Spiral autowaves in a round excitable medium

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It is shown that the dynamics of spiral autowaves in an excitable medium of finite size is much more variegated than in an unbounded medium. A simplified analytic description is proposed of the kinematics of a spiral autowave in a small round region. The results of a kinematic approach are compared with results of modeling equations of the "reaction-diffusion" type for the medium.

Phenomena in media called excitable have been attracting considerable interest of late. The rapid advances in this direction are due greatly to the fact the properties of excitable media are possessed by nerve and muscle tissues,[1,2] a number of chemical solutions,[3] electronic and solid-state systems,[4] magnetic superconductors,[5] and many other physical, chemical and biological systems.

A distributed excitable system consists of locally interconnected nonlinear active elements capable of producing a pulse in response to an incoming external signal. The pulses propagating in excitable media are frequently called autowaves.

A special type of elementary excitations in two-dimensional excitable media are rotating helical waves whose front line has the form of a helix moving with constant angular velocity ω_0 around an immobile center. The amplitude of the pulses near the rotation center is small.[6] This region of the medium is called the core of the helical wave. The form of the front and the rotation frequency of a helical wave in a medium of sufficient dimensions is an important property of the considered excitable medium.

A generally acceptable mathematical description of excitable media is provided by a system of nonlinear parabolic equations of the "reaction-diffusion" type:

$$\frac{\partial \mathbf{U}}{\partial t} = \hat{D}\Delta \mathbf{U} + \mathbf{F}(\mathbf{U}), \qquad (1)$$

where $\mathbf{U}$ is the state vector of a unit volume of the excited medium, $\hat{D}$ is a matrix of diffusion or heat-conduction coefficients, $\mathbf{F}(\mathbf{U})$ is a nonlinear function that specifies the dynamics of the "reactions" ocurring in each unit volume.

In the overwhelming majority of cases it suffices to describe processes in excitable media by investigating only one of the two equations:[7–9]

$$\frac{\partial E}{\partial t} = D_E \Delta E + F(E, g), \qquad \frac{\partial g}{\partial t} = D_g \Delta g + \varepsilon G(E, g), \qquad (2)$$

where E and g (frequently called activator and inhibitor) have the meaning of the densities of the reacting substances, the temperature, the electric potential, etc., while the "excitable" properties of the medium are due to the И-shaped zero-isocline of the first equation of (2) and the monotonic dependence of E on g, specified by the zero-isocline of the second equation of (2). It is usually assumed that $\varepsilon \ll 1$.

In the general case of two or three measurements, only a numerical solution of (2) can be obtained. It is therefore very important to develop approximate methods of analytically describing the autowave regimes. One such approximate method is the kinematic approach.[7,10]

Investigations of autowave regimes are usually confiend to infinite excitable media. However, the dynamics of autowaves (particularly spiral ones) in bounded media acquire qualitatively new features. Foremost is the drift of spiral waves along the boundary and their attraction to or repulsion from it.[11]

The present paper is devoted to an analytic investigation of the circulation of a spiral wave in a bounded excitable medium having the form of a circle of radius R. Within the framework of the kinematic approach, we derive expressions for the circulation frequency and the core radius, obtain the form of the autowave front, and investigate the stability of the circulation. We show, in particular, that the circulation can be either stable or unstable, depending on the radius of the circle. This effect, predicted by the kinematic description, is confirmed by a numerical integration of the "basic" system (2).

1. KINEMATIC APPROACH

According to the kinematic description, an autowave in a two-dimensional medium is fully specified by describing its front. Each section of the front moves in a normal direction with velocity $V = V(K)$ defined by the front curvature K on this section. The front can have a break (free edge) which, beside moving along the normal at a velocity $V(K_0)$ (where K_0 is the curvature of the front at the approach to the free edge), "sprouts" in a tangential direction with a certain velocity $C = C(K_0)$. At a certain critical curvature $K_0 = K_{cr}$ the velocity C vanishes and at $K_0 > K_{cr}$ the sprouting gives way to a shortening of the front. Near $K_0 = K_{cr}$ one can put $C = \gamma(K_{cr} - K_0)$, where $\gamma > 0$.

Accurate to the position on the plane, any curve can be specified by its natural equation $K = K(l)$ which determines the dependence of the curvature K on the length l of the arc of curve, which is best measured from the end point of the front. As the wave propagates, the curvature of its front can depend also on the time t. It is shown in Refs. 12 and 13 that the front curvature $K(l,t)$ satisfies the equation

$$\frac{\partial K}{\partial l}\Big(\int_0^l KV d\xi + C\Big) + \frac{\partial K}{\partial t} = -K^2 V - \frac{\partial^2 V}{\partial l^2}. \qquad (3)$$

As already stated, the natural equation describes a curve only accurate to its position on the plane. To describe fully the front displacement it suffices to cite the low of motion of one arbitrary front point. It is convenient to formulate this law for the end point of the front. Thus, if X_0 and Y_0 are the Cartesian coordinates of the end point on a plane, and α_0 is the angle between the vector tangent to the front at the

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point $l = 0$ and to the axis x, they are subject to the following equations:

$$\dot{X}_0 = -V(l=0)\sin\alpha_0 - C\cos\alpha_0,$$
$$\dot{Y}_0 = V(l=0)\cos\alpha_0 - C\sin\alpha_0, \qquad (4)$$
$$\dot{\alpha}_0 = \partial V/\partial l|_{l=0} + CK_0.$$

From Eqs. (3) and (4) we can determine the form and position of an autowave front on a plane.

We shall use hereafter a linear dependence of the velocity V on the curvature K:

$$V = V_0 - DK, \qquad (5)$$

where V_0 is the velocity of a planar front. At sufficiently small curvatures this dependence is quite realistic.[14,15]

An excitable medium is thus described in the framework of the kinematic approach by a small number of phenomenological parameters: V_0, K_{cr}, γ, and D. These parameters must be obtained from experiment or by solving Eqs. (2), which are "microscopic" from the standpoint of autowave kinematics. It turns out here that in a number of cases certain coefficients are independent of the actual forms of the functions F and G.[10]

A stationary solution of Eq. (3) describes the stationary regime in the form of a spiral wave in an infinite medium. The end point of the front moves here along a circle at a constant angular velocity ω_0. For sufficiently large l, the form of the front is close to the evolvent of a circle, and near the end point it is close to a Cornu spiral.[13] An approximate expression for ω approaching the results of the numerical calculation was obtained in Ref. 16:

$$\omega_0 = V_0 K_{cr}\psi(p), \qquad (6)$$

where $p = DK_{cr}/V_0$,

$$\psi(p) = c_1 p^{1/2} - c_2 p - c_3 p^2,$$

and the constants are $c_1 = 0.685$, $c_2 = 0.06$ and $c = 0.293$.

If the parameter $p \ll 1$ (this condition corresponds to media called "weakly excitable") it is possible to obtain also an approximate analytic solution.[13]

We proceed now to investigate the circulation of a helical wave in a bounded organic medium in the form of a circle.

2. STATIONARY CIRCULATION OF SPIRAL WAVE IN A CIRCLE

Let a spiral wave rotate in a circle of radius R. We consider first stationary rotation, when the center of the core coincides with that of the circle.

The shape of the front in stationary circulation is independent of the time $t(\partial K/\partial t = 0)$, there is no sprouting ($C = 0$), and the front's curvature K_0 on reaching the end point is equal to K_{cr}.

Under these conditions Eq. (3) takes the form

$$\frac{dK}{dl}\int_0^l KV\,d\xi = -K^2V - \frac{dV^2}{dl^2}. \qquad (7)$$

Equation (7) can be integrated once and reduced to the form

$$K\int_0^l KV\,d\xi + \frac{dV}{dl} = \omega, \qquad (8)$$

where the unknown integration constant ω is equal to the angular velocity of the spiral-wave rotation. In fact, it follows from (8) that $\omega = dV/dl$ for $l = 0$, but since the wave front approaches the core along the normal, dV/dl at the end point is the circulation angular velocity.

If the boundaries of the excited medium are impermeable to diffusion, the front of the autowave must also approach the boundary along a normal. It follows hence that dV/dl for $l = L$ (where L is the total length of the spiral-wave front) is also equal to ω. The substitution $l = L$ in (8) leads to the following condition on the front length L:

$$\int_0^L KV\,d\xi = 0. \qquad (9)$$

For a linear dependence of V on the curvature (5), the investigation of the stationary circulation reduces to solution of the equation

$$K\int_0^l K(V_0 - DK)d\xi - D\frac{dK}{dl} = \omega, \qquad (10)$$

supplemented by the conditions

$$K|_{l=0} = K_{cr},$$
$$D\frac{dK}{dl}\bigg|_{l=0} = D\frac{dK}{dl}\bigg|_{l=L} = \omega, \qquad (11)$$
$$\int_0^L K(V_0 - DK)d\xi = 0.$$

The problem (10), (11) was first solved numerically in Ref. 7. Approximate dependences were obtained of the angular velocity ω_R of the spiral wave on the circle radius R and on the angular velocity ω_0 in an unbounded medium.

No exact analytic solutuion of the problem (10), (11) is possible for an arbitrary circle radius. If, however, this radius is small enough, a kinematic approach admits of a quite simple and accurate analytic description.

The point is that for small radii of the circle, as shown by numerical integration of (10) and (11), the function $K(l)$ hardly differs from a linear one (see Fig. 1). It becomes possible therefore to seek the basic characteristics of a spiral wave (frequency of roation, shape of the front, etc.) by using an almost linear approximation of $K(l)$:

$$K = K_{cr} - \omega_R l/D. \qquad (12)$$

The succeeding analysis is quite simple.

Substitution of (12) in the third equation of the system (11) leads to the following dependence of the angular velocity ω_R on the total front length L:

$$\omega_R = V_0\,pf(p)/L, \qquad (13)$$

where V_0 is the velocity of a planar front, $p = DK_{cr}/V_0$, and

$$f(p) = 3\{(p - 1/2) + [(p - 1/2)^2 + 4(1-p)p/3]^{1/2}\}/2p. \qquad (14)$$

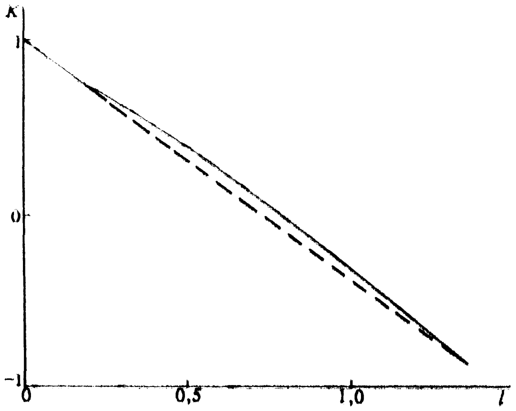

FIG. 1. Curvature K of spiral-wave front vs the arc length l, obtained by integration of Eq. (10) (solid line) and in accordance with the linear relation (12) (dashed) for $V_0 = 1$, $K_{cr} = 1$, $D = 0.5$, $\omega = 0.7$, $R = 2$.

Particular interest attaches, however, to the dependence of ω_R on the radius R of the circle. We obtain this dependence by starting from the following considerations. It follows from (12) and (13) that the front curvature at $l = L$ is equal to

$$K|_{l=L} = K_{cr}\,[1 - f(p)]. \tag{15}$$

The front velocity V_R near the circle boundary is then determined by expression (5), where K is given by Eq. (15). Recognizing that $\omega_R = V_R/R$, we obtain the dependence of ω_R on R

$$\omega_R = V_0[1 - p + pf(p)]/R. \tag{16}$$

The angular velocity increases thus with decrease of the circle radius. The inversely proportional dependence (16) is confirmed by numerical simulation using the "reaction-diffusion" model (2) (see below) and by numerical solutions of the kinematics equations.[7]

Substituting (16) in (12) we obtain the dependence of the front curvature on the arc length l. This linear dependence describes a Cornu spiral.

Using (5), (11), and (16) we can obtain also the dependence of the radius r of a spiral-wave core on the radius of the circle

$$r = \frac{1 - p}{1 - p + pf(p)}\,R. \tag{17}$$

It follows from (17) that the radius of the core increases linearly with the radius of the circle. In strongly excitable media ($1 - p \ll 1$), however, a spiral wave has a very small core compared with the circle radius. On the other hand, the core dimension in a weakly excitable medium ($p \ll 1$) practically coincides with the radius of the circle, since the front length is very small.

In fact, the front length L can be determined from (13) and (16)

$$L = \frac{pf(p)}{1 - p + pf(p)}\,R. \tag{18}$$

Another important feature of the shape of a spiral wave is an integral bend α of its front, equal to the difference between the slope angles α_0 and α_R of the tangent to the front on the core boundary and on the circle boundary, respectively. The value of the bend is determined by an integral of the local front curvature K. We have then from (12)

$$\alpha = \int_0^L K\,d\xi = K_{cr}L - \omega_R L^2/2D. \tag{19}$$

Substituting (18) and (16) in (19) we obtain for α the final expression:

$$\alpha = \frac{pf(p)[1 - f(p)/2]}{1 - p + pf(p)}\,RK_{cr}. \tag{20}$$

Thus, the linear approximation $K(l)$ has enabled us to investigate completely the problem of circulation of spiral waves in round media of small size. In contrast to the case of an infinite medium, there is not need here for the inequality $p \ll l$, i.e., we need not confine ourselves to weakly excitable media. This extends substantially the applicability of the resultant expressions.)

3. STABILITY OF CIRCULATION OF A SPIRAL WAVE

Assume that the instantaneous center of rotation of a spiral wave does not coincide with the center of the circle at the initial instant of time. The shape of the spiral wave will then be altered in the course of the circulation. By virtue of these changes, its instantaneous angular velocity ω will likewise not be constant.

It was shown in the preceding section that the angular velocity of a spiral wave is connected with features of its front, such as its length L and its integral bend α. Using (19), the angular velocity ω can be expressed explicitly in terms of the following parameters:

$$\omega = (K_{cr}L - \alpha)2D/L^2. \tag{21}$$

Using these expressions to describe the nonstationary spiral-wave motion, with the quantities L and α dependent on the time.

The change of the length of the front arc for nonstationary circulation is defined as

$$\dot L = \int_0^L KV\,d\xi. \tag{22}$$

In the stationary case this quantity vanishes [see (9)]. In the nonstationary regime, using the linear approximation (12), we obtain

$$\dot L = K_{cr}(V_0 - DK_{cr})L - \frac{\omega}{D}\left(\frac{V_0}{2} - DK_{cr}\right)L^2 - \frac{\omega^2}{3D}L^3. \tag{23}$$

The integral bend α of the front also changes in the nonstationary regime. The rate of change of the slope of the front at the point $l = 0$ will obviously be equal to the instantaneous angular velocity ω. The linear velocity of the end point along a circle of radius R is given by Eqs. (5) and (12) with $l = L$. Since the autowave front is always perpendicular to the boundary of the medium, its angular velocity at the point $l = L$ will be equal to the angular velocity of the boundary point along the circle. The rate of change of the integral bend α is thus determined by the expression

$$\dot\alpha = \omega - (V_0 - DK_{cr} + \omega L)/R. \tag{24}$$

The system (21), (23), (24) describes the dynamics the change of the form and of the angular velocity of the end point of the front. The end-point trajectory is then determined by the system (4). If we consider the evolution of the perturbations due to violation of central symmetry, the rate of the tangential displacement of the end point C must be set equal to zero.

We substitute (21) in (23) and (24) and linearize the resultant system near the equilibrium position. The linearized system is of the form

$$\dot{L}_1 = \frac{D}{3}\left(\frac{4\alpha_0^2}{L_0^2} - K_{cr}^2\right)L_1 + \left(V_0 + \frac{2}{3}DK_{cr} - \frac{8}{3}\frac{D\alpha_0}{L_0}\right)\alpha_1,$$
$$\dot{\alpha}_1 = \frac{2D}{L_0^2}\left(\frac{1}{2}\frac{\alpha_0}{L_0} - K_{cr} - \frac{\alpha_0}{R}\right)L_1 + \frac{2D}{L_0^2}\left(\frac{L_0}{R} - 1\right)\alpha_1, \tag{25}$$

where L_0 and α_0 is the length of the front and its integral bend for the regime of stationary rotation of a spiral wave, while L_1 and α_1 are the deviations from the equilibrium values. Using (18) and (20) to determine L_0 and α_0 it is easy to show that the system (25) is stable for all values of the parameter p. This means that in a small circle, for which the linear premise (12) is valid, a spiral wave is always stable.

A criterion for the applicability of the linear approximation can be easily obtained from (18). In fact, it is easy to verify that the consequence of (18) is the inequality

$$L_0/R < 1. \tag{26}$$

Equation (18), however, is a direct consequence of (12). The linear approximation (12) describes thus only spiral waves in which the front length is smaller than the radius of the circle.

The inequality (26) is of course violated in a circle with a sufficiently large radius. Strictly speaking, the treatment above cannot be used in this case. One can only see that when the ratio L_0/R increases, the stability reserve of the system (25) decreases. Consequently, a spiral wave can become unstable in a large circle.

It is of interest to compare the foregoing analysis with the results of numerical integration of Eqs. (3) and of the system (4), i.e., when the complete kinematic model is considered without the simplifying assumption (12).

Figure 2 shows the trajectories $X_0(t)$ and $Y_0(t)$ of the end point of a spiral wave in round media with different radii. Clearly, in a circle with a small enough radius (Fig. 2a) the trajectory of the end point approaches the circle asymptotically. This corresponds to a stable spiral wave, i.e., agrees with the results of the kinematic analysis employed.

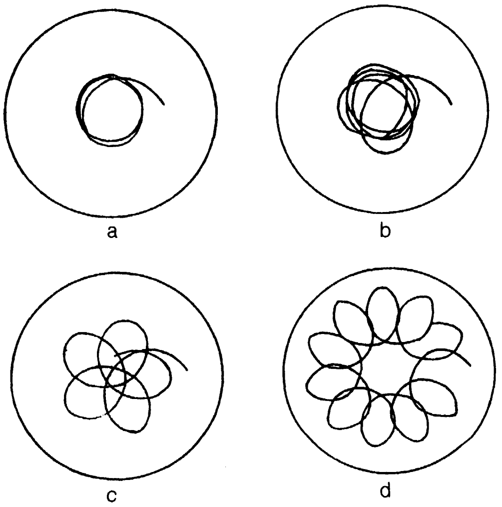

FIG. 2. Trajectory of end point of spiral wave in a circle of radius $R = 1.5$ (a), $R = 2.0$ (b), $R = 2.5$ (c), $R = 4.0$ (d).

An increase of the circle radius (Fig. 2b) leads to loss of stability in the complete kinematic model of a spiral wave. Modeling of the wave-motion kinematics in a circle of still larger radius (Figs. 2c,d) shows that the regime observed in this case is quite far from stationary rotation of a spiral wave around a center of a circle. This is in essence a drift of the core of a spiral wave along the boundary of the excited medium, similar to the drift, described in Ref. 11, along a straight-line boundary.

Is there a regime of stationary rotation of a spiral wave for media of large enough radius, which are shown in Figs. 2c and 2d? Yes, such regimes are observed in the kinematic model but they are unstable to shifts of the position of the spiral wave relative to the center of the circle. These regimes are therefore observed in very short time intervals, and the ensuing fluctuations lead then to motion of the core of the wave towards the circle boundary, and to a drift along this boundary. Thus the regime of spiral-wave drift along a boundary, shown in Figs. 2c and 2d, is stable, in contrast to the unstable central-symmetry regime.

In a small-radius circle (Fig. 2a) the situation is reversed: stationary rotation of a spiral wave around the center of the circle is a stable regime, in full accord with the above kinematic treatment, and a regime with drift along the boundary is impossible.

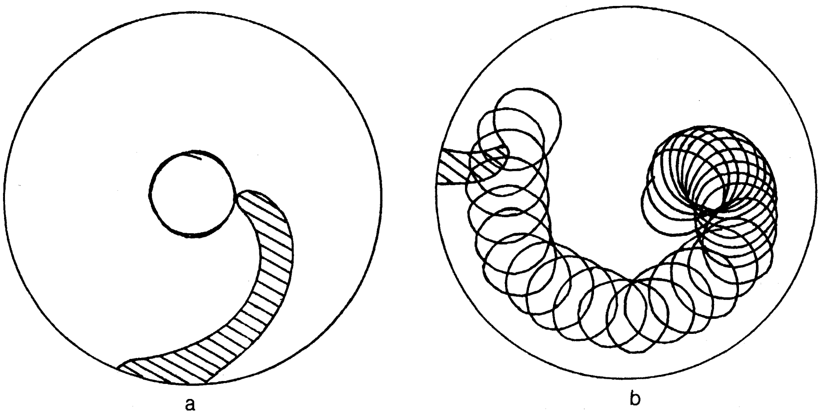

FIG. 3. Spiral wave in a round medium of radius $R = 30$. A region of the medium with $E > 0.6$ is hatched: a) centrosymmetric regime, b) two-period regime.

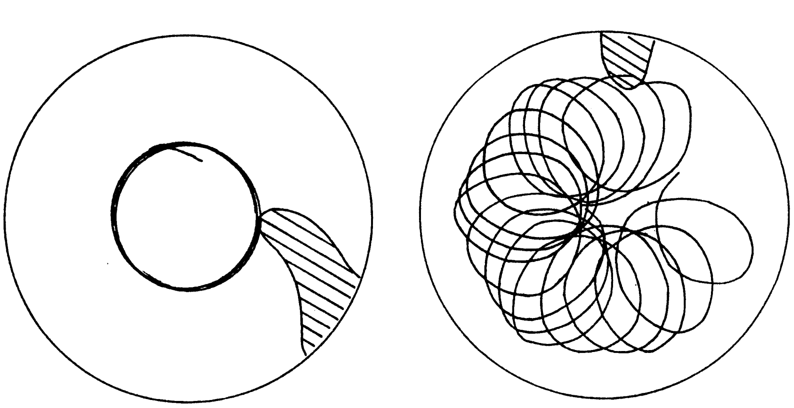

FIG. 4. Spiral wave in a round medium of radius $R = 18$. a) centrosymmetric regime, b) two-period regime.

4. SPIRAL WAVE IN THE REACTION-DIFFUSION MODEL

We consider now the results of computation experiments performed by us for a rather typical model of an excited medium of type (2). In this model the function $F(E,g)$ is defined as

$$F(E, g) = f(E) + g, \tag{27}$$

where

$$f(E) = \begin{cases} -Ek_1, & E < \sigma, \\ (E - a)k_f, & \sigma \le E \le 1 - \sigma, \\ (1 - E)k_2, & 1 - \sigma < E, \end{cases}$$

$$k_1 = \frac{a - \sigma}{\sigma} k_f, \quad k_2 = \frac{1 - \sigma - a}{\sigma} k_f.$$

The function of $G(E,g)$ is specified here as follows:

$$G(E, g) = \begin{cases} k_g E - g, & k_g E - g \ge 0, \\ k_\varepsilon (k_g E - g), & k_g E - g < 0. \end{cases} \tag{28}$$

In all the computer experiments whose results follow, the values of the model coefficients were: $k_f = 1.7$, $k_g = 2.0$, $a = 0.1$, $\sigma = 0.01$, $\varepsilon = 0.3$, $k_\varepsilon = 6.0$, $D_E = 1$, $D_g = 0$.

We begin the exposition of the results of computer experiments with a model of the "reaction-diffusion" type with a case when the dimension of the medium exceeds substantially the radius of the trajectory of the end point of the spiral wave. By special choice of the initial conditions we can obtain the conetrosymmetric solution shown in Fig. 3a. This regime, however, turns out to be unstable. To demonstrate this it suffices to change slightly the initial conditions by displacing the position of the center of the spiral-wave core towards the boundary of the circle (Fig. 3b). The trajectory of the end point of the spiral wave is then no longer a circle. It is open. The center of the core is displaced in the course of the circulation towards the boundary and begins along it.

A stationary regime of the spiral wave sets in gradually, with the core center displaced along the boundary but remaining at a fixed distance from it.

Thus, in full accord with the kinematic analysis, the stationary rotation of a spiral wave in a large-radius circle is unstable. Of course, the smaller the ratio of the core radius of the spiral wave to the circle radius the longer the time during which the controsymmetric solution is violated. It is therefore very difficult to observe the instability of the centrosymmetric solution in large-radius circle. The influence on the motion of a spiral wave becomes discernible only at a distance comparable with the radius of the core. In the remaining cases the situation is practically indistinguishable from an "indifferent" equilibrium.

The instability of a centrosymmetric solution is particularly clearly pronouned in simulation of an excited medium comparable in size with radius of the spiral-wave core. To obtain a centrosymmetric solution it is necessary to select very carefully the initial conditions (Fig. 4a). The smallest deviations leads to a two-period regime (Fig. 4b).

The dependence of the angular velocity of a spiral wave on the reciprocal of the circle radius is shown in Fig. 5 by a solid line. It can be seen that at small circle radii the dependence is practically linear in full accord with (16).

Figure 6a shows examples of the calculation for an excitable medium with a relatively small radius. The trajectory of the end point is seen in this case to differ significantly from the cycloid shown in Fig. 4b.

Comparison of Figs. 6a and 6b shows that when the radius of the medium is decreased the trajectory of the end point comes ever closer to the circle. Calculations show that a two-time regime becomes impossible with further decrease of the radius of the medium (Fig. 6c). All that takes place in

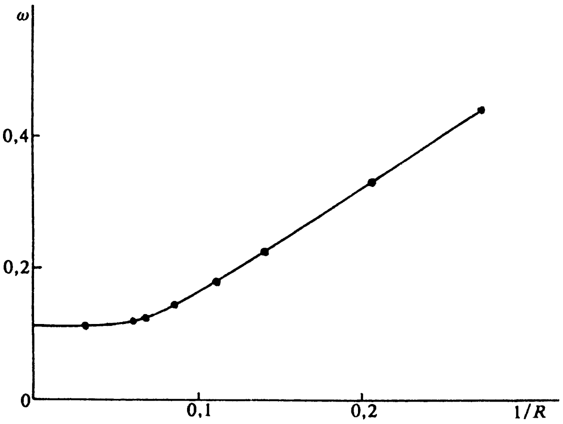

FIG. 5. Dependence of the angular velocity of a spiral wave on the dimension of a round excitable medium.

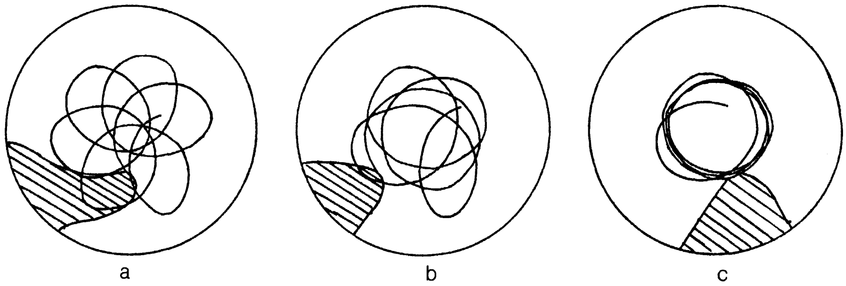

FIG. 6. Spiral line in a round medium of radius a) $R = 12.0$, b) $R = 9.6$, c) $R = 8.4$.

the medium in that case is centrosymmetric rotation of a spiral wave. Moreover, a centrosymmetric solution becomes in this case stable: sizable variations of the initial conditions lead to the appearance of an autowave that has central symmetry.

Further decrease of the size of the medium makes it impossible to obtain also a centrosymmetric solution. Thus, if the dimension of the medium is smaller than a certain cirtical value, no undamped autowave process can be induced in it.

Comparison of Figs. 6 and 2 shows that within the framework of the kinematic approach it is possible to account for all the qualitative two-period regime features observed when a system of the "reaction-diffusion" type is simulated.

5. CONCLUSION

The analysis in the present paper of the dynamics of a spiral autowave in a circular excitable medium has shown that the effect of the boundary is not limited to quantitative changes, but leads to appreciable qualitative actions on the spiral-wave circulation regimes. The kinematic approach has made possible appreciable progress in the study of these regimes. In particular, analytic estimates were obtained for the first time ever for the most important characteristics of the regime of stationary rotation of a spiral autowave, such as the angular velocity and radius of the rotation, the length of the front, and its integral bend. The previously obtained[7] numerical estimate of the angular velocity is redetermined much more accurately in the region $p \ll 1$.

Very promising is also the use of the kinematic approach to investigate the oberved two-period circulation regime of a spiral wave. It is important to note that complex trajectories of the end point of a spiral wave were observed at excitable-medium model parameters such that in an unbounded medium there exists only a stationary rotation regime.[17] The influence of the boundary conditions on the cycloidal regimes of the circulation of spiral waves, described for example in Refs. 9 and 17, is the subject of further research.

It would undoubtedly be interesting and important to obtain in experiment and investigate the observed effects, for example in the Belusov–Zhabotinskiĭ reaction.

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[1]G. R. Ivanitskiĭ, V. I. Krinskiĭ, and E. E. Sel'kov, *Mathematical Biophysics of the Cell* [in Russian], Nauka, 1978.
[2]A. T. Winfree, *When Time Breaks Down*, Princeton Univ. Press, (1987).
[3]V. A. Vasil'ev, Yu. M. Romanovskiĭ, and V. G. Yakhno, *Autowave Processes*, D. S. Chernyavskiĭ ed. [in Russian], Nauka, 1987.
[4]Yu. I. Balkareĭ, M. G. Nikulin, and M. I. Elinson, in *Autowave Processes in Systems with Diffusion*, A. V. Gaponov-Grekhov, ed. [in Russian], Gorky, Inst. Appl. Phys. (1981), p. 117.
[5]A. I. Buzdin and A. S. Mikhaĭlov, Zh. Eksp. Teor. Fiz. **90**, 394 (1986) [Sov. Phys. JETP **53**, 201 (1986)].
[6]S. C. Muller, T. Plesser, and B. Hess, Physica D **24**, 71 (1987).
[7]V. S. Zykov, *Simulation of Wave Processes in Excitable Media* [in Russian], Nauka, Moscow, 1984.
[8]J. J. Tyson and J. P. Keener, Physica D **32**, 327 (1988).
[9]A. T. Winfree, Chaos, Vol. 1, 303 (1991).
[10]V. A. Davydov, V. S. Zykov, and A. S. Mikhaĭlov, Usp. Fiz. Nauk **161**, 45 (1991) [Sov. Phys. Usp. **34**, 665 (1991)].
[11]E. A. Ermakova and A. M. Pertsov, Biofizika, **31**, No. 5, 855 (1986).
[12]V. S. Zykov, in *Control of Complex Systems* [in Russian], Nauka, Moscow, (1975), p. 59.
[13]V. A. Davydov and A. S. Mikhaĭlov in *Nonlinear Waves, Structures, and Bifurcation* [in Russian] A. V. Gaponov–Grekhov, ed., Nauka, Moscow (1987), p. 574.
[14]P. Foerster, S. C. Muller, and B. Hess, Proc. Natl. Acad. Sci. USA, **86**, 6831 (1989).
[15]A. S. Mikhaĭlov and V. S. Zykov, Physica D **52**, 379 (1991).
[16]V. S. Zykov, Biofizika **25**, 319 (1980).
[17]V. S. Zykov, Biofizika **31**, 862 (1986).

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