{\partial \tau} + \frac{\partial^2 \Phi}{\partial \tau^2} .$$

where $\tau_\ell$ is the unit antisymmetric tensor.

Equations (2.10) can be decoupled for waves with circular polarizations: $\dot{W}_\ell = \dot{W}_\ell + i \dot{W}_\ell$. Multiplying the second of Eqs. (2.10) by the imaginary unit, and adding it to the first (and then subtracting from it), we obtain

$$2i \frac{\partial}{\partial \tau} \dot{B}_{\phi \ell} + \frac{1}{k_0} \frac{\partial}{\partial \tau} \dot{B}_\phi = -\frac{2i}{k_0} \sqrt{\varepsilon_0} \tau_\eta \tau_\phi \tau_\ell \dot{W}_\ell = 0 .$$

The corrections of opposite signs to the effective refractive indices of waves with right and left circular polarizations establish a definite similarity between (2.11) and the equations for waves in optically active media (sugar solution, turpentine). One can say that the torsion of the propagation path leads to "quasi-" or "pseudogyrotropy" effects in media without spatial dispersion. We may mention in passing that in other situations, when $\Delta \approx \Delta_\chi$, it is more convenient to interpret the gyrotropy due to multiple scattering by macroscopic inhomogeneities as a manifestation of structural spatial dispersion of long-range order.

The skew nature of the coordinate system $(r, \eta, \phi)$ also leads to the replacement in (2.11) of the derivative with respect to $\tau$ by the operator $\dot{B}_{\phi \ell}$, which in the comoving cylindrical coordinates $(r_\ell, \phi)$ reduces to the form

$$\dot{B}_{\phi \ell} = \frac{2i}{k_0} \begin{bmatrix} \dot{\theta} \\ \dot{\chi} + \sqrt{\varepsilon_0} \phi \dot{\phi} \end{bmatrix} .$$

A similar operator occurs in the Schrödinger equation for an electron in a constant magnetic field; for its influence on the structure of the field, see Appendix 3.

3. BEAMS IN SYSTEMS ADMITTING SEPARATION OF THE VARIABLES

If it is admissible to go over to coordinates $(x_1, x_2)$ in which the quasioptical equation is invariant with respect to the substitutions $x_2 \rightarrow -x_2$ and $x_1 \rightarrow -x_1$, it is possible to use the method of separation of variables, representing the complex field amplitude in the form

$$W(x_1, x_2) = X(x_1)X(x_2) .$$

Substituting (3.1) in (2.8), where $\alpha_2=0$, we obtain two equations of the same kind. We write down only one of them, omitting the subscripts:

$$2i \frac{\partial X}{\partial \tau} + \frac{\partial^2 X}{\partial \tau^2} - k_0^2 \alpha(\tau) X^2 = 0 .$$

We go over in (3.2) to the dimensionless variables $(z, y)$ by means of the substitution

$$y(\tau) = \frac{x}{\alpha(\tau)} , \quad z(\tau) = \int_0^\tau \frac{d\tau}{k_0^2 \alpha(\tau)} ,$$

$$X = \sqrt{\frac{k_0}{\alpha}} \Phi(z, y) \exp \left( \frac{k_0}{2} \alpha \sigma \tau \right) ,$$

where $\alpha(\tau)$ is an as yet arbitrary function. As a result, Eq. (3.2) is reduced to the form

$$2i \frac{\partial V}{\partial \tau} + \frac{\partial^2 V}{\partial \tau^2} - k_0^2 \alpha^2 \tau^2 \Phi(z, y) \alpha \sigma \tau = 0 .$$

where $\Phi(z, y)$ is the effective refractive index of the medium.
It is obvious that (3.4) can be reduced in two ways to canonical forms by equating the coefficient of \(y^2\) to unity or zero. We consider each of these possibilities separately.

3.1. Expansion with respect to the modes of the discrete spectrum

Setting in (3.4)
\[
\tilde{\sigma} + \sigma (r) \sigma = \frac{1}{k_0^2 \sigma}, 
\]
we obtain the equation of a "quantum-mechanical oscillator":
\[
\left(2 \frac{\partial}{\partial z} + \frac{\partial^2}{\partial y^2} - y^2\right) V = 0.
\]

The general solution of this equation can be represented in the form of a series in Hermite functions:
\[
V = \frac{C_0}{\sqrt{(2\pi)^2 n!}} \exp\left[-\left(n + \frac{1}{2}\right) \frac{z^2}{2}\right] H_n(y),
\]
where
\[
H_n(y) = (-1)^n \exp(y^2) \frac{d^n}{dy^n} \exp(-y^2)
\]
is a Hermite polynomial.

The general solution of Eq. (3.5) for the characteristic width of the eigenfunctions contains two arbitrary constants, on the values of which the form of the expansion (3.7) in the dimensionless variables \((z,y)\) does not depend. However, in the "real" space \((r,x)\) the structure of the modes depends critically on the choice of the constants of integration (3.5). The possibility arises of making an optimum choice of the modes of the discrete spectrum in different applied problems. We shall return to this question and mention here only one detail—expansion of the wave field with respect to the modes of the discrete spectrum is possible not only in the case of focusing layers \([\sigma(r)>0]\) or, putting it differently, waveguide channels, but also in the case of defocusing inhomogeneities \((\sigma<0)\) (then, of course, the characteristic widths of the modes will increase exponentially along the reference ray).

3.2. Expansion in modes of the continuous spectrum

If we choose the normalization parameter \(\sigma(r)\) in such a way that it satisfies the linear equation
\[
\tilde{\sigma} + \sigma (r) \sigma = 0,
\]
then (3.4) can be reduced to the "vacuum" form
\[
\left(2 \frac{\partial}{\partial z} + \frac{\partial^2}{\partial y^2}\right) V = 0.
\]

In the quasioptics of homogeneous media, the solution of (3.9) is, as a rule, represented in the following two forms:
1) expansions in Green’s functions,
\[
V(z, y) = \frac{1}{\sqrt{2\pi i z}} \int_{-\infty}^{\infty} V_d(y') \exp\left(\frac{iy}{2z} (y-y')^2\right) dy';
\]
2) expansions in plane waves,
\[
V(z, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} V(q) \exp\left(-\frac{iq^2}{2} q^2 + iyq\right) dq.
\]
The linearity of the ray equations (3.13) makes it possible to introduce a fundamental system of solutions \( P(\tau) \) and \( S(\tau) \) that satisfy the initial conditions

\[
P(0) = 1, \quad \dot{P}(0) = 0, \quad S(0) = 0, \quad \dot{S}(0) = 1,
\]

with Wronskian \( \partial \ln P \partial \ln S = P S - P \dot{S} = 1 \). The solution of Eqs. (3.13) with initial conditions \( x(0) = x' \), \( \dot{x}(0) = \dot{x}' \) can be written in the form

\[
x = P(\tau)x' + S(\tau)p', \quad \dot{x} = \dot{P}(\tau)x' + \dot{S}(\tau)p'.
\]  

(3.14)

Here \( P(\tau) \) determines the rays of an initially plane wave, and \( S(\tau) \) determines rays that diverge as a fan from the origin (these circumstances determine the choice of notation in accordance with the initial letters of the words plane and source). However, \( P \) and \( S \) also have a different "geometrical" meaning—they are also a fundamental system of the equation of the normal sections of the ray tubes (this equation is obtained by differentiating the ray equations with respect to the parameter and, by virtue of the linearity of these equations, is identical to them). The solution of any problem in geometrical optics can be expressed in terms of the fundamental system of rays \( P \) and \( S \) (in the small-angle approximation).

For example, fixing \( x \) and \( x' \) in (3.14), we obtain (in the approximation of geometrical optics) an expression for the eikonal \( \Psi(x,x') \) of the two-point Green’s function:

\[
\frac{\partial \Psi}{\partial \tau} = -\dot{x}' = S^{-1}(P x' - x),
\]

\[
\frac{\partial \Psi}{\partial x} = p = S^{-1}(\dot{S} x' - x'),
\]

(3.15)

(3.15)

The amplitude of the Green’s function in the approximation of geometrical optics is determined from the law of conservation of the energy flux within a ray tube surrounding the ray and joining the points \( x' \) and \( x \). Consequently, the amplitude \( A \) at \( x_0 \) as follows:

\[
A(x_0) = \frac{1}{2 \pi} \int_{-\infty}^{\infty} f(x) \exp(-ik_0 x x') dx.
\]  

(3.16)

(3.16)

4. BEAMS IN SYSTEMS WITH "TORSION"

When the reference ray is an arbitrary curve with variable torsion, it is not possible to separate variables in Eq. (2.8). However, the important property of the aberrationless approximation noted above still holds—geometrical optics for the Green’s function of originally plane and focused waves gives the exact result. The ray equations remain (as in one-dimensional systems) linear:

\[
\frac{d x_m}{d \tau} = P_m(x), \quad \frac{d \phi_m}{d \tau} = -\alpha_m(x) x_m.
\]  

(4.1)

Their solution with the initial conditions

\[
x_m(0) = x_0, \quad \phi_m(0) = \phi_0,
\]

can be represented in the form

\[
x_m = P_m(x) \phi_m + S_m(x) \phi_0,
\]

\[
\phi_m = P_m(x) \phi_m + S_m(x) \phi_0.
\]  

(4.2)

The matrices \( P \) and \( S \) are themselves solutions of the equations for the cross sections of the ray tubes:

\[
\frac{d x_m}{d \tau} = P_m(x), \quad \frac{d \phi_m}{d \tau} = -\alpha_m(x) x_m,
\]

\[
\frac{d \phi_m}{d \tau} = \alpha_m(x) \frac{d x_m}{d \tau}.
\]  

(4.3)

The matrices \( P \) and \( S \) are themselves solutions of the equations for the cross sections of the ray tubes:

\[
\frac{d x_m}{d \tau} = P_m(x), \quad \frac{d \phi_m}{d \tau} = -\alpha_m(x) x_m,
\]

\[
\frac{d \phi_m}{d \tau} = \alpha_m(x) \frac{d x_m}{d \tau}.
\]  

(4.3)

It is readily shown that by virtue of the symmetry \( \alpha_m = \alpha_m \) the fundamental matrices are related to each other by

\[
S_m = -S_m \delta_m.
\]

The formalism of matrix algebra makes it possible to generalize the two-dimensional results of the previous section to three-dimensional systems. For this, it is necessary to
replace the operations of multiplication and division of a scalar by a scalar by the operations of multiplication of a vector by a matrix and by the inverse matrix. Thus, the integral representations of the field analogous to (3.16) and (3.17) are transformed as follows:

1) Expansion in Green’s functions:

\[
W(r,x) = \frac{k_0}{2\pi i D_I} \int \int W_0(x') \exp(ik_0\Psi(x,x')) dx_1dx_2,
\]

\[
\Psi = \frac{1}{2} \sum_{\alpha} \left( P_{\alpha xv} x_v + S_{\alpha xv} x_v \right). \quad D_I = \text{det}[\mathcal{S}].
\]

2) Expansion in originally plane waves:

\[
W(r,x) = \frac{1}{2\pi i D_J} \int \int W_0(q) \exp(ik_0\Psi(x,q)) dq_1dq_2,
\]

\[
\Psi = \frac{1}{2} \sum_{\alpha} \left( -S_{\alpha qv} q_v + 2x_\alpha q_v + P_{\alpha xv} x_v \right). \quad D_J = \text{det}[\mathcal{P}].
\]

Note that the expansion in Green’s functions is completely invertible, i.e., in (4.3) it is possible to interchange the distribution of the complex field amplitude in the initial section and in the plane of observation:

\[
W_0(x') = \frac{i k_0}{2\pi i D_J} \int \int W(r,x) \exp(-ik_0\Psi(x,x')) dx_1dx_2.
\]

In the aberrationless approximation, a wave beam with Gaussian distribution of the complex field amplitude in the initial section \(r=0\) always remains Gaussian. Substituting into (4.3)

\[
W_0(x') = \frac{i k_0}{2\pi i D_J} \int \int W(r,x) \exp\left[-\frac{1}{2} \left( \sigma_{\alpha v} x_v - ik_0 \gamma_{\alpha v} \right)^2 \right],
\]

we obtain

\[
W(r,x) = \frac{i k_0}{2\pi i D_J} \int \int \left[ \gamma_{\alpha v} + \frac{i}{k_0} \gamma_{\alpha v} x_\alpha \right] \exp\left[ -\frac{1}{2} \left( \sigma_{\alpha v} x_v - ik_0 \gamma_{\alpha v} \right)^2 \right].
\]

(\(D_J\) and \(D_I\) are the determinants of the corresponding matrices). The symmetric matrices \(\sigma^{\alpha v}\) and \(\sigma^{-1}\) give the characteristic dimensions of the beam and the curvature of the phase front in the initial section \(r=0\). In any other normal section, the corresponding quantities are determined by

\[
\gamma_{\alpha v} = \alpha_{\alpha v} (P_{\alpha xv} + S_{\alpha xv}) + \frac{i}{k_0} S_{\alpha v},
\]

\[
\sigma_{\alpha v}^{-1} = \text{Im}(k_0 \gamma_{\alpha v}), \quad R_{\alpha v} = \text{Re}(\gamma_{\alpha v} \gamma_{\alpha v}).
\]

The matrices \(\left[\sigma_{\alpha v}^{-1}\right]\) and \(\left[R_{\alpha v}\right]\) are symmetric, but in the general case they cannot be simultaneously reduced to diagonal form by a rotation since the ellipses of equal intensities and of equal phases are rotated relative to each other, and the angle of rotation changes along the propagation path.

5. CONCLUSIONS

The procedure described above for integrating wave equations in the quasioptical (aberrationless) approximation can be conveniently implemented on a computer. It reduces to the solution of the system of ordinary differential equations (2.2), (2.4), and (4.1) and subsequent calculation of evolutions of the type of (4.3) or (4.4). We emphasize that the first block of calculations is completely independent of the actual structure of the required wave field and reflects only universal properties of the propagation path. This is an important difference between our approach and the traditional generalizations of geometric optics (Maslov’s method, the Kravtsov–Ludwig method of reference functions, etc.), in which the wave field is found in an appropriate ray approximation. In the second stage, one can simultaneously calculate the fields of wave beams with different initial distributions of the amplitude, phase, and polarization in all cross sections that are of interest (without calculating of the field in the intermediate regions); moreover, these distributions need not be smooth (their integrability is sufficient).

APPENDIX I: “HELICAL” COORDINATE SYSTEM

The geometrical properties of the comoving coordinate system introduced in Sec. 2 does not depend on whether or not the reference curve is a ray. In this Appendix, we illustrate some of these properties for the example of a helical reference curve having in a cylindrical coordinate system \((r,\phi,z)\) an equation of the form

\[
r = a, \quad \phi = \phi_0 z = \Phi_\phi,
\]

where \(a\) is the radius of the surface around which the reference curve is “wound”; \(\Phi_\phi = h/2\pi\), where \(h\) is the pitch of the spiral. The curvature and torsion of the helical curve are

\[
K = 2a, \quad T = \frac{h}{a^2 + \frac{1}{4}h^2}.
\]

The principal normal \(n\) is directed along the radius; the vector \(\ell\) tangent to the curve makes an angle \(\gamma = \tan^{-1}(a/h)\) with the \(z\) axis. The element of arc length is \(ds = d\phi/\sqrt{a^2 + h^2}\). The natural trihedral revolves about the \(z\) axis and makes a complete revolution when the variable \(\phi\) changes by \(2\pi\). The orthogonal basis \((\ell,\epsilon_1,\epsilon_2)\) rotates to the left (for a right-handed spiral \(h>0\)) with respect to the natural trihedral:

\[
\frac{d\theta}{d\phi} = -\frac{h}{2a^2 + h^2},
\]

where \(\theta\) is the angle between \(n\) and \(\epsilon_1\). In one period of the helical curve, the lag of the orthogonal basis behind the natural dihedral is measured by the angle \(\theta_\phi = h\sqrt{a^2 + \frac{1}{4}h^2}\).

It can be seen that for a compact spiral \((h=a)\) the angle \(\theta_\phi \to 0\), i.e., the orthogonal basis hardly lags behind the natural trihedral. In the other limiting case of a significantly stretched spiral \((h>a)\), we have \(\theta_\phi \to -2\pi\), i.e., the lag is greatest, and the orthogonal basis hardly rotates relative to...
the $z$ axis. However, over extended paths (over many turns of the spiral) it is necessary to take into account the integrated effects, which lead to a rotation of the orthogonal basis about the $z$ axis through an angle $\Delta \phi = \sqrt{\frac{a^2}{2} \beta}$.

**APPENDIX 2: ON THE GENERALIZATION OF RYTOV’S LAW IN QUASIOPTICS**

Rytov’s law for the rotation of the polarization vector (2.4) is obtained in the geometrical optics approximation. In wave optics, the field at the point of observation is formed by the interference of signals that arrive along different rays from the region of the sources. The torsion of these interference rays (indices $i$) differs in principle from the torsion of the reference ray, and one can speak of their relative torsion $\psi_i = \psi_i - \psi_0$. Therefore, the polarizations of the partial signals differ from each other. However, in quasioptics (in the small-angle approximation) the interference rays consist of segments of strongly elongated spirals that wind around the reference ray. As was shown in Appendix 1, in this case the orthogonal basis (and, therefore, the polarization vector) for the partial ray will not twist (up to terms of order $\sqrt{\frac{a^2}{2} \beta}$) relative to the orthogonal basis of the reference ray. Therefore, for paths with moderate extension $r$, Rytov’s law is also fairly well satisfied in the quasioptical approximation.

**APPENDIX 3: INFLUENCE OF TORSION OF THE REFERENCE RAY ON THE FOCUSING PROPERTIES OF THE EQUIVALENT OPTICAL LINE**

Torsion of the reference ray not only leads to the Rytov polarization effect (pseudogyrotropy) but also changes the focusing properties of the equivalent optical line. In this Appendix, we consider the special case with constant torsion ($T = \text{const}$) of the reference ray (a helical curve) and torsion-independent (in the frame of reference attached to the natural trihedral) parameters of the medium:

$$\beta_1 = \text{const}, \quad \beta_2 = \text{const}, \quad \beta_3 = 0 \quad (A.3.1)$$

[see (2.9)]. Without loss of generality, we can also set $\theta_0 = 0$. Such properties are possessed, for example, by a gradient-index glass fiber that spirals round a cylindrical surface.

As was noted in Sec. 4, the properties of the equivalent optical line are completely determined by the behavior of the geometrical-optics rays, the equations of which in the co-moving skew coordinate system $(r, \gamma_1, \gamma_2)$ have the form

$$\frac{d\gamma_1}{dT} = p_1 + T q_1, \quad \frac{d\gamma_2}{dT} = p_2 + T q_2,$$

$$\frac{dp_1}{dT} = -\beta_1 q_1 - T p_2, \quad \frac{dp_2}{dT} = -\beta_2 q_1 + T p_1. \quad (A.3.2)$$

The solution of the system of equations (A.3.2) can be represented in the form

$$\eta_0 = \text{Re} \sum_{j=1}^s c_{ij} \exp(i H_0),$$

where $c_{ij}$ are constant complex coefficients determined from the initial conditions. The "spatial frequencies" $H_0$ of the ray oscillations satisfy the characteristic relation

$$H_0^2 = T^2 + \frac{\beta_1 + \beta_2}{2} r + \frac{1}{2} \sqrt{(\beta_1 - \beta_2)^2 + 8(\beta_1 + \beta_2)T^2}.$$

(A.3.3)

Figure 1 is the graph of the dependence $H_0^2(T^2)$. It can be seen from Fig. 1 that torsion of the reference ray leads to an enhancement of the focusing properties of the effective optical line in certain directions and to a weakening in others. Moreover, there exists a range of values for the torsion, $\beta_2 < T < \beta_1$, in which one of the ray modes is unstable ($H_0^2 < 0$), and, therefore, the optical line acquires defocusing properties, even though it is made of collecting lenses for all directions ($\beta_3 > 0$).

For this effect, there is a two-dimensional analog. Let us consider, for example, an axisymmetric parabolic wave channel whose focusing properties vary along the axis in accordance with a harmonic law:

$$\nu = \nu_0 - \beta(1 + a \sin \Omega r)r^2.$$

In such a channel, the rays are described by the Mathieu equation

$$\frac{d^2r_j}{dT^2} + \beta(1 + a \sin \Omega r)r_j = 0,$$

which has a discrete set of instability bands (the first band $\Omega_2 - 4\beta$). In contrast to an asymmetric system, in a twisted...
elliptical wave channel there exists only one continuous band of parametric instability. An example of such an instability is given in Fig. 2.

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1) Systems are generally called optical if they obey the laws of geometrical optics sufficiently well.

2) Quasioptics as a branch of science has a fairly ancient tradition. The investigations of Fresnel on diffraction by an opening in a screen and at the edge of a screen were made in the approximation of quasioptics. Schrödinger wrote down an equation that, essentially, is the quasioptical approximation of a more rigorous relativistic equation. Of course, in all these (and other) "prehistoric" cases a different terminology was used—the small-angle, for example, or paraxial approximation.1

3) It should be noted that in optics effects associated with astigmatism of the refracting surfaces are usually classed as aberration effects. In quasioptics effects associated with astigmatism of the refracting surfaces) are usually classed as aberration effects. In quasioptics there is a somewhat different definition—the aberrationless approximation is assumed to be one in which a Gaussian wave beam remains Gaussian (with parabolic phase front).

4) Apart from the terminology and different notation, all results will, of course, be valid for waves of arbitrary nature—acoustic, seismic, information, etc.

5) The choice of the reference ray is not unique and can vary depending on the type of problem to be solved. In problems in which the wave beam is formed by a system of emitters or collimators, it is sensible to choose as the reference ray the ray that emanates from the center of the beam-forming aperture (or other device) in the direction of the maximum of the beam pattern.

6) The polarization degeneracy inherent in (2.6) is lifted in the general case in the following order in \( \alpha \) and \( \mu \); this can lead to significant effects over very extended propagation tracks.

7) Note that wave beams in a discrete optical line can be described by the same equation (2.8) if we set \( a_{\alpha\mu}(r) = \delta_{\alpha\mu} - \gamma r - \alpha_{\alpha\mu} \) where \( R(r) \) is the Dirac delta function and \( k \) is the number of the corrector.

8) In optics and electrodynamics, the directions of the circular polarizations are defined differently. In the given case, the "plus" sign corresponds to right-circular polarization in the electrodynamical sense—the direction of rotation of the polarization vector is related to the wave vector by the right-hand screw rule.

9) This symmetry can be nominally called mirror symmetry, though it is only such for the equivalent (rectified) optical line. In the real space, mirror symmetry may not be present.

10) The completeness and orthogonality of such an expansion can be proved as follows. An ideal quadratic corrector with optical strength \( 1 / \ln N \) is positioned in the section \( r = 0 \). At its "output," the field is expanded in a Fourier integral, and one then positions a compensating corrector with optical strength \( 1 / \ln N \), which transforms each plane wave of the expansion into a focused wave.

11) The profile of the permittivity of the fiber glass has the form \( \epsilon = \epsilon_0 - \gamma r \) and it is wound onto a cylinder of radius \( a \) with pitch \( 2 \pi b \), then \( \beta_b = \gamma r 2 \pi b [a^2 + (r/a)^2]^{1/2} \), \( \beta_r = \gamma r \).

12) For discrete optical lines and in cavities, astigmatism can also lead to instability of quasioptical modes.