Hydrodynamic action for orbital and superfluid dynamics of ^{3}He-*A* at $T=0$

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The anomaly in the dynamics of the orbital vector $\hat{\mathbf{l}}$ in superfluid ^{3}He-*A*, which corresponds to the chiral anomaly and zero-charge effect in 3 + 1 quantum field theory (QFT) with chiral fermions, is considered in the presence of superflow. The hydrodynamic action which describes the dynamics of both the $\hat{\mathbf{l}}$ vector and the superfluid velocity $\mathbf{v}_s$ field, is derived at $T=0$ in the collisionless limit using the fact that in the vicinity of the gap nodes the Bogoliubov quasiparticles behave as chiral relativistic fermions. While the quantity $\mathbf{B}=k_F\hat{\mathbf{l}}$ plays the part of the vector potential of the effective electromagnetic field acting on the chiral fermions, the combination $B_4=k_F(\mathbf{v}_s\hat{\mathbf{l}})$ corresponds to the scalar component of the electromagnetic field. As a result the $\mathbf{v}_s$ field contributes to the Wess–Zumino term in the action which is responsible for the chiral anomaly. The derived hydrodynamic action, which appears to be Galilean invariant, leads to a closed set of hydrodynamic equations for the reversible motion of $\hat{\mathbf{l}}$ and $\mathbf{v}_s$ at $T=0$. The problem of the conservation of the linear momentum (mass current) of the liquid is discussed. In the collisionless limit the hydrodynamic momentum is conserved but is not well-defined because of the chiral anomaly: due to the flow of the quasiparticle energy levels through the gap nodes during the dynamics of $\hat{l}$, the linear momentum of the liquid depends on the prehistory of the $\hat{\mathbf{l}}$ field and therefore cannot be expressed in terms of the instantaneous values of $\hat{\mathbf{l}}$ and $\mathbf{v}_s$. In the opposite limit (the case of a slowly moving $\hat{\mathbf{l}}$ vector) the quasiparticle scattering on the container wall becomes important. As a result the quasiparticles which fill the energy levels crossing the gap nodes are removed from the coherent motion of liquid and their linear momentum transfer to the walls at a rate dictated by the level flow process. In this limit the linear momentum is well defined in terms of $\hat{\mathbf{l}}$ and $\mathbf{v}_s$ but is not conserved. This nonconservation of the hydrodynamic linear momentum is analogous to the nonconservation of the chiral charge in QFT with chiral anomaly. It leads to an additional force acting on the moving quantized vortices, which resembles the Iordanskii force.

1. INTRODUCTION

The remarkable phenomenon which characterizes the superfluid ^{3}He-*A* is gapless superfluidity (see the review in Refs. 1 and 2). The gap nodes exist in the quasiparticle spectrum at the momenta $\mathbf{k}=\pm k_F\hat{\mathbf{l}}$, where k_F is the Fermi momentum. The Bogoliubov quasiparticles in the vicinity of the gap nodes are chiral, which leads to anomalies in the static and dynamic properties. These anomalies, which are essentially analogous to the chiral anomaly and to the zero-charge effect in quantum field theory (QFT) with massless chiral fermions, should appear at low temperatures (~ 0.1 T_c), when the low-energy chiral excitations start to be important. Experimentally this temperature region in the *A*-phase is accessible now, which requires extensive theoretical investigations of the problems related to low-temperature anomalies.

Our purpose here is to continue the derivation of the hydrodynamic equations for ^{3}He-*A* developed in Refs. 2 and 3, which take into account the anomalies. We are restricting ourselves to the limiting case of $T=0$ where we can use the Lagrangian technique for the description of the *A*-phase dynamics, because this method naturally treats the chiral anomaly which is represented by the Novikov–Wess–Zumino term in the hydrodynamic action.[2,3] We consider here the collisionless limit, in which one can neglect the dynamics of the normal component; however the opposite limit case of intensive scattering of the quasiparticles on the walls of the container is also discussed. We consider first the anomaly-free dynamics (Sec. 3), which is appropriate for the liquid with the same symmetry as the *A*-phase but without gap nodes in the quasiparticle spectrum. The hydrodynamic action for the anomaly-free *A*-phase can be obtained in a phenomenological way using the gauge and Galilean invariances. This is a generalization of the well known action for the conventional superfluidity of ^{4}He discussed in Sec. 2.

The derivation of the anomalous terms in the hydrodynamic action for the real *A*-phase with the gap nodes requires that the fermionic states of the liquid be considered. The topological contribution to the action from the chiral anomaly effect resulting from the gapless fermions in the vicinity of gap nodes is found in Sec. 4, while the logarithmically divergent term due to the zero-charge effect is calculated in Sec. 5. This scheme of separation of the anomaly-free hydrodynamic action and the anomalous action resulted from the fermionic zero modes in the vicinity of the gap nodes is equivalent to the superhydrodynamics developed by Andreev and Kagan,[4] which describes the bosonic and fermionic zero modes. They first noticed that the Lagrangian for the fermions in the vicinity of the gap nodes can be derived phenomenologically, so that the total hydrodynamic action can be obtained solely on the basis of symmetry consideration without any microscopic calculations. Integrating over the fermionic degrees of freedom within their approach, they found the contribution corresponding to the zero-charge effect. In Refs. 2 and 3 it was stressed that the topological similarity between the fermionic zero modes in ^{3}He-A and chiral fermions in quantum electrodynamics leads to a similar effect in these two systems. In this approach, also phenomenological, both the zero-charge and chiral anomaly effects have been calculated.

In addition to the results obtained in Refs. 2 and 3, the superfluid velocity $\mathbf{v}_s$ field is now included into the anomalous contributions to the hydrodynamic action. The $\mathbf{v}_s$ field represents one of the components of the effective gauge field acting on the chiral fermions and thus contributes to the general 5-form of the topological Novikov–Wess–Zumino term in the hydrodynamic action, which describes the chiral anomaly. Introduction of the superfluid velocity into the general scheme of chiral anomaly restores the Galilean invariance of the Novikov–Wess–Zumino term. In Sec. 5 the v_s contribution to the zero-charge effect is considered, which restores the Galilean invariance of the logarithmically divergent term in the action.

The derived Galilean invariant action allows us to consider the problem of the conservation of the linear momentum related to the chiral anomaly. The dynamics of the linear momentum in the process in which the levels of the chiral quasiparticles flow through the gap nodes, which is the essence of the chiral anomaly in the A-phase, is discussed in Sec. 6. In the collisionless regime, due to the level flow the linear momentum is not a function of the instant values of $\hat{\mathbf{l}}$ and $\mathbf{v}_s$, but instead depends on the prehistory of the $\hat{\mathbf{l}}$ field. In the opposite limiting case of the intensive quasiparticle scattering on the container wall, discussed in Sec. 7, the quasiparticles on the energy levels, which cross the gap nodes, are scattered by the walls with their linear momenta being transferred to the normal component. In this limit the hydrodynamic linear momentum is not conserved, but now is well defined in terms of $\hat{\mathbf{l}}$ and $\mathbf{v}_s$. The nonconservation of the momentum leads to an additional force acting on the moving quantized vortices, which resembles the Iordanskii force.

2. HYDRODYNAMIC ACTION FOR CONVENTIONAL SUPERFLUIDS

Here we recall the Lagrangian for the superfluid dynamics at $T=0$ for liquids with conventional superfluid properties (^{4}He and ^{3}He-B), which are described by a single Goldstone field Φ, the phase of the Bose condensate, and by the mass density ρ. The condensate phase forms the potential α for the superfluid velocity $\mathbf{v}_s$:

$$\mathbf{v}_s=\nabla\alpha, \qquad \alpha=\frac{\hbar}{M}\Phi\,, \tag{2.1}$$

where M is the mass of the boson in the Bose-condensate, $M=m_4$ for ^{4}He and $M=2m_3$ for ^{3}He-B. The variable α changes under the Galilean transformation as

$$\alpha(\mathbf{r},t)\to\alpha(\mathbf{r}-\mathbf{u}t,t)+(\mathbf{ur})-{}^1/_2u^2t. \tag{2.2}$$

The Galilean invariance of the hydrodynamic action and the facts that the mass density ρ and α are dynamically conjugate variables and that the energy of the liquid contains kinetic energy and internal energy $\varepsilon(\rho)$ dictate the following form of the hydrodynamic action (see e.g., Appendix in Review 5):

$$S=\int d^4x\Big\{\rho\Big(\partial_t\alpha-\frac{q}{m_3}A_4\Big)+\frac{1}{2}\rho\Big(\nabla\alpha-\frac{q}{m_3}\mathbf{A}\Big)^2+\varepsilon(\rho)\Big\}. \tag{2.3}$$

Here we have also introduced the electromagnetic gauge field for the case of the electrically charged superfluids. Equation (2.3) is invariant under the gauge transformation $\alpha\to\alpha+\chi/m_3$, $A_\mu\to A_\mu+\partial_\mu\chi/q$. For a neutral system the charge satisfies $q=0$; nevertheless, introducing the gauge field is the easiest way to find the contribution of the $\mathbf{v}_s$ field to the anomaly in the hydrodynamic action for ^{3}He-A using the gauge invariance arguments. The scalar potential of the gauge field is transformed as $A_4\to A_4-(\mathbf{uA})$ under the Galilean transformation.

The Euler–Lagrange equations for electrically neutral superfluids include the equation for the phase,

$$\frac{\delta S}{\delta\rho}=\mu=\partial_t\alpha+\frac{1}{2}(\nabla\alpha)^2+\frac{\delta\varepsilon}{\delta\rho} \tag{2.4}$$

and the continuity equation

$$\frac{\delta S}{\delta\alpha}=0=\partial_t\rho+(\nabla\mathbf{j}).\qquad \mathbf{j}=\rho\nabla\alpha. \tag{2.5}$$

In Eq. (2.4) we have introduced the spatial independent Lagrange multiplier μ, which ensures conservation of the total number of particles. The mass current may be also defined as the response to the vector potential $\mathbf{A}$ in the limit $q\to0$:

$$\mathbf{j}=-\frac{\delta S}{q\delta\mathbf{A}}=\rho\Big(\nabla\alpha-\frac{q}{m_3}\mathbf{A}\Big)\to\rho\nabla\alpha. \tag{2.6}$$

The linear momentum is defined as the response to the coordinate transformation $\mathbf{r}\to\mathbf{r}-\mathbf{u}t$:

$$\mathbf{P}=-\int d^3x\frac{\delta S}{\delta\mathbf{u}}=\int d^3x\rho\,\nabla\alpha \tag{2.7}$$

and coincides with the total mass current. Since $\mathbf{P}$ is obtained from the general Lagrange formalism, this is a conserved quantity, which can also be checked directly using Eqs. (2.4) and (2.5).

3. HYDRODYNAMIC ACTION FOR ANOMALY-FREE ^{3}He-A

The hydrodynamics of the A-phase without anomalies resulted from the gap nodes was first discussed by Lebedev and Khalatnikov.[6] The superfluid with the A-phase symmetry but without gap nodes can be realized in the strong-coupling limit when the liquid is close to the state in which the Cooper pairing occurs in real space. When the interaction in the Cooper channel increases the transition from the A-phase state with gap nodes to the A-phase state without nodes occurs at some critical value of the interaction parameter as a Lifshitz (zero-temperature) phase transition.[2] In this section we discuss the hydrodynamic action for the A-phase above the transition.

In ^{3}He-A the superfluid and orbital motions are described by two orthogonal unit vectors, $\hat{e}^{(1)}$ and $\hat{e}^{(2)}$, with

$$[\hat{\mathbf{e}}^{(1)}\hat{\mathbf{e}}^{(2)}]=\hat{\mathbf{l}}.$$

The superfluid velocity, which in conventional superfluids is $\mathbf{v}_s = \nabla\alpha$, in the case of ^{3}He-$A$ is expressed in terms of these vectors:

$$(\mathbf{v}_s)_i = \frac{\hbar}{M}\hat{\mathbf{e}}^{(1)}\nabla_i\hat{\mathbf{e}}^{(2)} \tag{3.1}$$

and is not curl-free. In the same manner the time derivative $\partial_t\alpha$ in conventional superfluids transforms to the quantity

$$v_4 = \frac{\hbar}{M}\hat{\mathbf{e}}^{(1)}\partial_t\hat{\mathbf{e}}^{(2)} \tag{3.2}$$

in ^{3}He-A and is not the total time derivative. As a result we have four hydrodynamic quantities, $v_\mu = (\mathbf{v}_s, \mathbf{v}_4)$, which correspond to the gradients of the single potential α in conventional superfluids: $v_\mu = \nabla_\mu\alpha = (\nabla\alpha, \partial_t\alpha)$. These quantities are not independent, but are connected with the $\hat{\mathbf{l}}$ field by two kinematic relations which can be derived from Eqs. (3.1) and (3.2). The Mermin–Ho relation expresses the vorticity in terms of the gradients of the $\hat{\mathbf{l}}$ field:

$$[\nabla\mathbf{v}_s] = \frac{\hbar}{2M}e_{ikl}\hat{l}_i[\nabla\hat{l}_k\,\nabla\hat{l}_l]. \tag{3.3}$$

The Josephson equation

$$\partial_t\mathbf{v}_s - \nabla v_4 = \frac{\hbar}{M}e_{ijk}\hat{l}_i\partial_t\hat{l}_j\,\nabla\hat{l}_k \tag{3.4}$$

describes the phase slip process in terms of the $\hat{\mathbf{l}}$ dynamics.

The action for anomaly-free A-phase, S^{af} can be obtained by a generalization of Eq. (2.3) which takes into account the constraints (3.3) and (3.4) using Lagrange multipliers $\mathbf{f}$ and $\mathbf{g}$ (see also Ref. 4):

$$\begin{aligned} S^{af}\{v_\mu, \rho, \hat{\mathbf{l}}\} = \int d^4x\Big\{\rho\Big(v_4 - \frac{q}{m_3}A_4\Big) \\ + \frac{1}{2}\rho\Big(\mathbf{v}_s - \frac{q}{m_3}\mathbf{A}\Big)^2\ \varepsilon + (\rho, \hat{\mathbf{l}})\Big\} \\ + \int d^4x\Big(\mathbf{f}\Big([\nabla\mathbf{v}_s] - \frac{\hbar}{2M}e_{ikl}\hat{l}_i[\nabla\hat{l}_k\,\nabla\hat{l}_l]\Big)\Big) \\ + \int d^4x\Big(\mathbf{g}\Big(\partial_t\mathbf{v}_s + \nabla v_4 - \frac{\hbar}{M}e_{ijk}\hat{l}_i\partial_t\hat{l}_j\,\nabla\hat{l}_k\Big)\Big). \end{aligned} \tag{3.5}$$

The Euler–Lagrange equations are

$$\frac{\delta S}{\delta\rho} = \mu = v_4 - \frac{q}{m_3}A_4 + \frac{1}{2}\Big(\mathbf{v}_s - \frac{q}{m_3}\mathbf{A}\Big)^2 + \frac{\delta\varepsilon}{\delta\rho}, \tag{3.6}$$

$$\frac{\delta S}{\delta v_4} = 0 = \rho + (\nabla\mathbf{g}), \tag{3.7}$$

$$\frac{\delta S}{\delta\mathbf{v}_s} = 0 = \rho\Big(\mathbf{v}_s - \frac{q}{m_3}\mathbf{A}\Big) + [\nabla\mathbf{f}] - \partial_t\mathbf{g}, \tag{3.8}$$

$$\frac{\delta S}{\delta\hat{\mathbf{l}}} = 0 = \frac{\delta\varepsilon}{\delta\hat{\mathbf{l}}} + \frac{\hbar}{M}[\hat{\mathbf{l}}((\nabla\mathbf{g})\partial_t\hat{\mathbf{l}} + (([\nabla\mathbf{f}] - \partial_t\mathbf{g})\nabla)\hat{\mathbf{l}})],$$

$$\frac{\delta S}{\delta\mathbf{g}} = \frac{\delta S}{\delta\mathbf{f}} = 0. \tag{3.9}$$

Expressing the Lagrange multipliers $\mathbf{f}$ and $\mathbf{g}$, and function v_4 in terms of the other hydrodynamic variables we find that at $T=0$ the dynamics of the anomaly-free A-phase is described by the Euler–Lagrange equations which include the continuity equation

$$\partial_t\rho + (\nabla\mathbf{j}) = 0. \tag{3.10}$$

with $j = \rho v_s$; the Josephson equation which follows from (3.6) and the kinematic equation (3.4):

$$\partial_t\mathbf{v}_s + \nabla\Big(\frac{1}{2}\mathbf{v}_s^2 + \frac{\delta\varepsilon}{\delta\rho}\Big) = \frac{\hbar}{M}e_{ijk}\hat{l}_i\,\partial_t\hat{l}_j\,\nabla\hat{l}_k; \tag{3.11}$$

and the equation for the orbital momentum $\hat{\mathbf{l}}$:

$$\frac{\hbar}{M}(\rho\partial_t\hat{\mathbf{l}} + (\mathbf{j}\nabla)\mathbf{l}) = -\Big[\hat{\mathbf{l}}\frac{\delta\varepsilon}{\delta\hat{\mathbf{l}}}\Big]. \tag{3.12}$$

The mass current and the linear momentum of neutral liquid are defined in the same way as in Sec. 2:

$$\mathbf{j} = -\frac{\delta S}{q\delta\mathbf{A}} = \rho\Big(\mathbf{v}_s - \frac{q}{m_3}\mathbf{A}\Big) \to \rho\mathbf{v}_s, \tag{3.13}$$

$$\begin{aligned} \mathbf{P} = -\int d^3x\frac{\delta S}{\delta\mathbf{u}} = \int d^3x(g_i\nabla(\mathbf{v}_s)_i - [\mathbf{g}[\nabla\mathbf{v}_s]]) = \int d^3x(\mathbf{g}\nabla)\mathbf{v}_s \\ = -\int d^3x(\nabla\mathbf{g})\mathbf{v}_s = \int d^3x\,\rho\mathbf{v}_s. \end{aligned} \tag{3.14}$$

The linear momentum is a conserved quantity.

4. CHIRAL ANOMALY ACTION FOR ^{3}He-A

The anomaly-free hydrodynamics of the A-phase, like liquids without gap nodes, does not depend on the type of quantum statistics of the atoms of the liquid. It is determined only by the symmetry of the order parameter and can be constructed using a phenomenological approach. The anomalies in the real A-phase are essentially related to the spectrum of the fermionic excitations and are defined by the topological properties of the spectrum. The anomalies appear below the Lifshitz transition when the topologically stable gap nodes appear in the Bogoliubov quasiparticle spectrum. The Lagrangian which describes the Bogoliubov quasiparticles is

$$\begin{aligned} \Lambda = -i\partial_t + \tau_3\Big(\varepsilon(\mathbf{k}) - \mu - qA_4 + \frac{q^2}{2m_3}\mathbf{A}^2\Big) \\ - \frac{q}{m_3}(\mathbf{kA}) + c_\perp(\tau_1\hat{\mathbf{e}}^{(1)}\mathbf{k} - \tau_2(\hat{\mathbf{e}}^{(2)}\mathbf{k})), \end{aligned} \tag{4.1}$$

where τ_i are the Pauli matrices in the Bogoliubov–Nambu particle-hole space, $\varepsilon(\mathbf{k})$ is the quasiparticle energy in the normal ^{3}He, and $c_\perp = \Delta/k_F$ where Δ is the amplitude of the gap. In contrast with the general scheme of Ref. 4 for the fermionic action we use here the simplified model with $\varepsilon(\mathbf{k}) = k^2/2m_3$. The structure of the anomalous terms in the hydrodynamic action, obtained after integration over the fermions, should not depend on the model if the topological properties of the spectrum and gauge and Galilean invariances are relevant. Also we do not consider the spin structure of the fermions, which, however, becomes important if we are interested in the spin dynamics.

This Lagrangian is invariant under the gauge transformation

$$\hat{e}^{(1)}+i\hat{e}^{(2)}\to e^{2i\chi}(\hat{e}^{(1)}+i\hat{e}^{(2)}),\quad \Lambda\to\exp[-i\tau_3\chi\Lambda]\exp[i\tau_3\chi],$$
$$A_\mu\to A_\mu+\frac{1}{q}\partial_\mu\chi, \tag{4.2a}$$

and under the Galilean transformation

$$\mathbf{r}\to\mathbf{r}-\mathbf{u}t,\quad \hat{e}^{(1)}+i\hat{e}^{(2)}\to\exp[2im_3((\mathbf{ur})-u^2t/2)](\hat{e}^{(1)}+i\hat{e}^{(2)}),$$
$$\Lambda\to\exp[-i\tau_3m_3((\mathbf{ur})-u^2t/2)]\Lambda\exp[i\tau_3m_3((\mathbf{ur})-u^2t/2)],$$
$$A_4\to A_4-(\mathbf{uA}). \tag{4.2b}$$

The chiral anomaly is the consequence of the flow of energy levels through the gap nodes. Therefore the anomaly is completely defined by the fermionic spectrum in the vicinity of the nodes and depends on the topological charge of the node. In the case of Eq. (4.1), where the spin structure of the fermions is not taken into account, the energy spectrum becomes zero at $\mathbf{k}=ek_F\hat{l}$, $e=+1$ for the north pole of the Fermi-sphere and $e=-1$ for the south pole, and

$$k_F^2=2m_3(\mu-qA_4)-q^2\mathbf{A}^2. \tag{4.3}$$

The topological charge of each node coincides with e (see Ref. 2). The Bogoliubov Lagrangian (4.1) can be linearized and reduced to the following form, which is general for the fermions with the topological charge $+1$ or -1:

$$\Lambda=\sum_{N=1}^{4}\tau_N e_N{}^\mu(k_\mu-eB_\mu). \tag{4.4}$$

Here τ_4 is the unit matrix, $k_4=-i\partial_t$, and the coefficients e_N^μ form a four-vector with the following components in the case of the A-phase:

$$e_1{}^i=c_\perp\hat{e}_i^{(1)},\quad e_2{}^i=-c_\perp\hat{e}_i^{(2)},\quad e_3{}^i=ec_\parallel\hat{l}_i,$$
$$e_4{}^i=-\frac{q}{m_3}A_i,\quad e_4{}^4=1,\quad e_i{}^4=0, \tag{4.5a}$$

where $c_\parallel=v_F$. The components of the effective $U(1)$ gauge field are

$$\mathbf{B}=k_F\hat{\mathbf{l}},\quad B_4=\frac{q}{m_3}k_F(\hat{\mathbf{l}}\mathbf{A}). \tag{4.6}$$

So the Bogoliubov quasiparticles in the vicinity of the topologically stable gap nodes with unit topological charge are equivalent to massless relativistic chiral fermions moving in a curved space described by the four-vector e_N^μ, under the $U(1)$ field B_μ. The corresponding metric tensor

$$g^{\mu\nu}=-e_4{}^\mu e_4{}^\nu+\sum_{N=1}^{3}e_N{}^\mu e_N{}^\nu \tag{4.5b}$$

has the following components:

$$g^{44}=-1,\quad g^{ij}=c_\parallel^2\hat{l}_i\hat{l}_j+c_\perp^2(\delta_{ij}-\hat{l}_i\hat{l}_j)-\left(\frac{q}{m_3}\right)^2A_iA_j,$$
$$g^{i4}=\frac{q}{m_3}A_i. \tag{4.5c}$$

Note that the vector potential $\mathbf{A}$ of the electromagnetic field influences the quasiparticles both through the $U(1)$ field B_μ in Eq. (4.6) and through the effective gravity fields in Eq. (4.5).

The anomalous contribution to the hydrodynamic action is completely defined by the analogy with the axial anomaly for relativistic chiral fermions, supplemented by the arguments of the gauge and Galilean invariance. The chiral anomaly implies the creation of the chiral charge from the superfluid vacuum. The rate of chiral charge production from the vacuum is defined by the level flow through the zeroes in the particle spectrum induced by the "electric" and "magnetic" fields of the gauge field B_μ: $\mathbf{E}_B=\nabla B_4-\partial_t\mathbf{B}$ and $\mathbf{H}_B=[\nabla\mathbf{B}]$:

$$\partial_\mu j_\mu=\frac{e^2}{4\pi^2\hbar^2}e^{\mu\nu\alpha\beta}\partial_\mu B_\nu\partial_\alpha B_\beta=\frac{e^2}{2\pi^2\hbar^2}(\mathbf{E}_B\mathbf{H}_B). \tag{4.7}$$

The right hand side of Eq. (4.7) can be represented as $\partial_\mu j_\mu^{\rm vac}$ where

$$j_\mu{}^{\rm vac}=\frac{e^2}{4\pi^2\hbar^2}e^{\mu\nu\alpha\beta}B_\nu\partial_\alpha B_\beta \tag{4.8}$$

is an additional anomalous term in the vacuum chiral current.

Our aim is to construct the action which leads to such a current: $j_\mu^{\rm vac}=\delta S^{\rm vac}/\delta eB_\mu$. The variation of the action is thus

$$\delta S^{\rm vac}=\frac{1}{4\pi^2\hbar^2}e^{\mu\nu\alpha\beta}B_\nu\partial_\alpha B_\beta\,\delta B_\mu \tag{4.9}$$

while the action itself is not local. One of the ways to construct the nonlocal action is the Wess–Zumino approach in which a new fictitious 5th dimension is introduced (see e.g., Ref. 7). The 5-dimensional "disc" $x_\mu=(\tau,t,\mathbf{r})$ is considered whose boundary coincides with the physical $(t,\mathbf{r})$ space. The Novikov–Wess–Zumino action, $S^{\rm NWZ}$, is defined as the integral over the 5-d disc:

$$S^{NWZ}=\frac{e^3}{12\pi^2\hbar^2}\int d^5x\,e^{\mu\nu\alpha\beta\gamma}B_\nu\,\partial_\alpha B_\beta\,\partial_\gamma B_\mu$$
$$=\frac{e^3}{6\pi^2\hbar^2}\int d^5x\{\partial_5B_4(\mathbf{B}[\nabla\mathbf{B}])-B_4([\nabla\mathbf{B}]\partial_5\mathbf{B})$$
$$+(\partial_5(\mathbf{B}[\mathbf{B}(\partial_4\mathbf{B}-\nabla B_4)])\}. \tag{4.10}$$

The variation of this action is a total derivative:

$$\delta S^{NWZ}=\frac{e^3}{4\pi^2\hbar^2}\int d^5x\,e^{\mu\nu\alpha\beta\gamma}\partial_\gamma(\delta B_\mu B_\nu\,\partial_\alpha B_\beta), \tag{4.11}$$

which transforms to the integral over the boundary of the 5-dimensional disc, i.e., over the physical space, giving rise to Eq. (4.9).

In the case of the A-phase with the gauge field given by Eq. (4.6) and with $B_5=0$ the Eq. (4.10) becomes

$$S^{NWZ}=\frac{\hbar}{2}\int d^3x\,dt\,d\tau\frac{k_F^3}{3\pi^2\hbar^3}\{(\hat{\mathbf{l}}[\partial_t\hat{\mathbf{l}}\,\partial_\tau\hat{\mathbf{l}}])-3(q\mathbf{A}\hat{\mathbf{l}})([\nabla\hat{\mathbf{l}}]\partial_\tau\hat{\mathbf{l}})\}$$
$$+\frac{\hbar}{2}\int d^3x\,dt\frac{k_F^3}{3\pi^2\hbar^3}(q\mathbf{A}\hat{\mathbf{l}})(\hat{\mathbf{l}}[\nabla\hat{\mathbf{l}}]). \tag{4.12}$$

This action is not invariant under the Galilean transformation and under the gauge transformation associated with the electromagnetic field $A_\mu\to A_\mu+(1/q)\partial_\mu\chi$. This occurred because we still did not take into account the anomalous contribution to the action from the fields $\mathbf{v}_s$ and v_4. The latter transform under the gauge transformation as $v_\mu \to v_\mu + (1/m_3)\partial_\mu \chi$.

To find this contribution one could use the gauge invariance requirement which dictates that one must replace A_μ by the gauge invariant quantity

$$q\mathbf{A} \to q\mathbf{A} - m_3\mathbf{v}_s$$

and

$$qA_4 \to qA_4 - m_3 v_4.$$

This is in agreement with the observation made by Volovik[10] that the superfluid velocity adds the contribution $-k_F(\hat{\mathbf{l}}\mathbf{v}_s)$ to the scalar potential B_4, which together with the gauge field $\mathbf{A}$ contribution to B_4 [see Eq. (4.6)] produces the gauge-invariant result

$$B_4 = k_F\left(\hat{\mathbf{l}}\left(\frac{q}{m_3}\mathbf{A} - \mathbf{v}_s\right)\right).$$

Restoration of the gauge invariance simultaneously restores Galilean invariance. So the correct Wess–Zumino term is

$$S^{NWZ} = \frac{\hbar}{2} C_0 \int d^3x\, dt\, d\tau \Big\{ (\hat{\mathbf{l}}[\partial_t \hat{\mathbf{l}}\, \partial_\tau \hat{\mathbf{l}}]) + 3\left(\left(\mathbf{v}_s - \frac{q}{m_3}\mathbf{A}\right)\hat{\mathbf{l}}\right)([\nabla \hat{\mathbf{l}}]\partial_\tau \hat{\mathbf{l}})\Big\} - \frac{\hbar}{2} C_0 \int d^3x\, dt \left(\left(\mathbf{v}_s - \frac{q}{m_3}\mathbf{A}\right)\hat{\mathbf{l}}\right)(\hat{\mathbf{l}}[\nabla \hat{\mathbf{l}}]). \tag{4.13}$$

Here C_0 is the anomaly parameter:

$$C_0 = \frac{k_F^3}{3\pi^2\hbar^3}. \tag{4.14}$$

The Fermi momentum is defined by Eq. (4.3) modified by introducing the v_4 field to restore the gauge invariance:

$$k_F^2 = 2m_3(\mu - m_3 v_4 + qA_4) - (m_3\mathbf{v}_s - q\mathbf{A})^2 = 2m_3\tilde{\mu}, \tag{4.15}$$

where according to the motion equation (3.6) $\mu = \delta\varepsilon/\delta\rho$ is the local chemical potential. The parameter C_0 continuously decreases as a function of the coupling strength in the Cooper channel and becomes zero at the Lifshitz transition when the gap nodes disappear. In Eq. (4.13) we neglected the spatial dependence of C_0.

One can check that the Eq. (4.13) is invariant under the Galilean transformation

$$\mathbf{r} \to \mathbf{r} - \mathbf{u}t, \quad \mathbf{v}_s \to \mathbf{v}_s + \mathbf{u}. \tag{4.16}$$

The NWZ action transforms as

$$S^{NWZ} \to S^{NWZ} + \frac{\hbar}{2} C_0 \int d^3x\, dt\, d\tau \times \{-(\hat{\mathbf{l}}[(\mathbf{u}\nabla)\hat{\mathbf{l}}\partial_\tau \hat{\mathbf{l}}]) + 3(\mathbf{u}\hat{\mathbf{l}})([\nabla \hat{\mathbf{l}}]\partial_\tau \hat{\mathbf{l}})\} - \frac{\hbar}{2} C_0 \int d^3x\, dt\, (\mathbf{u}\hat{\mathbf{l}})(\hat{\mathbf{l}}[\nabla \hat{\mathbf{l}}]). \tag{4.17}$$

The part which depends on the transformation parameter u can be rewritten in the notations of Appendix and disappears when Eq. (A1) is applied:

$$\frac{\hbar}{2} C_0 u_i \int d^3x\, dt\, d\tau (W_{5i} + 3U_{5i} - \partial_5 q_i) = -u_i \frac{\hbar}{2} C_0 \int d^3x\, dt\, d\tau\, \partial_k q_{i5k} = 0. \tag{4.18}$$

The anomalous contribution, Eq. (4.13), is to be added to the anomaly-free hydrodynamic action (3.5).

5. ZERO-CHARGE ACTION IN ^{3}He-A

Due to the gapless nature of fermions the zero-charge term should also be added to $\varepsilon\{\hat{\mathbf{l}}\}$. This is the nonanalytic gradient term which contains an imaginary part corresponding to the pair creation if the effective electric field $\mathbf{E}_B = \nabla B_4 - \partial_t \mathbf{B}$ exceeds the effective "magnetic" field $\mathbf{H}_B = [\nabla \mathbf{B}]$ (see Ref. 2). Here we consider how this term is modified due to the $\mathbf{v}_s$ field.

Since the zero-charge effect is also defined by the fermions in the vicinity of the gap nodes, we again can use the linearized equation (4.4) and write down the general expression for the zero-charge term, S^{zc}, in action in analogy with QFT:

$$S^{zc} = \frac{e^2}{\hbar} \frac{g^{1/2}}{24\pi^2} \frac{1}{2} g^{\mu\nu} g^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta} \ln \frac{\Lambda^2}{(\frac{1}{2} g^{\mu\nu} g^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta})^{1/2}}, \tag{5.1}$$

where

$$F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu. \tag{5.2}$$

Using expressions (4.5) and (4.6) for $g^{\mu\nu}$ and $F_{\mu\nu}$ one obtains

$$g^{\mu\nu} g^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta} = 2k_F^2 c_\perp^2 \Big[-\Big(\partial_t \hat{\mathbf{l}} - \frac{q}{m_3}(\mathbf{A}\nabla)\hat{\mathbf{l}} - \frac{q}{m_3} \hat{l}_i \nabla_\perp \mathbf{A}_i\Big)^2 + c_\parallel^2((\hat{\mathbf{l}}\nabla)\hat{\mathbf{l}})^2 + c_\perp^2(\hat{\mathbf{l}}[\nabla \hat{\mathbf{l}}])^2 \Big], \tag{5.3}$$

where $\nabla_\perp = \nabla - \hat{\mathbf{l}}(\hat{\mathbf{l}}\nabla)$. The gauge invariance of this expression is restored by introduction of the v_s field. By using the equation

$$g^{1/2} = (c_\parallel c_\perp^2)^{-1} \tag{5.4}$$

one obtains the following zero-charge contribution to the hydrodynamic action:

$$S^{zc} = \frac{k_F^2}{48\pi^2 \hbar v_F} \int d^3x\, dt\, Q \ln \frac{1}{Q},$$

$$Q = -\Big\{\Big(\partial_t + \Big(\Big(\mathbf{v}_s - \frac{q}{m_3}\mathbf{A}\Big)\nabla\Big)\Big)\hat{\mathbf{l}} + \hat{l}_i \nabla_\perp\Big((\mathbf{v}_s)_i - \frac{q}{m_3} A_i\Big)\Big\}^2 + v_F^2((\hat{\mathbf{l}}\nabla)\hat{\mathbf{l}})^2 + c_\perp^2(\hat{\mathbf{l}}[\nabla \hat{\mathbf{l}}])^2. \tag{5.5}$$

This term is invariant under the Galilean transformation (4.16). Another important symmetry property is also satisfied, which corresponds to the Larmor theorem: the constant magnetic field is equivalent to rotation. If one takes $\mathbf{A} = \frac{1}{2}[\mathbf{H}\mathbf{r}]$ then the time derivative enters in the combination $\partial_t \hat{\mathbf{l}} - [\boldsymbol{\Omega}\hat{\mathbf{l}}]$ where $\boldsymbol{\Omega} = (q/2m_3)\mathbf{H}$ is the Larmor frequency. Thus we have obtained the total hydrodynamic action (3.5) + (4.13) + (5.5) for combined orbital and superfluid dynamics of the A-phase, which satisfies all the symmetry requirements and reflects all the anomalous properties of the dynamics (chiral anomaly + zero-charge effect):

$$S = S^{af} + S^{NWZ} + S^{zc}. \tag{5.6}$$

6. LINEAR MOMENTUM PARADOX

We have obtained the hydrodynamic action Eq. (5.6), which satisfies all the symmetries. This means that all the conservation laws, including the conservation of linear momentum, should be valid. The mass current and linear momentum are defined in the same way as in Sec. 2. The mass current is obtained as a variation over the vector potential $\mathbf{A}$:

$$\mathbf{j} = -\frac{\delta S}{q\delta \mathbf{A}} = \rho \mathbf{v}_s + \mathbf{j}^{an} + \mathbf{j}^{lf} + \mathbf{j}^{zc},$$

$$\mathbf{j}^{an} = -\frac{\hbar}{2} C_0 \hat{\mathbf{l}} (\hat{\mathbf{l}} [\nabla \hat{\mathbf{l}}]), \quad \mathbf{j}^{lf} = \frac{3\hbar}{2} C_0 \int d\tau \hat{\mathbf{l}} ([\nabla \hat{\mathbf{l}}] \partial_\tau \hat{\mathbf{l}}),$$

$$\mathbf{j}^{zc} = -\frac{\delta S^{zc}}{q\delta \mathbf{A}}. \tag{6.1}$$

The first term in (6.1) represents the response of the action in Eq. (3.5), which gives Eq. (3.13). The second and third terms in (6.1) represent the response of the Novikov–Wess–Zumino action (4.13) where $\mathbf{j}^{an}$ represents an anomalous current and $\mathbf{j}^{lf}$ is the current related to the level flow. The last term comes from Eq. (5.5).

The linear momentum is defined as action response to the coordinate transformation $\mathbf{r} \rightarrow \mathbf{r} - \mathbf{u}t$. Due to the Galilean invariance the variation over $\mathbf{u}$ is equivalent to the variation over $e\mathbf{A}$ if the full derivatives are neglected in the mass current. Therefore the total momentum of the liquid coincides with the total mass current:

$$\mathbf{P} = \int d^3x\, \mathbf{j}, \tag{6.2}$$

which also can be checked in direct way using Eq. (A1) of the Appendix.

The second term in (6.1) is known as the anomalous current, while the third one is the level-flow term. This term is not a well defined quantity: it is not expressed in terms of instant values of the hydrodynamic variables, but depends on the history of the $\hat{\mathbf{l}}$ dynamics. To understand the physical meaning of this term let us consider the integration over τ as the integration over the history of the $\hat{\mathbf{l}}$. We take $\hat{\mathbf{l}}(t,\tau) = \hat{\mathbf{l}}(t-\tau)$, so the parameter $\tau > 0$ describes the contribution of the $\hat{\mathbf{l}}$ field to $\mathbf{j}(t)$ at the previous moment $\tilde{t} = t - \tau < t$. Then one obtains for this term

$$\begin{aligned}\mathbf{j}^{lf} &= \frac{3\hbar}{2} C_0 \int_{-\infty}^{t} d\tilde{t}\, \hat{\mathbf{l}}(\tilde{t}) ([\nabla \hat{\mathbf{l}}(\tilde{t})]\, \partial_{\tilde{t}} \hat{\mathbf{l}}(\tilde{t})) \\ &= \int_{-\infty}^{t} d\tilde{t}\, k_F \hat{\mathbf{l}}(\tilde{t}) \frac{k_F^2}{2\pi^2} ([\nabla \hat{\mathbf{l}}(\tilde{t})]\, \partial_{\tilde{t}} \hat{\mathbf{l}}(\tilde{t})) \\ &= \int_{-\infty}^{t} d\tilde{t}\, k_F \hat{l}(\tilde{t}) \frac{1}{2\pi^2} (\mathbf{E}_B(\tilde{t})\, \mathbf{H}_B(\tilde{t})).\end{aligned} \tag{6.3}$$

The term $(1/2\pi^2)[\mathbf{E}_B(\tilde{t})\mathbf{H}_B(t)]$ is the rate of the quasiparticle flow through the gap nodes at the time $\tilde{t}$ [see Eq. (4.7)] induced by the "electric" field $\mathbf{E}_B$, while $k_F\hat{\mathbf{l}}(\tilde{t})$ is the quasiparticle linear momentum. Therefore the whole integral is the total momentum reversibly transferred by the quasiparticles through the gap nodes during the whole history of the orbital dynamics.

In the collisionless limit the quasiparticle scattering is absent and the empty levels which under the "electric" field enter from the region above the chemical potential ($E > 0$) through the gap nodes into the region below the chemical potential ($E < 0$) remain empty. In the same manner the occupied levels with $E < 0$, which enter the region with $E > 0$, remain occupied. As a result the state of the liquid at the moment t contains a nonequilibrium number of free or occupied levels, which depends on the history. Therefore the total momentum of the liquid depends on the history of the orbital dynamics.

In the collisionless dynamics, described by the hydrodynamic action, the total momentum $\mathbf{P}$ in Eq. (6.2) is a conserved quantity according to the Noether theorem

$$\partial_t \mathbf{P} = 0, \tag{6.4}$$

which also can be checked from the equations of motion. The momentum is conserved but is not well-defined. Correspondingly, the well-defined (equilibrium) part of the hydrodynamic mass current is not conserved:

$$\begin{aligned}\partial_t \mathbf{P}^{eq} = \partial_t \int d^3x [\rho \mathbf{v}_s + \mathbf{j}^{an} + \mathbf{j}^{zc}] &= -\int d^3x\, \partial_t \mathbf{j}^{lf}(t) \\ &= -\frac{3\hbar}{2} C_0 \int d^3x\, \hat{\mathbf{l}}([\nabla \hat{\mathbf{l}}]\, \partial_t \hat{\mathbf{l}}),\end{aligned} \tag{6.5}$$

which represents the linear momentum paradox in the A-phase. The rhs of (6.5) is the rate of the momentum transfer through the gap nodes. In the derivation of Eq. (6.5) we can use the simplified Eq. (6.3) for $\mathbf{j}^{lf}$. For the general case of $\mathbf{j}^{lf}$ in Eq. (6.1) the Eq. (A6) of Appendix has been used which gives

$$\begin{aligned}\partial_t \int d^3x\, d\tau \hat{\mathbf{l}}([\nabla \hat{\mathbf{l}}]\, \partial_\tau \hat{\mathbf{l}}) &= \int d^3x\, d\tau\, \partial_t \mathbf{U}_5 \\ &= \int d^3x\, d\tau\, \partial_\tau \mathbf{U}_0 = \int d^3x \hat{\mathbf{l}} (\partial_t \hat{\mathbf{l}} [\nabla \hat{\mathbf{l}}]).\end{aligned} \tag{6.6}$$

7. INTERACTION WITH WALLS: ANALOG OF THE IORDANSKII FORCE

If the quasiparticle scattering on the wall is taken into account, then the level flow is accompanied by a redistribution of the quasiparticles between the levels. In general this process should be described by the complicated kinetic theory; however the problem is simplified in the limiting case when the equilibrization is fast compared with the $\hat{\mathbf{l}}$ dynamics. In this low-frequency limit the liquid is always in a local equilibrium and the hydrodynamic momentum is $\mathbf{P} = \mathbf{P}^{eq}$. This momentum is not conserved according to Eq. (6.5) and the rhs of Eq. (6.5) is the rate of the momentum exchange with the walls of container: the walls absorb or release all the momentum transferred through the gap nodes in the process of the level flow. The hydrodynamics of the liquid in this limit is also reversible, since we suppose that there is no time delay in the absorption of the momentum by the wall, and therefore no dissipation.

The hydrodynamic action in this limit can be obtained by introducing the normal velocity $\mathbf{v}_n$, the velocity of the heat bath of the walls, which interacts with the quasiparticle momenta $\mathbf{j}^{lf}$, transferred through the gap nodes. The hydrodynamic action is modified by addition of the momentum exchange term, S^{ex}, describing this interaction:

$$S^{ex}=\int d^3x\,dt\left(\left(\mathbf{v}_n-\mathbf{v}_s+\frac{q}{m_3}\mathbf{A}\right)\mathbf{j}^{lf}\right). \tag{7.1}$$

As a result the NWZ action is to be modified:

$$S^{NWZ}+S^{ex}=\frac{\hbar}{2}C_0\int d^3x\,dx\,d\tau(\hat{\mathbf{l}}[(\partial_t+(\mathbf{v}_n\nabla))\hat{\mathbf{l}}\partial_\tau\hat{\mathbf{l}}])$$
$$-\frac{\hbar}{2}C_0\int d^3x\,dt\left(\left(\mathbf{v}_s-\frac{q}{m_3}\mathbf{A}\right)\hat{\mathbf{l}}\right)(\hat{\mathbf{l}}[\nabla\hat{\mathbf{l}}]). \tag{7.2}$$

This action is invariant under the Galilean transformation, which now includes also the transformation of the normal velocity: $\mathbf{v}_n\to\mathbf{v}_n+\mathbf{u}$. The variation of the total action over $\mathbf{A}$ gives an equilibrium value $\mathbf{P}=\mathbf{P}^{eq}$ of the hydrodynamic momentum. Note that the normal velocity which enters the first term in Eq. (7.2) defined on 5-d disc is the velocity of the walls and therefore is assumed to be constant. One can check that under this condition the variation of this term is defined on the disc boundary representing the physical space:

$$\delta\int d^3x\,dt\,d\tau(\hat{\mathbf{l}}[(\partial_t+(\mathbf{v}_n\nabla))\hat{\mathbf{l}}\,\partial_\tau\hat{\mathbf{l}}])$$
$$=\int d^3x\,dt(\hat{\mathbf{l}}[(\partial_t+(\mathbf{v}_n\nabla))\hat{\mathbf{l}}\,\delta\hat{\mathbf{l}}]). \tag{7.3}$$

Therefore the equations of motion are completely local in this regime. In particular the motion equation for $\hat{\mathbf{l}}$ is [compare with Eq. (3.13)]

$$\frac{\hbar}{M}[(\rho\partial_t\hat{\mathbf{l}}+(\mathbf{j}\nabla)\hat{\mathbf{l}})-m_3C_0(\partial_t\hat{\mathbf{l}}+(\mathbf{v}_n\nabla)\hat{\mathbf{l}})]=-\left[\hat{\mathbf{l}}\frac{\delta S^{\rm loc}}{\delta\hat{\mathbf{l}}}\right], \tag{7.4}$$

where the rhs is the variation of the local part of the action, which also includes the zero-charge contribution containing the second order time derivative. If one considers this equation as a kind of conservation law for the orbital angular momentum $\partial_t\mathbf{L}=-[\hat{\mathbf{l}}(\delta S^{\rm loc}/\delta\hat{\mathbf{l}}]$ then the dynamical orbital momentum appears to be $\mathbf{L}=(\hbar/M)(\rho-m_3C_0)\hat{\mathbf{l}}$.

It is instructive to consider the forces acting on the doubly quantized vortex in the regime of the slow vortex motion, when the Eq. (7.2) is valid. For this vortex the $\hat{\mathbf{l}}$ field is continuously distributed within the soft core without any singularity in the order parameter (see the review in Ref. 8) and therefore the hydrodynamic equations can be applied to describe the vortex motion. The vortex is characterized by the topological invariant for the $\hat{\mathbf{l}}$ field in the soft core:

$$\frac{1}{8\pi}\int_\sigma dS_k e^{ijk}(\hat{\mathbf{l}}[\partial_i\hat{\mathbf{l}}\,\partial_j\hat{\mathbf{l}}])=1, \tag{7.5}$$

where the integral is over the cross section σ of the core. If the vortex moves with a velocity $\mathbf{v}_L$ different from the asymptotic value of the superfluid velocity far from the vortex, the conventional Magnus force acts on the vortex:

$$\mathbf{F}^M=\left(\frac{2h}{M}\right)\rho[\hat{\mathbf{v}}(\mathbf{v}_s-\mathbf{v}_L)], \tag{7.6}$$

where $\hat{\mathbf{v}}$ is the unit vector tangential to the vortex line.

In addition, if the vortex moves with respect to the walls, then according to Eq. (6.5) another force should arise which corresponds to momentum exchange between the vortex and the walls due to the level flow. The integration of the rate of momentum transfer (the rhs of Eq. (6.5) modified by introduction of the normal velocity $\mathbf{v}_n$) over the vortex cross section σ gives the following force per unit length of the vortex line:

$$\mathbf{F}^{lf}=-\frac{3\hbar}{2}C_0\int_\sigma\hat{\mathbf{l}}([\nabla\hat{\mathbf{l}}](\partial_t+(\mathbf{v}_n\nabla))\hat{\mathbf{l}}). \tag{7.7}$$

Inserting the $\hat{\mathbf{l}}$ field of the moving vortex

$$\hat{\mathbf{l}}(\mathbf{r},t)=\hat{\mathbf{l}}(\mathbf{r}-\mathbf{v}_Lt), \tag{7.8}$$

and using Eq. (A7) of Appendix one obtains:

$$\mathbf{F}^{lf}=-\frac{3\hbar}{2}C_0\int_\sigma\hat{\mathbf{l}}([\nabla\hat{\mathbf{l}}]((\mathbf{v}_n-\mathbf{v}_L)\nabla))\hat{\mathbf{l}})$$
$$=-\frac{\hbar}{4}C_0\left[(\mathbf{v}_n-\mathbf{v}_L)\int_\sigma e_{ijk}\hat{l}_i[\nabla\hat{l}_j\,\nabla\hat{l}_k]\right]$$
$$=\left(\frac{2h}{M}\right)m_3C_0[\hat{\mathbf{v}}(\mathbf{v}_n-\mathbf{v}_L)]. \tag{7.9}$$

The last expression is obtained using the Eq. (7.5) for the topological charge of the vortex. This reactive (nondissipative) force resembles the reversible Iordanskii force acting on the vortex from the normal component of the liquid.[9] The Iordanskii force in conventional superfluids disappears in the $T\to0$ limit, when the normal component is frozen out, while in the A-phase this force extrapolates to the finite value proportional to the anomaly parameter C_0. This is the result of the gapless superfluidity of the A-phase with topological zeroes.

8. CONCLUSION

We have obtained here the hydrodynamic action for the $T=0$ orbital and superfluid dynamics of the liquid with the topologically stable zeroes in the fermionic quasiparticle energy spectrum. This action is gauge- and Galilean-invariant and also describes the anomalies related to the existence of the topological gap nodes. This allowed us to consider the linear momentum paradox. It is shown that there are two limiting cases of low frequency and high frequency as compared with the inverse scattering time of the quasiparticles on the walls. In the high-frequency limit the hydrodynamic linear momentum of the liquid is conserved but is not defined in terms of the instant values of the hydrodynamic variables. Due to the effect of the level flow, related to the chiral anomaly, the momentum depends on the history of the orbital $\hat{\mathbf{l}}$ vector which defines the position of the gap nodes. In the low-frequency limit the linear momentum is well-defined in terms of the local and instant hydrodynamic variables, but is not conserved since the momentum which is transferred through the gap nodes is absorbed by the walls. The momentum transfer rate is described by the level flow. In this limit the momentum nonconservation leads to the additional reactive force acting on the quantized vortex, the analog of Iordanskii force, which in the A-phase exists even at $T = 0$ and is proportional to the anomaly parameter C_0.

This Lagrange approach to the hydrodynamics can in principle be applied to more complicated situations with the gap nodes. For example the gap nodes exist within the cores of the B-phase vortices,[8] and these nodes will govern the vortex dynamics at low T.

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APPENDIX

Invariance of the NWZ action (4.17) under the Galilean transformation was checked using the following relation:

$$\nabla_\nu q_m - \nabla_k q_{m\nu k} = 3U_{\nu m} + W_{\nu m}, \tag{A1}$$

where

$$q_m = \hat{l}_m (\check{\mathbf{l}}[\nabla \check{\mathbf{l}}]) = q_{mkk}, \quad q_{m\nu k} = \hat{l}_m [\nabla_\nu \hat{\mathbf{l}} \hat{\mathbf{l}}]_k.$$
$$U_{\nu m} = \hat{l}_m (\nabla_\nu \hat{\mathbf{l}} [\nabla \hat{\mathbf{l}}]), \quad W_{\mu\nu} = -W_{\nu\mu} = (\hat{\mathbf{l}} [\nabla_\mu \hat{\mathbf{l}} \nabla_\nu \hat{\mathbf{l}}]). \tag{A2}$$

The Eq. (A1) can be found from the following chain of equations:

$$\nabla_\nu q_m - \nabla_k q_{m\nu k} = V_{\nu m} + 2U_{\nu m} + Q_{\nu m},$$

$$V_{\nu m} = \nabla_\nu \check{l}_m (\check{\mathbf{l}}[\nabla \check{\mathbf{l}}]), \quad Q_{\nu m} = (\nabla \check{l}_m [\check{\mathbf{l}} \nabla_\nu \check{\mathbf{l}}]),$$

$$Q_{\nu m} - W_{\nu m} = [[\nabla \hat{\mathbf{l}}][\hat{\mathbf{l}} \nabla_\nu \hat{\mathbf{l}}]]_m = U_{\nu m} - V_{\nu m}. \tag{A3}$$

Here in the last line we have twice used the equation $[a \times (b \times c)] = b(a \cdot c) - c(a \cdot b)$.

Using Eq. (A1) with $\nu = 4$ one may find the time derivative of the anomalous current in Eq. (6.1):

$$\begin{aligned}\partial_t (\hat{l}_m (\hat{\mathbf{l}}[\nabla \check{\mathbf{l}}]) &= \partial_t q_m = 3U_{4m} + W_{4m} + \nabla_k q_{m4k} \\ &= 3\hat{l}_m (\partial_t \hat{\mathbf{l}} [\nabla \hat{\mathbf{l}}]) + (\hat{\mathbf{l}} [\partial_t \hat{\mathbf{l}} \nabla_m \hat{\mathbf{l}}]) - \nabla_k q_{m4k},\end{aligned} \tag{A4}$$

and its variation if one changes $\partial_\nu \to \delta$:

$$\delta \int \hat{l}_m (\hat{\mathbf{l}}[\nabla \hat{\mathbf{l}}]) = \int 3\hat{l}_m (\delta \hat{\mathbf{l}} [\nabla \hat{\mathbf{l}}]) + \int (\hat{\mathbf{l}} [\delta \hat{\mathbf{l}} \nabla_m \hat{\mathbf{l}}]). \tag{A5}$$

Another relation:

$$\partial_\mu U_{\nu i} - \partial_\nu U_{\mu i} = -(\nabla (\hat{l}_i \hat{\mathbf{l}} W_{\mu\nu})) \tag{A6}$$

was used to find the time derivative of the level-flow mass current in Eq. (6.1).

The Eq. (7.9) is obtained if the Eq. (A1) with both spatial indices is integrated over the cross section σ of the vortex core of the nonsingular doubly quantized vortex:

$$\begin{aligned}\int_\sigma \hat{l}_m ((\mathbf{a}\nabla) \hat{\mathbf{l}} [\nabla \hat{\mathbf{l}}]) &= a_n \int_\sigma U_{nm} = \frac{1}{3} a_n \int W_{mn} \\ &= \frac{1}{6} \left[\mathbf{a} \int_\sigma e_{ijk} \hat{l}_i [\nabla \hat{\mathbf{l}}_j \nabla \hat{\mathbf{l}}_k] \right]_m = \frac{4\pi}{3} [\mathbf{a} \hat{\mathbf{v}}]_m,\end{aligned} \tag{A7}$$

where $\mathbf{a} = \mathbf{v}_n - \mathbf{v}_L$.

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