Amplification of a Schrödinger soliton in a bounded frequency band

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Using the adiabatic approximation we consider the process of the nonlinear propagation of a basic optical soliton in a fiber with a cubic nonlinearity and losses. The latter are compensated by periodic or distributed amplification with a band width comparable to the width of the soliton spectrum. We obtain analytical expressions describing the evolution of the dispersions of the fluctuations in the momentum and in the position of the soliton peak. We establish that when the amplification band width and the width of the soliton spectrum are approximately equal they are very close to the quantum momentum-coordinate indeterminacy relations of a basic soliton propagating in an ideal lossless nonlinear waveguide. We give the correction to the Gordon–Haus quantum limit taking into account the frequency drag of the soliton carrier frequency under the amplification line contour.

INTRODUCTION

Rather many papers (see, e.g., Refs. 1 to 25) have in recent years already been devoted to the problem of the storage and transformation of classical and quantum noise by an optical Schrödinger soliton propagating in a light conductor with a cubic nonlinearity. Whereas the features of the behavior of solitons in an ideal lossless fiber have been studied very thoroughly[1–16] it seems that an elucidation of the quantum aspects peculiar to their propagation in a waveguide with dissipation and of the possibility of its compensation through distributed or periodically positioned amplifiers is only in its initial stage.[17–21] A pioneering paper in this respect is Ref. 17 in which a semi-classical description is used to estimate the random drift of the carrier frequency of the soliton caused by noise photons produced in the amplifiers. The change in the propagation velocity then leads to fluctuating shifts of the soliton peak. The distance over which these shifts become comparable with the soliton length has been called the Gordon-Haus quantum limit. A more rigorous consistent quantum treatment of this effect has not yielded any important corrections to the quantitative relation.[18,19] However, in experiments on the observation of groups of solitons propagating along paths even exceeding the Gordon–Haus quantum limit the corresponding changes in the distances between the solitons have as yet not been detected.

What causes this violation of the theoretical prediction? The fact is, these prognoses were clearly based on assumptions about an unbounded frequency band of the amplifiers. However, it is clear even on a qualitative level that the finite width of the amplification line must limit the growth of the indeterminacy of the carrier frequency of the soliton and of the random shifts corresponding to it.[19] The first quantitative estimates of this "saturation effect" were given in Ref. 20, the main result of which is the conclusion that there is not only a slowing down of the growth in the fluctuations of the carrier frequency but also the presence of a "bending point" after which these fluctuations decrease without limits. The choice of a sufficiently large propagation length can thus apparently lead to an arbitrarily stable soliton. This result induces definite misgivings about the adequacy of our considerations for two reasons. Firstly, the distributed acceleration noise acts along the whole of the path and does not "disturb" the soliton at its end worse than at its start. It would thus be reasonable to expect a gradual establishment of some stationary value of the frequency indeterminacy determined by the amplification band width. The second reason for doubt is not connected with physical subtleties of the effect but is based upon a purely logical contradiction.

Suppose that at the entrance to the waveguide an ideal soliton without any indeterminacies (we forget temporarily the purely quantum effects, the more so as they are neglected in the semiclassical considerations applied by the authors of Ref. 20). We shall then have practically the same stable soliton, according to Ref. 20, at sufficiently large distances. What will happen after that? According to the authors of Ref. 20, the soliton will become even more stable. But how is it better than the original ideal soliton in which in the initial stage of the propagation there occurs a very strong growth of the frequency drift? If, however, the two solitons are identical, then why does the noise grow at the beginning of the path but decrease at the end?

The above conclusion of Ref. 20 is thus erroneous. Moreover, the following fact causes a definite dissatisfaction. In that paper the basic parameter determining the effect of the finite amplification line width is the excess of the maximum amplification growth rate in the peak of the contour over the losses. Since the dissipation is assumed to be uniform over the frequency no growth in the soliton amplitude will occur in that case since there is no amplification (or loss) when integrated over the frequency. Therefore, if the amplification and absorption lines are identical (of course, apart from the sign) this excess will be equal to zero and the evolution of the noise will, according to Ref. 20, be described by the Gordon–Haus functions.[17] However, it is very clear that also in that case the effect of

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a finite amplification linewidth must show up (if only because the noise photons will occur only in that bounded frequency band) and cause saturation of the growth of the frequency fluctuations. We note that in practice such a situation may occur in an active fiber (simultaneously playing the role of the propagation medium and of the amplifier) which is boosted externally by exciting radiation (see, e.g., Refs. 1 and 26 and the literature cited there).

The solution proposed in Ref. 20 is thus not a comprehensive one and the problem under consideration needs a more detailed study, which is the subject of the present paper.

1. BASIC MODEL AND MAIN EQUATIONS

For our considerations we start from the following Schrödinger equation with a nonvanishing right-hand side in the Heisenberg representation:[9,10]

$$\frac{\partial^2\phi(x,s)}{\partial x^2}+i\frac{\partial\phi(x,s)}{\partial s}+2C\phi^+(x,s)\phi^2(x,s)+\frac{i}{2}(\varkappa-\gamma)\otimes\phi(x,s)=i[\Gamma_\varkappa(x,s)+\Gamma_\gamma(x,s)]. \quad (1.1)$$

Here

$$x=vt-z$$

is the distance measured from the soliton peak in a moving system of coordinates moving along the z-axis together with the soliton with a group velocity

$$v=(k')^{-1}\equiv(\partial k_0/\partial\omega_0)^{-1},$$

k_0 is the wavenumber and ω_0 the carrier frequency,

$$s=\frac{1}{2}\frac{k''}{|k'|^2}z$$

is the normalized path of the soliton, and the parameter

$$k''=\partial^2k_0/\partial\omega_0^2$$

characterizes the dispersion of the group velocity; in order not to deviate from the usual notation for Schrödinger equation, in what follows we replace the independent variable s by t, considering it as the normalized time for the soliton propagation; $C>0$ is the nonlinearity parameter which is proportional to the cubic susceptibility of the fiber, and $\varkappa$ and γ are the absorption and amplification growth rates (with respect to the intensity) which in general depend on x for absorption and amplification lines which are inhomogeneous along the spectrum, and in that case a convolution occurs, denoted by $\otimes$. The field operators ϕ and ϕ^+ satisfy the commutation relations

$$[\phi(x',t),\phi^+(x,t)]=\delta(x-x'),$$
$$[\phi(x',t),\phi(x,t)]=[\phi^+(x',t),\phi^+(x,t)]=0. \quad (1.2)$$

In fact, these normalized operators are the positive- and negative-frequency parts of the field. The normalization is such that

$$\int_{-\infty}^{\infty}\langle\phi^+(x,t)\phi(x,t)\rangle dx=n_0, \quad (1.3)$$

where n_0 is the average number of photons in the soliton, while the averaging is over its quantum state.

The right-hand side of Eq. (1.1), which is also an operator, is a Langevin noise force with a zero average. If we change to the Fourier spectrum of the operators, for instance,

$$\tilde{f}(p,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x,t)e^{ipx}dx, \quad (1.4)$$

the statistical properties of its terms have the form

$$\langle\tilde{\Gamma}_\gamma^+(p',t')\tilde{\Gamma}_\gamma(p,t)\rangle=\gamma(p)\delta(p'-p,t'-t),$$
$$\langle\tilde{\Gamma}_\gamma\tilde{\Gamma}_\gamma^+\rangle=\langle\tilde{\Gamma}_\gamma\tilde{\Gamma}_\gamma\rangle=\langle\tilde{\Gamma}_\gamma^+\tilde{\Gamma}_\gamma^+\rangle=0. \quad (1.5)$$

$$\langle\tilde{\Gamma}_\varkappa(p',t')\tilde{\Gamma}_\varkappa^+(p,t)\rangle=\varkappa(p)\delta(p'p,t'-t),$$
$$\langle\tilde{\Gamma}_\varkappa^+\tilde{\Gamma}_\varkappa\rangle=\langle\tilde{\Gamma}_\varkappa\tilde{\Gamma}_\varkappa\rangle=\langle\tilde{\Gamma}_\varkappa^+\tilde{\Gamma}_\varkappa^+\rangle=0. \quad (1.6)$$

Here and henceforth we neglect thermal and technical noise for simplicity, restricting ourselves thereby to finding the minimum possible purely quantum contribution to the fluctuations. For a two-level amplifier this means, in particular, that the inversion is complete: $N_2/(N_2-N_1)=1$ where the N_j are the level populations, i.e., the effective amplifier temperature is equal to zero. If these conditions are not satisfied, one can easily take this into account by introducing a correction factor (see, e.g., Ref. 10).

We have also introduced in Eqs. (1.4) to (1.6) the soliton momentum p and taken into account the spectral dependence of its growth rates. The frequency ω is connected with p through the simple relation $\omega=pv$.

If there is no amplification or loss, when we have $\gamma\equiv\varkappa\equiv\Gamma_{\varkappa,\gamma}\equiv0$, Eq. (1.1) changes to the well known nonlinear Schrödinger equation (see, e.g., Refs. 3 to 11). The classical analog of the latter will be obtained by replacing the operators ϕ and ϕ^+ by a pair of complex slowly varying amplitudes $\phi(x,t)$ and $\phi^*(x,t)$. The classical solution for a single basic soliton, which is valid for $C>0$, then has the form[16–19]

$$\phi_0(x,t)=\frac{n_0\sqrt{C}}{2}\exp\left[i\frac{n_0^2C^2}{4}t-ip_0^2t+ip_0(x-x_0)+i\varphi_0\right]\times\operatorname{sech}\left[\frac{n_0C}{2}(x-x_0-2p_0t)\right], \quad (1.7)$$

where we have introduced the initial momentum p_0 determining the carrier frequency and, accordingly, the soliton propagation velocity, as well as the initial coordinate x_0 and phase φ_0. Without loss of generality we can put those equal to zero which is equivalent to the choice of suitable initial conditions. The condition $p_0=0$ corresponds to a frequency mismatch with the carrier frequency: $\Omega\equiv\omega-\omega_0=pv$. According to (1.4) the classical spectrum of the basic soliton then has the form

$$\tilde{\phi}_0(p,t)=\frac{\pi}{\sqrt{C}}\exp\left(\frac{in_0^2C^2t}{4}\right)\operatorname{sech}\left(\frac{\pi}{n_0C}p\right). \quad (1.8)$$

We note that

$$\int_{-\infty}^{\infty} |\phi_0(x)|^2 dx \equiv \int_{-\infty}^{\infty} |\tilde{\phi}(p)|^2 dp = n_0. \quad (1.9)$$

Since it is not possible to obtain an exact analytical solution of the initial operator Eq. (1.1) we discuss in what follows the possibilities of a simple and clear approximate description of the effects which arise when the basic soliton propagates in an active nonlinear channel.

2. EVOLUTION OF THE MOMENTUM INDETERMINACY IN THE ADIABATIC APPROXIMATION

It is well known[22] that the adiabatic approximation assumes that the shape of the soliton remains unchanged while it propagates. This is not contradicted by experimental results or by data from numerical calculations even for the extremely long paths which are at present possible. This, of course, does not exclude at all the possibility in principle that the adiabatic approximation is violated, which may, for instance, occur in fibers with random inhomogeneities[24] but in the framework of our model when the conditions stipulated below are satisfied the use of the adiabatic approximation is completely appropriate.

We select in the waveguide an elementary section characterized by the time Δt for it to be traversed. The effects occurring in it we can conveniently split into three successive stages characterizing the action of various factors: a) the perturbation of the soliton by the action of the quantum noise; b) dissipation with amplification neglecting the noise produced by it; c) nonlinear (soliton) propagation. In the next Δt interval the picture is repeated, and so on.

A similar separation of the factors was used in Ref. 18 for the case of unbounded amplification and loss bands. It was then shown that if $\Delta t \ll T = 8\pi/n_0^2 C^2$, where T is the soliton period, i.e., the time during which the nonlinear phase advance has accumulated to 2π, such a model is equivalent to the situation in which all factors are manifested simultaneously and reduces in practice to the original Gordon–Haus result.[17] It is then completely unimportant whether the lumped active sections of the fiber (or simply the amplifiers) alternate periodically (at intervals Δt) or whether the amplification is distributed uniformly over the whole of its length. The condition $\Delta t \ll T$ also ensures that the soliton shape is unchanged,[27] which guarantees the adequacy of the adiabatic approximation. It is thus logical to expect that when it is satisfied our considerations are also valid.

Thus, in the first stage there occurs a "noisification" of the soliton due to the Langevin forces:

$$\begin{aligned}
&e^{-i\varphi(t+\Delta t)}\tilde{\phi}(p-p',t+\Delta t) \\
&\quad = e^{-i\varphi(t)}\tilde{\phi}(p-p',t) + \int_0^{\Delta t} \tilde{\Gamma}(p-p',t)dt \\
&\quad \approx e^{-i\varphi(t)}\tilde{\phi}(p-p',t) + \tilde{\Gamma}(p-p',t)\Delta t, \\
&\tilde{\Gamma}(p,t) = \tilde{\Gamma}_\varkappa(p,t) + \tilde{\Gamma}_\gamma(p,t). \quad (2.1)
\end{aligned}$$

Here we have taken into account a possible shift of the soliton spectrum by an amount p' and a phase advance φ which do not alter the soliton shape, i.e., we work in the adiabatic approximation.

We multiply both sides by $\exp[i\varphi(t)]$, expand $\exp[-i\varphi(t+\Delta t)+i\varphi(t)]$ in a power series of the argument of the exponential, restricting ourselves to the first order, and we add the equation obtained to the corresponding Hermitian conjugate one. As a result we get

$$\begin{aligned}
&\mathrm{Re}[\tilde{\phi}(p-p',t+\Delta t)] \\
&\quad \approx \mathrm{Re}[\tilde{\phi}(p-p',t)] + \mathrm{Re}[e^{i\varphi(t)}\tilde{\Gamma}(p-p',t)]\Delta t, \quad (2.2)
\end{aligned}$$

where $\mathrm{Re}\, f = (f+f^+)/2$.

We are first of all interested in the random drift Δp of the momentum on an elementary interval Δt which arises due to the action of the random forces Γ. To first order in Δp we can thus write

$$\begin{aligned}
\tilde{\phi}(p-p',t+\Delta t) &\approx \tilde{\phi}(p-p',t) + \frac{\partial\tilde{\phi}(p-p',t)}{\partial p}\Delta p \\
&\equiv \tilde{\phi}(p-p',t) + \phi'(p-p',t)\Delta p. \quad (2.3)
\end{aligned}$$

It is completely obvious that the external force Γ causes not only a shift Δp of the spectrum but also leads to a number of other distortions of the soliton. To distinguish its contribution to the random drift Δp of the momentum we should take the "projection" of $\mathrm{Re}\,\tilde{\Gamma}$ on $\mathrm{Re}\,\phi'$, i.e., a scalar product of the form

$$\int_{-\infty}^{\infty} \mathrm{Re}\,\phi'\,\mathrm{Re}\,\tilde{\Gamma}dp.$$

After substituting (2.3) into (2.2), multiplying both sides of the relation obtained from the left by $\phi'(p-p',t)$, and integrating over p between infinite limits, we have thus

$$\Delta p = \frac{\int_{-\infty}^{\infty}\mathrm{Re}[\phi'(p-p',t)]\mathrm{Re}[e^{i\varphi(t)}\tilde{\Gamma}(p-p',t)]dp}{\int_{-\infty}^{\infty}\{\mathrm{Re}[\phi'(p,t)]\}^2 dp}\Delta t. \quad (2.4)$$

By virtue of the linearity of the problem considered [Eqs. (2.1) to (2.4) are linear in $\tilde{\phi}$] it must be adequate to describe it using a classical approach.[28] The operator $\tilde{\phi}$ can then be replaced by the classical single-soliton solution $\tilde{\phi}_0$. According to (1.8) we have in this case

$$\phi'(p) = -\frac{\pi^{3/2}}{\sqrt{2}n_0C^{3/2}}\,\mathrm{th}\left(\frac{\pi p}{n_0 C}\right)\mathrm{sech}\left(\frac{\pi p}{n_0 C}\right). \quad (2.5)$$

The "quanticity" of the problem, on the other hand, will be guaranteed by introducing in it moments of the random force of the form

$$\begin{aligned}
&\langle \mathrm{Re}[e^{i\varphi}\tilde{\Gamma}(p',t')]\mathrm{Re}[e^{i\varphi}\tilde{\Gamma}(p,t)]\rangle \\
&\quad \equiv \langle \mathrm{Re}\,\tilde{\Gamma}(p',t')\mathrm{Re}\,\tilde{\Gamma}(p,t)\rangle \\
&\quad = [\gamma(p)+\varkappa(p)]\delta(p-p',t-t')/4. \quad (2.6)
\end{aligned}$$

We have used here (1.5) and (1.6).

The dispersion of the fluctuations in the soliton momentum, produced by the external forces on the elementary section Δt, is thus equal to

$$\langle\Delta p^2\rangle=\frac{9C}{8\pi}\Delta t\int_{-\infty}^{\infty}\text{th}^2\left(\frac{\pi p}{n_0C}\right)\text{sech}^2\left(\frac{\pi p}{n_0C}\right)[\gamma(p+p')+\varkappa(p+p')]dp. \quad (2.7)$$

In the particular case of identical absorption and amplification lines which are uniform over the spectrum ($\gamma\equiv\varkappa=\text{const}$) we have

$$\langle\Delta p^2\rangle=3\varkappa n_0C^2\Delta t/2\pi^2. \quad (2.8)$$

The only difference between this last result and the corresponding expression obtained in Ref. 19 on the basis of a completely different approach is that we have not yet taken into account the vacuum fluctuations at the entrance to the waveguide (this will be done in what follows) and also by a practically insignificant quantitative difference: instead of π in Ref. 19 we have three.

We turn to the description of the next stage of the passing of the soliton through the elementary section Δt which directly takes into account that the amplification and loss spectral lines are not identical. In that case we have

$$\tilde\phi(p-p',t+\Delta t)=\tilde\phi(p-p',t)\exp\{[\gamma(p)-\varkappa(p)]\Delta t/2\}\approx\tilde\phi(p-p',t)[1+\gamma(p)-\varkappa(p)]\Delta t/2, \quad (2.9)$$

where, as before, we understand by $\tilde\phi$ not operators but complex amplitudes. For their real parts we have, using (2.3),

$$\Delta p\approx\Delta t\int_{-\infty}^{\infty}\tilde\phi_0(p-p')\phi'(p-p')\times[\gamma(p)-\varkappa(p)]dp/2\int_{-\infty}^{\infty}\phi'^2(p-p')dp. \quad (2.10)$$

Here we have again used the scalar product of the perturbing function and $\phi'(p-p')$.

It is convenient for the further calculations to change to a dimensionless variable,

$$q=\pi p/n_0C. \quad (2.11)$$

Then we have

$$\Delta q=-\frac{3}{4}\Delta t\int_{-\infty}^{\infty}\text{th}\,q\,\text{sech}^2q\times[\gamma(q+q')-\varkappa(q+q')]dq. \quad (2.12)$$

In fact, Eq. (2.12) describes the evolution of the drift of the normalized momentum q', i.e., Δq is the increase in q'. In what follows we can thus omit the primes as q and q' exchange places.

One can estimate the integral in (2.12) analytically by stipulating the line shapes. It is logical to assume them to be Lorentzian:

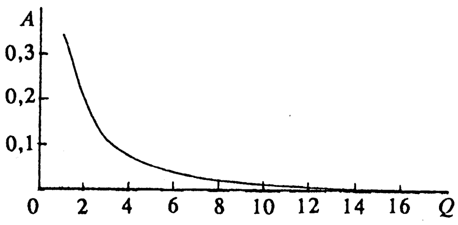

FIG. 1. The function $A(Q)$ constructed as the result of a numerical calculation of the integral in (2.16).

$$\gamma(p)=\frac{\gamma_0}{1+p^2/P^2}=\frac{\gamma_0}{1+q^2/Q^2},\quad \varkappa(p)=\frac{\varkappa_0}{1+p^2/L^2}=\frac{\varkappa_0}{1+q^2/M^2}. \quad (2.13)$$

In the $M\gg1$ case with the absorption line width much larger than the width of the soliton spectrum we have

$$\frac{dq}{dt}\approx-\frac{3\gamma_0}{4}\int_{-\infty}^{\infty}\frac{\text{th}\,y\,\text{sech}^2\,y dy}{1+(y-q)^2/Q^2}, \quad (2.14)$$

where we have changed from finite to differential increments and introduced the integration variable y.

We expand the Lorentzian in powers of q and, assuming the drift of the normalized momentum to be small, we restrict ourselves to the second order terms. Using the fact that $\tanh y\,\text{sech}^2\,y$ is odd we then find

$$\frac{dq}{dt}\approx-\frac{3\gamma_0q}{2Q^2}\int_{-\infty}^{\infty}\frac{y\,\text{sh}\,y\,dy}{\text{ch}^3y(1+y^2/Q^2)^2}, \quad (2.15)$$

i.e.,

$$q=q_0e^{-A\gamma_0\Delta t},\quad A=\frac{3}{Q^2}\int_0^{\infty}\frac{y\,\text{th}\,y\,\text{sech}^2\,y}{(1+y^2/Q^2)^2}dy. \quad (2.16)$$

We show the function $A(Q)$ in Fig. 1. In particular, $A(Q=1)\approx0.3405$.

Finally in the third and last stage of the passage of the soliton through the element Δt there is a nonlinear interaction and dispersive spreading. To describe it we restrict ourselves to simple qualitative considerations avoiding calculations. The fact is that neither of these processes should affect the shift of the soliton momentum (or its carrier frequency). A similar statement could follow even from the classical solution of the nonlinear Schrödinger equation without amplification or loss. This also does not contradict quantum treatments.[3–11]

The specific action in this stage affects the soliton shape but it can be only particularly "favorable," i.e., making the actual envelope, which is distorted by noise and by amplification and losses which are non-uniform along the spectrum, more closely resembling an ideal one in the form of a hyperbolic secant. Indeed, the soliton can retain its exceptional stability just thanks to its remarkable "self-cleaning" property[1,2,14] with regards to the noise which has penetrated it.

Of course, these arguments do not pretend to give an exhaustive proof. Their adequacy can only be ascertained by experimental verification, even if only a numerical one performed in the framework of the semiclassical approach.

Provisionally summarizing the transformation of a soliton on a section Δt of the waveguide we conclude that it acquires an indeterminacy of its momentum determined by the dispersion of the fluctuations in the form (2.7) multiplied by $\exp(-2A\gamma_0\Delta t)$, i.e., there occur two contrasting processes: a "build-up" of the carrier frequency due to noise which is external in relation to the soliton and its stabilization due to the finiteness of the amplification band.

If we choose Δt to be greater than the inverse width of the amplification line the noise on the next elementary section can be assumed to be statistically independent of that on the previous one. Hence, the dispersion of the fluctuations in the momentum from each successive section of the fiber is ultimately summed up. This fact enables us to express $\langle\Delta p^2(t+\Delta t)\rangle$ in terms of $\langle\Delta p^2(t)\rangle$ or, in normalized form, $\langle\Delta q^2(t+\Delta t)\rangle$ in terms of $\langle\Delta q^2(t)\rangle$:

$$\langle\Delta q^2(t+\Delta t)\rangle = \langle\Delta q^2(t)\rangle\exp(-2A\gamma_0\Delta t)+\frac{9\Delta t}{8n_0}\int_{-\infty}^{\infty}\text{th}^2 y\,\text{sech}^2 y \times[\gamma(y+\sqrt{\langle\Delta q^2(t)\rangle}) +\varkappa(y+\sqrt{\langle\Delta q^2(t)\rangle})]dy. \tag{2.17}$$

Here we have equated the drift of the normalized soliton momentum from its average zero value equal to the mean square deviation $\sqrt{D(t)} = \sqrt{\langle\Delta q^2(t)\rangle}$.

We note also that whereas in Ref. 20 a parabolic approximation of the amplification line was used in the "whole volume," in our case it appears only in the first term of Eq. (2.17) and does not affect the second one.

Using (2.13) we have for $M\gg1$

$$D(t+\Delta t)=D(t)\exp(-2A\gamma_0\Delta t)+\frac{3\Delta t}{4n_0}\times\left\{\varkappa_0+\frac{3}{2}\gamma_0\int_{-\infty}^{\infty}\frac{\text{th}^2 y\,\text{sech}^2 y\,dy}{1+[y+\sqrt{D(t)}]^2/Q^2}\right\}. \tag{2.18}$$

We introduce $\Delta D=D(t+\Delta t)-D(t)$, expand the exponent, and replace the finite increments by differential ones:

$$\frac{1}{\gamma_0}\frac{dD}{dt}=\frac{3}{4n_0}\left[\frac{\varkappa_0}{\gamma_0}+\frac{3}{2}\int_{-\infty}^{\infty}\frac{\text{th}^2 y\,\text{sech}^2 y dy}{1+(y+\sqrt{D})^2/Q^2}\right] -2A(Q)D. \tag{2.19}$$

We write the solution of this equation in the form

$$\gamma_0 t=\int_{D_0}^{D}[J(D)]^{-1}dD, \tag{2.20}$$

where $J(D)$ is the right-hand side of Eq. (2.19) and D_0 the initial dispersion of the fluctuations of the normalized soliton momentum at $t=0$. If the cause of those fluctuations is the vacuum quantum indeterminacy, we have $D_0\approx3/4n_0$ (Refs. 16, 18, and 19), where the approximate equality sign reflects that the number π has been replaced by three.

Instead of the required solution $D(t)$ we calculate the inverse function $t(D)$ according to (2.20), i.e., the time during which the soliton acquires the dispersion D.

As one should expect the growth of the momentum fluctuations with time saturates and the subsequent propagation of the soliton is characterized by the asymptotic approach to a stationary value of the dispersion, given by the equation

$$J(D)=0. \tag{2.21}$$

In this connection we remind ourselves once again that this conclusion does not agree with the results of Ref. 20, as was discussed in detail in the Introduction.

Concluding this section we note that the proposed approach enables us also to analyze the particular case where the amplification and loss lines are identical: $\gamma(p)\equiv\varkappa(p)$. In accordance with (2.12) the coefficient $A(Q)$ is equal to zero, i.e., one of the factors stabilizing the carrier frequency is absent. Nonetheless the growth in the soliton momentum fluctuations has also a tendency to saturate because the noise (the Langevin force Γ) acts only in a bounded frequency band.

According to (2.7), Eq. (2.20) remains valid for $\gamma\equiv\varkappa$, only with the difference that $J(D)$ now means

$$J(D)=\frac{9}{4n_0}\int_{-\infty}^{\infty}\frac{\text{th}^2 y\,\text{sech}^2 y dy}{1+(y+\sqrt{D})^2/Q^2}. \tag{2.22}$$

It is clear that the increase in $D(t)$ with t slows down gradually, i.e., saturation occurs again. This fact is illustrated by the curves in Fig. 2. It is also not included in the framework of the results of Ref. 20 where the $\gamma\equiv\varkappa$ case corresponds to the Gordon–Haus approximation when $D(t)$ grows without bound in proportion to t. Moreover, the differences between our considerations and Ref. 20 appear not only in the far zone where saturation turns up but even at the very start of the path for small t. Indeed, in accordance with (2.20) and (2.22) finite amplification and (or) loss bandwidths lead to a decrease in the growth rate of the dispersion D, i.e., in dD/dt. This effect is connected with the integrated (over the spectrum) reduction of the amplification and loss noise as compared to the case of unbounded uniform lines. For instance, for $Q=1$ the dispersion D increases on the initial section approximately twice as slowly as for $Q\to\infty$.

The next section is devoted to a more detailed analysis of the final relations (2.19) to (2.22).

3. FURTHER APPROXIMATIONS

We attempt to simplify the results obtained and to give them more clarity. First of all it would be desirable to get rid of the convolution integrals in the $J(D)$ function. This can be done for $Q\gtrsim1$. The Lorentzian contour is in this case already quite gentle in comparison with the graphical form of the other factor in the integrand, $\tanh^2 y\,\text{sech}^2 y$, which can be replaced approximately by two δ-functions. Indeed, in the range from 0 to ∞ $\tanh^2 y\,\text{sech}^2 y$ has its center of gravity at the point $y_c\approx1.08$. We then have

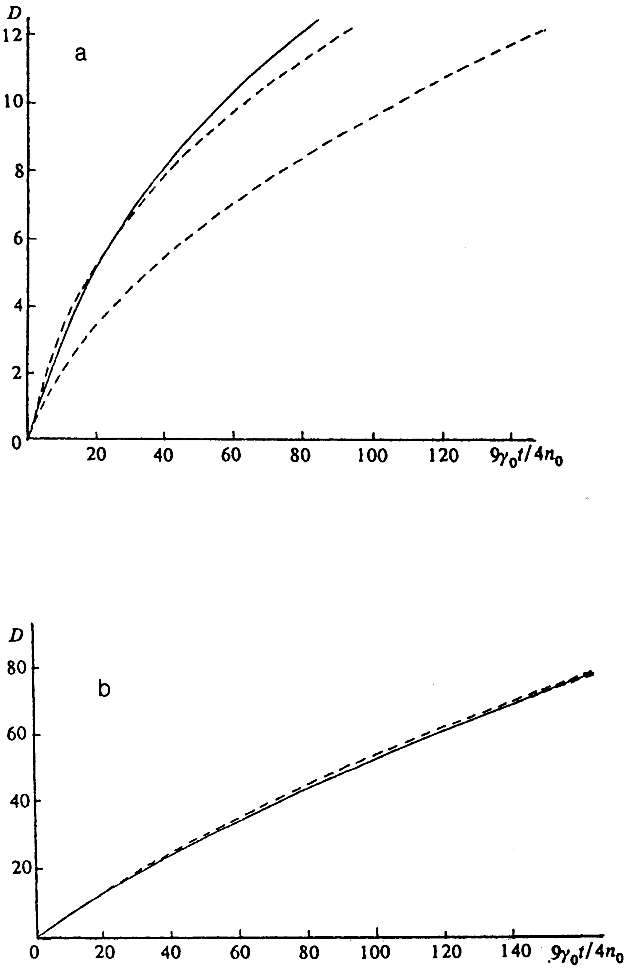

FIG. 2. The normalized dispersion of the soliton momentum fluctuations as a function of the normalized propagation time for $\gamma\equiv\varkappa$. The solid curves are the result of a numerical solution of Eq. (2.20) taking (2.22) into account. The dashed curves are various degrees of approximation corresponding to the first equation in (3.2) and the rougher relation (3.3), with the latter occurring below the former: a: $Q=1$; b: $Q=10$. Everywhere we have $D_0=0$.

$$\int_{-\infty}^{\infty}\frac{\operatorname{th}^2 y \operatorname{sech}^2 y\,dy}{1+(y+\sqrt{D})^2/Q^2}\approx\frac{2Q^2}{3}\frac{Q^2+D+y_c^2}{(Q^2+D+y_c^2)^2-4y_c^2D}. \tag{3.1}$$

We show in Fig. 3 the curves of the exact and the approximate values of the integral. It is clear that even for $Q=1$ we find good agreement.

In the $\gamma\equiv\varkappa$ case we have from (2.20) and (2.22)

$$\gamma_0 t\approx\frac{n_0}{3Q^2}\left[D^2+(2Q^2-7)D+9.332(Q^2+1.166)\times\ln\left(1+\frac{D}{Q^2+1.166}\right)\right]\approx\frac{n_0}{9Q^2}[3D^2+(6Q+7)D] \tag{3.2}$$

for $D_0=0$. If, however, we have $D_0\neq 0$ we must subtract from the right-hand sides of (3.2) the corresponding expressions for $D=D_0$. As a result we get

$$D(t)\simeq[(D_0+Q^2+7/6)^2+3Q^2\gamma_0 t/n_0]^{1/2}-Q^2-7/6. \tag{3.3}$$

For $Q\to\infty$ we have $D\approx D_0+3\gamma_0 t/2n_0$ which corresponds to the model of unbounded amplification and loss bands.[16–19]

The curves constructed in accordance with the approximate expressions (3.2) and (3.3) are also shown in Fig. 2. It is clear that they satisfactorily approximate the more exact calculations. However, for $Q=1$ one observes nevertheless a considerable divergence which is connected with the approximate representation of the logarithm in the second part of Eq. (3.2). For a comparison we therefore show curves constructed without using the expansion of the logarithm [first part of Eq. (3.2)]. We then reach much better agreement.

We also note that in the far zone (for large t) the dispersion of the fluctuations of the soliton momentum (and its carrier frequency) continues to increase as

$$Q\sqrt{3\gamma_c t/n_0}\propto\sqrt{t}.$$

We now turn to an analysis of the $\gamma\not\equiv\varkappa$ case for $\varkappa=$const. We can obtain a rough estimate by assuming the amplification noise to be uniform over the spectrum (white noise) and by taking the boundedness of the spectrum into

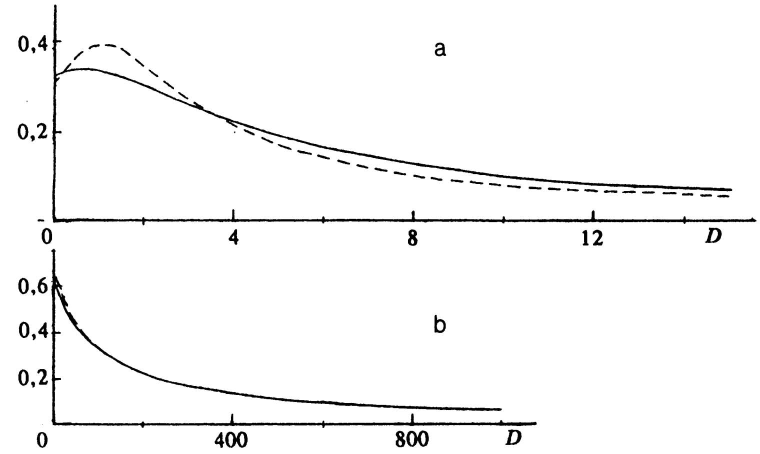

FIG. 3. Values of the integral (3.1) established as the result of a numerical calculation (solid curves) and of a rough approximation (dashed curves): a: $Q=1$; b: $Q=10$.

account only through the last terms in (2.19). We then obtain somewhat overestimated values of the dispersion which satisfy the equation

$$\gamma_0^{-1} dD/dt \approx 3/2n_0 - 2AD, \tag{3.4}$$

where one can easily estimate the stationary value D_s which the dispersion approaches asymptotically:

$$D_s \approx 3/4An_0. \tag{3.5}$$

Further, according to (3.4) we have

$$D(t) = D_0 \exp(-2A\gamma_0 t) + 3[1-\exp(-2A\gamma_0 t)]/4An_0. \tag{3.6}$$

If, however, the input fluctuations are determined only by the vacuum quantum indeterminacy ($D_0 = 3/4n_0$) we have

$$D(t) = 3[1-(1-A)\exp(-2A\gamma_0 t)]/4An_0. \tag{3.7}$$

The basic conclusion from these results is the following. If the width of the soliton spectrum is comparable with that of the amplification band ($Q \approx 1$) the momentum fluctuations of this soliton will be comparable with its quantum indeterminacy. For instance, for $Q=1$ the maximum dispersion D_s is not more than three times larger than D_0.

Moreover, for $An_0 \gg 1$, which is practically always satisfied by virtue of the large number of photons in the soliton, we have, according to (3.5), $D_s \ll 1$, i.e., the random drift of the carrier frequency is considerably smaller than the width of the soliton spectrum. Apart from its fundamental importance, this fact allows us to make our approximation slightly more precise. Indeed, for small D we have

$$\int_{-\infty}^{\infty} \frac{\mathrm{th}^2 y \,\mathrm{sech}^2 y \, dy}{1+(y+\sqrt{D})^2/Q^2} \approx \int_{-\infty}^{\infty} \frac{\mathrm{th}^2 y \,\mathrm{sech}^2 y dy}{1+y^2/Q^2} \equiv \frac{2}{3} B(Q) \approx \frac{2Q^2}{3(Q^2+1.166)}. \tag{3.8}$$

In the last approximate expression we used (3.1). For $Q \to \infty$ the coefficient $B(Q) \to 1$, and for $Q=1$ we have $B \approx 1/2$.

We thus have

$$\gamma_0^{-1} \frac{dD}{dt} \approx \frac{3}{4n_0} \left[\frac{\varkappa_0}{\gamma_0} + B(Q) \right] - 2A(Q) D \approx \frac{3(1+B)}{4n_0} - 2AD, \tag{3.9}$$

where

$$D(t) = D_0 \exp(-2A\gamma_0 t) + \frac{3(1+B)[1-\exp(-2A\gamma_0 t)]}{8An_0} = \frac{3}{8An_0} [1+B-(1-2A+B)\exp(-2A\gamma_0 t)], \tag{3.10}$$

and the stationary value is

$$D_s \equiv D(t\to\infty) = 3(1+B)/8An_0. \tag{3.11}$$

It is interesting that D_s is independent of γ_0. One can understand this result simply by recalling that the growth of γ_0 stimulates two mutually contradictory processes: an increase in the noise and stabilization of the carrier frequency as it approaches the maximum of the amplification line. In final reckoning they cancel one another and there remains no dependence on γ_0.

Since for any finite Q the coefficient B is smaller than unity we get even closer to the quantum limit D_0 in our estimates of D.

Thus, the optimum soliton propagation regime is apparently that for which the width of its spectrum is approximately equal to the amplification line width. In that case the drift of the carrier frequency with time is less than a factor $\sqrt{2}$ since we have $D_s \approx 2D_0$. However, if there are small excess fluctuations at the entrance, $D(t)$ will practically not evolve at all. A broadening of the amplification band leads to an increase in the noise and a growth of $D(t)$. On the other hand, its narrowing causes the soliton to lengthen (spread) due to the monochromaticity of its spectrum. We can only conjecture this last effect since it goes beyond the framework of our adiabatic assumption that the soliton shape remains unchanged on its journey.

4. INDETERMINANCY OF THE SOLITON POSITION

The soliton momentum fluctuations considered above cause a corresponding spread in its propagation velocity which cannot be caused by random shifts. Apart from its obvious fundamental importance it is important to know the indeterminacy of the soliton coordinate for purely practical considerations also. Indeed, if we are dealing with optical communication lines a soliton which is randomly shifted by an amount on the order of its eigenwidth transforms the signal into noise since at the moment it should be present, it is not.

Thus, if we know the momentum indeterminacy we can write

$$\langle \Delta x^2(t) \rangle \approx 4 \left(\int_0^t \langle \Delta p^2(t) \rangle^{1/2} dt \right)^2 + \langle \Delta x_0^2 \rangle = \left(\frac{2n_0 C}{\pi} \int_0^t \sqrt{D(t)} dt \right)^2 + \langle \Delta x_0^2 \rangle, \tag{4.1}$$

where $\langle \Delta x_0^2 \rangle = \langle \Delta x_0^2(0) \rangle$ is the dispersion of the coordinate indeterminacy at the start of the path. Its minimum value is caused by the quantum limit (vacuum fluctuations)[19]

$$\langle \Delta x_0^2 \rangle = \pi^2/3n_0^3 C^2, \tag{4.2}$$

and under actual circumstances ($n_0 \gg 1$) it is insignificant.

Generally speaking, the relation between Δx and Δp is not very rectilinear. In principle one can invent an action of Γ such that a shift Δx does not lead to a drift of the carrier frequency and of the momentum Δp (and vice versa). However, in the adiabatic approximation used by us when the shape of the soliton and its spectrum remain unaltered, soliton shifts Δx lead inevitably to a transformation of the spectrum with the necessary change in the

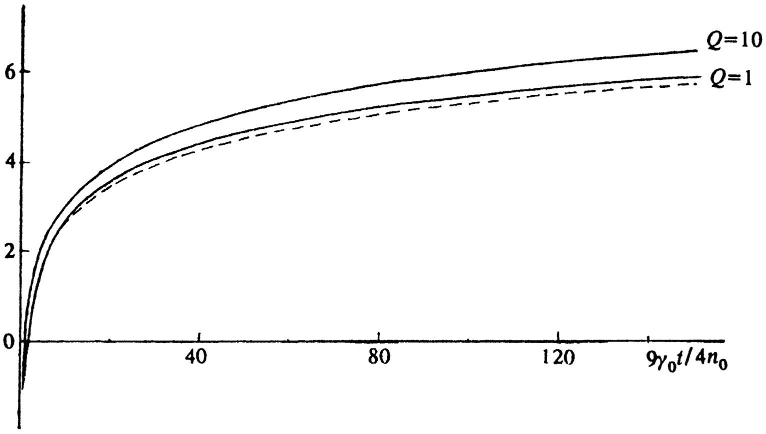

FIG. 4. Logarithm of the normalized dispersion of the soliton position indeterminacy, $\lg[(9\gamma_0\pi/4n_0^2C)\langle\Delta x^2\rangle]$, as a function of the normalized propagation time for $\gamma\equiv\varkappa$. The solid curves are constructed as the result of a numerical calculation and the dashed ones using the approximation (4.5) to (4.7). Everywhere we have $D_0=\langle\Delta x_0^2\rangle=0$. The calculated and approximating curves are the same for $Q=10$.

carrier frequency and a nonvanishing Δp (and vice versa). A single-valued connection of the form (4.1) is thus valid in the present approach.

Simplest of all is the estimate of the position indeterminacy when we put $D(t)=D_s$:

$$\langle\Delta x^2(t)\rangle\approx\left(\frac{2n_0Ct}{\pi}\right)^2 D_s+\langle\Delta x_0^2\rangle$$

$$=\frac{3(1+B)n_0C^2t^2}{2A\pi^2}+\frac{\pi^2}{3n_0^3C^2}\approx\frac{n_0C^2}{\pi A}t^2. \tag{4.3}$$

We have here used (3.11), (3.5), and (4.2). A more rigorous estimate can be obtained by using (3.10). However, both in that case and in other cases the growth of the dispersion of the indeterminacy is quadratic in t in the far zone when we have $A\gamma_0t\gg1$ and it is not linear at all, as was predicted by the authors of Ref. 20. It is also clear that the dispersion obtained, $\langle\Delta x^2(t)\rangle$, is comparable to the quantum limit[16,19]

$$\langle\Delta x^2(t)\rangle\approx n_0C^2t^2/3+\langle\Delta x_0^2\rangle, \tag{4.4}$$

calculated for an ideal fiber without amplification or loss.

The result (4.3) given above refers to a variant of soliton propagation in a waveguide with $\varkappa(p)\approx$const ($M\gg1$). If, however, we have $\varkappa(p)\equiv\gamma(p)$, then according to (3.3) and (4.1),

$$\langle\Delta x^2(t)\rangle=\frac{2n_0C}{\pi}{}^2[F(t)-F(0)]^2+\langle\Delta x_0^2\rangle \tag{4.5}$$

$$F(t)\approx\frac{4}{15a}\left[(at+b)^{1/2}-c\right]^{3/2}\left[3(at+b)^{1/2}+2c\right], \tag{4.6}$$

$$a=\frac{3Q^2\gamma_0}{n_0},\quad b=\left(D_0+Q^2+\frac{7}{6}\right)^2,\quad c=Q^2+\frac{7}{6}. \tag{4.7}$$

In the limit of an unbounded amplification band ($Q\to\infty$) we get the Gordon–Haus approximation with account of the initial indeterminacy:

$$\langle\Delta x^2(t)\rangle=\langle\Delta x_0^2\rangle+8\gamma_0n_0C^2t^3/3\pi^2. \tag{4.8}$$

On the other hand, the boundedness of the amplification and loss bands leads to a slowing down in the growth of the coordinate indeterminacy. In particular, in the far zone $\langle\Delta x^2(t)\rangle$ increases proportional to $t^{5/2}$, but again not linearly.

We show in Fig. 4 curves illustrating the dynamics of $\langle\Delta x^2\rangle$. They have been constructed in two variants: using Eq. (4.1) in the result of a numerical solution of Eq. (2.20) for $\varkappa\equiv\gamma$, and also with the approximate analytical description (4.5) to (4.7). It is clear that the agreement is good.

5. CONCLUSION

We estimate the maximum possible path of a soliton for which its random mean square displacement $\langle\Delta x^2\rangle^{1/2}$ reaches its half-width. According to Refs. 8 and 19 the spreading of the average envelope $\langle\phi^+\phi\rangle$ then causes it to broaden by a factor of two. Thus, assuming the momentum indeterminacy $\langle\Delta p^2\rangle$ to be approximately constant during the nonlinear propagation (as follows from the results of § 3 for $Q\approx1$) the characteristic time t_x during which such a spreading takes place is equal to[7,8,19]

$$t_x\approx2/n_0C\sqrt{\langle\Delta p^2\rangle}. \tag{5.1}$$

This relation is valid for

$$\sqrt{\langle\Delta p^2\rangle}\gg C. \tag{5.2}$$

According to (2.11) and (3.11) we have

$$\sqrt{\langle\Delta p^2\rangle}\approx\frac{C}{5}\sqrt{\frac{(1+B)n_0}{A}}, \tag{5.3}$$

where it follows that (5.2) is automatically satisfied for $n_0\gg1$.

We introduce the soliton period $T=8\pi/n_0^2C^2$, i.e., the time during which a nonlinear phase advance equal to 2π is accumulated. Using (5.1) and (5.2) we then get

$$t_x\approx\frac{2T}{5}\sqrt{\frac{An_0}{1+B}}. \tag{5.4}$$

If the width of the soliton spectrum is approximately equal to the width of the amplification line ($Q\approx1$, $A\approx0.34$, $B\approx0.5$), we have

$$t_x \simeq \frac{T}{5}\sqrt{n_0}. \tag{5.5}$$

This is also the limiting time for soliton propagation during which it can "escape" from its own legitimate position by an amount equal to its characteristic half-width. Since for optical solitons we have, as a rule, $n_0 \sim 10^6$, it is clear that we find $t_x \gg T$ and the length of the path may reach very impressive magnitudes, considerably exceeding the Gordon–Haus quantum limit for which the boundedness of the amplification band was neglected.

This main result is confirmed also by the results of a numerical experiment[21] where in the framework of the semiclassical description the behavior of two interacting solitons was studied while they were amplified periodically. Notwithstanding the difference in the statement and the solution of the problem, in that case also conditions were found under which it is possible to exceed the Gordon–Haus limit.

Definite parallels can also be traced with the results of Ref. 25 in which a completely different problem was solved: on the basis of a classical approach the effect was studied of the spectral composition of fluctuations acting on a soliton propagating in a stochastic medium. The tendency observed in Ref. 25 for the soliton fluctuations to saturate (the establishment of a stationary regime) and the significant dependence of its statistical characteristics on the spectral composition of the perturbation indicate, on the one hand, the importance of taking the latter into account and, on the other hand, are the basis for drawing analogies of a very general nature to the role of the spectral properties of stochastic and regular perturbations. This fact was drawn to my attention by a referee, to whom I am grateful also for some very useful comments.

We also note the following fact. The behavior of a soliton in media with random perturbations have recently been the subject of studies undertaken in the classical approximation (see, e.g., Refs. 22 to 24) and, apparently, can at once be used for the equation for the position of the center of gravity of the soliton and (or) the drift of its carrier frequency. This work has, indeed, been carried out by V. A. Vysloukh.[1)] The result turned out to be practically identical with that given above. The existence of the model proposed in the present paper could really be justified by its simplicity and clarity and also by its suitability for a rigorous quantum description, in contrast to the more formal methods a generalization of which to the corresponding procedures with quantum mechanical operators is a very complicated problem which requires a special justification.

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