Vacuum polarization and nuclear multipole moments

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We consider the contribution of electron vacuum polarization to the magnetic dipole and electric quadrupole moments of heavy nuclei. The leading contributions, proportional to the large logarithm $\ln(\lambdabar_e/R)$ ($\lambdabar_e$ is the Compton wavelength of the electron, R is the the nuclear radius), are calculated exactly in the parameter $Z\alpha$ ($Z|e|$ is the nuclear charge, α is the fine structure constant) with the help of the Green's function for the Dirac equation in a Coulomb field. The feasibility of experimental observation of the effect is discussed.

1. INTRODUCTION

As is well known, the discussion of certain quantum electrodynamic processes in a strong Coulomb field requires the exact contribution of that field, not an expansion in powers of the parameter $Z\alpha$ ($Z|e|$ is the nuclear charge, $\alpha = e^2 = 1/137$ is the fine structure constant, e is the electron charge, $\hbar = c = 1$). For example, Coulomb corrections noticeably reduce the coherent scattering cross section of a photon in the field of a nucleus.[1,2]

An exact treatment of the Coulomb field is also needed in the analysis of the effect of vacuum polarization on the structure of the energy spectrum of muonic atoms (see Ref. 3 and the literature cited therein). Usually in such an analysis only the spherically symmetric part $\rho(r)$ of the induced vacuum charge distribution is taken into account. In particular, in the pioneering work of Wichmann and Kroll,[4] the Laplace transform of the product $\rho(r)r^2$ was calculated. The density $\rho(r)$ itself was found in Ref. 5. In Ref. 6 the behavior of the spherically symmetric part of the density at small distances was studied by operator methods. In addition particular contributions to $\rho(r)$ were found with the help of numerical methods (see Refs. 3 and 7). Knowledge of the potential due to the induced charge permits the calculation of shifts of atomic energy levels (see, for example, Ref. 8).

However, there exist heavy nuclei possessing large multipole moments. The fields of such nuclei induce corresponding moments in the vacuum. In contrast to the total induced charge, equal to zero in view of the electric neutrality of the vacuum, higher multipole moments of the vacuum have nonzero values. More than that, it turns out that the leading contributions to these moments have additional enhancement factors in the form of the large logarithm $\ln(\lambdabar_e/R)$ of the ratio of the electron Compton wavelength $\lambdabar_e$ to the nuclear radius R. The present work is devoted to the calculation of these contributions to the induced magnetic dipole and electric quadrupole moments.

The logarithmic enhancement of the induced moments was first discovered in Refs. 9 and 10 in the calculation of the three-loop contribution of the electron vacuum polarization to the anomalous magnetic moment of the muon (the role of R in this case is played by the muon Compton wavelength). One of the authors of Ref. 11 has proposed a simple method for the calculation of the logarithmic contribution in lowest order of perturbation theory. The muon is viewed as a point source of the Coulomb field and the magnetic dipole field. As a result there appears in the calculation of the induced moment a logarithmically divergent integral over the distances from the field centers, which is cut off at short distances at the muon Compton wavelength $\lambda_\mu = 1/m_\mu$ and at large distances at $\lambdabar_e = 1/m_e$. We use a similar approach in the case of the nucleus. However, for nuclei with large charge Z it is no longer legitimate to consider their Coulomb field by means of perturbation theory (the expansion parameter is now $Z\alpha \sim 1$). For this reason we perform the calculations with the help of the exact Green's function for the electron in the Coulomb field.

2. MAGNETIC DIPOLE MOMENT

We go to the calculations. Outside the nucleus its magnetic moment $\boldsymbol{\mu}$ produces a magnetic field with the vector potential $\mathbf{A} = [\boldsymbol{\mu}\mathbf{r}]/r^3$. This magnetic field induces a vacuum electron current

$$\mathbf{J} = -ie\int\frac{d\varepsilon}{2\pi}\,\mathrm{Tr}\,\boldsymbol{\gamma}G(\mathbf{r},\mathbf{r}'|\varepsilon), \tag{1}$$

where $G(\mathbf{r},\mathbf{r}'|\varepsilon)$ is the electron Green's function, which we represent in the form

$$G(\mathbf{r},\mathbf{r}'|\varepsilon) = \left\langle \mathbf{r}\left|\frac{1}{\gamma_0(\varepsilon+Z\alpha/r)-\boldsymbol{\gamma}(\mathbf{p}-e\mathbf{A}/c)-m}\right|\mathbf{r}'\right\rangle. \tag{2}$$

Here γ_μ are the Dirac matrices. According to the Feynman rules the contour of integration over the energy ε in (1) goes from $-\infty$ to $+\infty$ below the real axis in the left ε-halfplane and above the axis in the right halfplane. The magnetic moment corresponding to the vacuum current (1) is

$$\mathbf{M} = \frac{1}{2}\int d\mathbf{r}[\mathbf{r}\mathbf{J}]. \tag{3}$$

This moment is directed along $\boldsymbol{\mu}$: $\mathbf{M} = g\boldsymbol{\mu}$. Expanding the Green's function in a series in powers of $\mathbf{A}$ we obtain from (1) and (3) the following expression for the coefficient g:

$$g = \frac{i\alpha}{12\pi}\int d\varepsilon\int\frac{d\mathbf{r}}{r^3}\,\mathrm{Tr}\left\langle\mathbf{r}\left|\frac{1}{\hat{P}-m}[\mathbf{r}\boldsymbol{\gamma}]\frac{1}{\hat{P}-m}[\mathbf{r}\boldsymbol{\gamma}]\right|\mathbf{r}\right\rangle, \tag{4}$$

where $\hat{P} = \gamma_0(\varepsilon + Z\alpha/r) - \boldsymbol{\gamma}\mathbf{p}$. It will be shown below that the first term in the expansion of the renormalized quantity g in powers of $Z\alpha$ is proportional to $(Z\alpha)^2$. For this reason it is necessary to subtract from the integrand in (4) its value at

$Z=0$. In what follows this subtraction is to be understood, it will be carried out explicitly in the final result. After the subtraction we still must regularize the integral since it is logarithmically divergent at small distances. We perform this regularization by taking the nuclear radius R as the lower limit in the integration over r. Since we confine ourselves in the following to logarithmic accuracy we may set the electron mass equal to zero. Further, the radial integration should be cut off above by the electron Compton wavelength $\lambdabar_e$. In this fashion we obtain an expression for the logarithmic contribution to the quantity g that is exact in $Z\alpha$.

Simple manipulations using the identity $1/\hat{P}=\hat{P}/\hat{P}^2$ bring (4) to the form

$$g=\frac{i\alpha}{12\pi}\int d\varepsilon\int\frac{d\mathbf{r}}{r^3}\operatorname{Tr}\Big(2r^2\Big\langle\mathbf{r}\Big|\frac{1}{\hat{P}^2}\Big|\mathbf{r}\Big\rangle+\Big\langle\mathbf{r}\Big|(2\mathbf{j}^2-\mathbf{\Sigma l})\frac{1}{(\hat{P}^2)^2}\Big|\mathbf{r}\Big\rangle+\Big\langle\mathbf{r}\Big|\Big(\frac{3}{2}\mathbf{n}(\mathbf{\Sigma n})-\mathbf{\Sigma}\Big)\frac{1}{\hat{P}^2}\mathbf{\Sigma}\frac{1}{\hat{P}^2}\Big|\mathbf{r}\Big\rangle\Big),\tag{5}$$

where $\mathbf{l}$ is the orbital angular momentum operator, $\mathbf{j}=\mathbf{l}=\mathbf{\Sigma}/2$, $\mathbf{\Sigma}=\gamma_0\gamma_5\boldsymbol{\gamma}$, $\mathbf{n}=\mathbf{r}/r$. The analyticity properties of the Green's function allow rotating the contour of integration by $\pi/2$ into coincidence with the imaginary axis. An integral representation for the electron Green's function in the Coulomb field was obtained in Ref. 12, valid in the entire complex plane of the variable ε. This representation is very convenient for applications. Equation (16) from Ref. 12 for $D(\mathbf{r},\mathbf{r}'|\varepsilon)=\langle\mathbf{r}|1/\hat{P}^2|\mathbf{r}'\rangle$ for $\varepsilon=iE$ can be represented as

$$D(\mathbf{r},\mathbf{r}'|\pm i|E|)=-\frac{1}{4\pi rr'|E|}\int_0^\infty ds\exp[\pm 2iZ\alpha s-|E|(r+r')\operatorname{cth}s]\times\sum_{l=1}^{\infty}\Big\{\frac{1}{2}(1-\boldsymbol{\gamma}\mathbf{n}\cdot\boldsymbol{\gamma}\mathbf{n}')ByI_{2\nu}'(y)+[(1+\boldsymbol{\gamma}\mathbf{n}\cdot\boldsymbol{\gamma}\mathbf{n}')lA+iZ\alpha\gamma_0\boldsymbol{\gamma}(\mathbf{n}+\mathbf{n}')B]I_{2\nu}(y)\Big\}.\tag{6}$$

Here $I_{2\nu}(y)$ is the modified Bessel function of the first kind,

$$\mathbf{n}'=\mathbf{r}'/r',\quad y=2|E|(rr')^{1/2}/\operatorname{sh}s,\quad x=\mathbf{nn}',$$
$$A(x)=\frac{d}{dx}[P_l(x)+P_{l-1}(x)],\quad B(x)=\frac{d}{dx}[P_l(x)-P_{l-1}(x)],$$

P_l are the Legendre polynomials, and $\nu=[l^2-(Z\alpha)^2]^{\frac{1}{2}}$. Passing from integration over r to integration over $r|E|$ we obtain the factored logarithmically divergent integral, which gives the large logarithm mentioned above. It is now convenient to express (5) in the form

$$g=\frac{2\alpha}{3\pi}\ln\Big(\frac{\lambdabar_e}{R}\Big)f_M(Z\alpha).\tag{7}$$

It follows from the results of Refs. 10 and 11 that $f_M(Z\alpha)\to(Z\alpha)^2$ for $Z\alpha\to 0$. Using the representation (6) it is not hard to calculate the first term in the brackets in (5):

$$f_1=\sum_{l=1}^{\infty}\Big\{2l[\psi(l)-\operatorname{Re}\psi(\nu+iZ\alpha)]+1-\frac{\nu}{l}\Big\}.\tag{8}$$

Here $\psi(x)=d\ln\Gamma(x)/dx$, where $\Gamma(x)$ is the gamma function.

For the calculation of the second and third terms we represent the matrix element $\langle\mathbf{r}|(1/\hat{P}^2)\mathbf{\Sigma}(1/\hat{P}^2)|\mathbf{r}\rangle$ in the form of the integral

$$\int d\mathbf{r}'\langle\mathbf{r}|(1/\hat{P}^2)|\mathbf{r}'\rangle\mathbf{\Sigma}\langle\mathbf{r}'|(1-\hat{P}^2)|\mathbf{r}\rangle$$

and similarly for $\langle\mathbf{r}|(1/\hat{P}^2)^2|\mathbf{r}\rangle$. We then take the trace over the products of γ matrices and integrate over the angles of $\mathbf{n}$ and $\mathbf{n}'$. This is easiest done by making use directly of Eq. (16) from Ref. 12, where the function $D(\mathbf{r},\mathbf{r}'|\varepsilon)$ is represented as a sum over partial waves. The corresponding projection operators satisfy the conditions of orthonormality, making it easy to calculate the necessary traces and integrals. As a result there remain in the double sum over l and l' just the terms with $l'=l$ and $l'=l\pm 1$ ("diagonal" and "off-diagonal" transitions respectively). We have

$$f_2=\sum_{l,l'=1}^{\infty}\sum_{\sigma\sigma'=\pm 1}\iint_0^\infty\frac{ds\,dt}{\operatorname{sh}s\operatorname{sh}t}\cos(2Z\alpha T)\iint_0^\infty\frac{dr}{r^2}r'\,dr'\times\exp\Big[-\frac{(r+r')\operatorname{sh}T}{\operatorname{sh}s\operatorname{sh}t}\Big]I_{2\nu+\sigma}(y)I_{2\nu'+\sigma'}(y')G.\tag{9}$$

Here

$$y=2(rr')^{1/2}/\operatorname{sh}s,\quad y'=2(rr')^{1/2}/\operatorname{sh}t,$$
$$T=s+t,\quad \nu'=[l'^2-(Z\alpha)^2]^{1/2},$$
$$G=\frac{\delta_{l,l'}}{\nu^2}\frac{4l^3}{4l^2-1}[l^2(1-\nu\sigma-2\nu^2)\delta_{\sigma,\sigma'}-(Z\alpha)^2\delta_{-\sigma,\sigma'}]+\frac{\delta_{l',l+1}}{2}\frac{ll'}{l+l'}\Big[1+\frac{l\sigma}{\nu}-\frac{l'\sigma'}{\nu'}-\frac{ll'-3(Z\alpha)^2}{\nu\nu'}\sigma\sigma'\Big].\tag{10}$$

In (9) we made use of the symmetry under the exchange of l and l'. We describe briefly the subsequent calculations, as they are not entirely trivial.

We start with a discussion of the contribution f_{21} of the diagonal transitions. In this case all integrals are calculated analytically. The integration over r' and then over r is performed with the help of familiar formulas (Ref. 13, p. 321 and 303):

$$\int_0^\infty dx\,x\exp(-cx^2)I_\alpha(ax)I_\alpha(bx)=\frac{1}{2c}I_\alpha\Big(\frac{ab}{2c}\Big)\exp\Big(\frac{a^2+b^2}{4c}\Big),$$
$$\int_0^\infty\frac{dx}{x}\exp(-px)I_\alpha(x)=\frac{1}{\alpha}[p-(p^2-1)^{1/2}]^\alpha,\tag{11}$$

and also recursion relations for Bessel functions. Introducing then new variables $T=s+t$ and $\tau=s-t$ and integrating over τ and over T we finally obtain

$$f_{21}=4(Z\alpha)^2\sum_{l=1}^{\infty}\frac{l^5}{\nu^3(4l^2-1)}\left[\frac{\operatorname{Re}\psi'(\nu+iZ\alpha)}{2\nu-1}+\frac{\operatorname{Re}\psi'(\nu+1+iZ\alpha)}{2\nu+1}-\frac{1}{l^2}\right], \qquad (12)$$

where $\psi'(x)=d\psi(x)/dx$. Calculation of the contribution f_{22} of the off-diagonal transitions requires a somewhat bigger effort. We introduce the variables

$$T=s+t, \quad y=\operatorname{sh} s/\operatorname{sh} T, \quad \rho=(rr')^{1/2}, \quad u=(r/r')^{1/2}.$$

We integrate over u and then over T, making use of the relation (Ref. 13, p. 358)

$$\int_0^{\infty} K_0([a^2+b^2+2ab\operatorname{ch} T]^{1/2})\operatorname{ch}(\mu T)\,dT=K_\mu(a)K_\mu(b). \qquad (13)$$

Here $K_\mu(x)$ is the modified Bessel function of the third kind. After integration over ρ and y the function f_{22} can be represented as

$$f_{22}=\frac{1}{2}\sum_{l=1}^{\infty}\frac{ll'}{l+l'}\int_0^1\frac{dx}{x^2}\left(\frac{2}{x}-1\right)\times\{[\Phi'(\nu,x)-l\Phi(\nu,x)][\Phi'(\nu',x)+l'\Phi(\nu',x)]+3(Z\alpha)^2\Phi(\nu,x)\Phi(\nu',x)-\Phi_0^2(x)\}, \qquad (14)$$

where

$$\Phi(\nu,x)=x^\nu\frac{\Gamma(\nu+iZ\alpha)\Gamma(\nu-iZ\alpha)}{\Gamma(2\nu+1)}F(\nu+iZ\alpha,\nu-iZ\alpha;2\nu+1;x),$$

$F(a,\ b;\ c;\ x)$ is the hypergeometric function, $\Phi'(x)=\partial\Phi(x)/\partial x$, and

$$l'=l+1, \quad \nu=[l^2-(Z\alpha)^2]^{1/2},$$

$$\Phi_0=x^{l'}\frac{(l!)^2}{(l+l')!}F(l',l';2l';x).$$

We recall that the function $f_M(Z\alpha)$, in terms of which the sought-for quantity g (7) is expressed, is equal to $f_1+f_{21}+f_{22}$ [see (8), (12), and (14)]. Hence we have evaluated the logarithmic part of the induced magnetic moment in the field of the nucleus. In Fig. 1 we show the dependence of the ratio $f_M(Z\alpha)/(Z\alpha)^2$ on $Z\alpha$. It turns out that the contribution of the off-diagonal transitions to this ratio is less than 3% and depends very weakly on $Z\alpha$. At the same time it is clear from Fig. 1 that the ratio in question increases rapidly in the neighborhood of the point $Z\alpha=\sqrt{3}/2$. The reason for this behavior is to be found in the presence of a pole at $\nu=1/2$ ($l=1$, $Z\alpha=\sqrt{3}/2$) in the expression for f_{21}. As is well known, the pole at $\nu=1/2$ is present in the matrix element of the hyperfine interaction operator, evaluated with respect to eigenfunctions of the Dirac equation in the Coulomb field (see, for example, Ref. 14). The origin of this

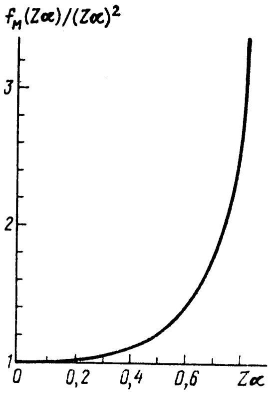

FIG. 1. Dependence of the ratio $f_M(Z\alpha)/(Z\alpha)^2$ on $Z\alpha$.

pole has to do with the singularity in the vector potential **A** at small distances. Indeed, the wave function behaves at small distances like $r^{\nu-1}$, and the matrix element of the interaction with the vector potential **A** is proportional to $(2\nu-1)^{-1}$. It is clear that near the point $Z\alpha=\sqrt{3}/2$ finite-nuclear-size effects must be taken more accurately into account. It is not hard to see that Eq. (12) is valid for $2\nu-1>1/\ln(\lambda_e/R)$. At the very point $Z\alpha=\sqrt{3}/2$ the calculation leads to the integral

$$\int_R\frac{dx}{x}\ln\frac{\lambda_e}{x}=\frac{1}{2}\ln^2\frac{\lambda_e}{R}.$$

We discuss now the question of the contribution to the induced moment of lowest order perturbation theory. The well-known connection between the "bare" potential and the potential induced by vacuum polarization at the one-loop level,[15] permits one to write the following expression for the induced vector potential in the momentum representation:

$$\mathbf{A}(\mathbf{k})=-i[\boldsymbol{\mu}\mathbf{k}]\frac{\mathcal{P}(-\mathbf{k}^2)}{(-\mathbf{k}^2)}\frac{4\pi}{\mathbf{k}^2}, \qquad (15)$$

where $\mathcal{P}(\mathbf{k}^2)$ is the polarization operator. Making use of formulas (114.2) and (114.5) from Ref. 15 in the coordinate representation we obtain

$$\mathbf{A}(\mathbf{r})=-[\boldsymbol{\mu}\nabla]\frac{\phi(r)}{Z|e|}. \qquad (16)$$

Here $\phi(r)$ is the Uehling potential.[16] At short distances $r\ll\lambda_e$ this potential differs from the Coulomb potential by an additional factor $2\alpha\cdot\ln(\lambda_e/r)/3\pi$, while at large distances it differs by the factor $(\alpha/4\pi^{\frac{1}{2}})\exp(-2mr)/(mr)^{3/2}$. In view of this last circumstance lowest order perturbation theory gives no contribution to the total induced moment, although it modifies the nuclear potential at short distances.

3. ELECTRIC QUADRUPOLE MOMENT

The quadrupole part of the electrostatic potential of a deformed nucleus

$$\varphi=Q_{ij}n_in_j/2r^3 \qquad (17)$$

(Q_{ij} is the nuclear quadrupole moment tensor, $\mathbf{n}=\mathbf{r}/r$) induces a corresponding moment in the electron vacuum:

$$\tilde{Q}_{ij} = \int d\mathbf{r}\, \rho(\mathbf{r}) r^2 (3n_i n_j - \delta_{ij}), \tag{18}$$

with $\tilde{Q}_{ij} = qQ_{ij}$. Similarly to the case of the magnetic dipole we obtain for the coefficient q the expression

$$q = -\frac{i\alpha}{10\pi} \int d\varepsilon \int \frac{d\mathbf{r}\, d\mathbf{r}'}{r^3} \times (r')^2 P_2(x) \operatorname{Tr} \gamma_0 G_c(\mathbf{r}, \mathbf{r}' | \varepsilon) \gamma_0 G_c(\mathbf{r}', \mathbf{r} | \varepsilon). \tag{19}$$

Here

$$G_c(\mathbf{r}, \mathbf{r}' | \varepsilon) = \langle \mathbf{r} | [\gamma_0 (\varepsilon + Z\alpha/r) - \boldsymbol{\gamma}\mathbf{p} - m]^{-1} | \mathbf{r}' \rangle$$

is the electron Green's function in the Coulomb field, $x = \mathbf{nn}'$, $P_2(x) = (3x^2 - 1)/2$ is a Legendre polynomial. It is now more convenient to use for G_c the representation in Eq. (19) of Ref. 12 (for $m = 0$ and $\varepsilon = iE$):

$$G_c(\mathbf{r}, \mathbf{r}' | \pm i|E|) = -\frac{1}{4\pi r r'} \int_0^\infty ds \exp[\pm 2iZ\alpha s - |E|(r + r') \operatorname{cth} s]$$
$$\times \sum_{l=1}^{\infty} \Big\{ \gamma_0 (1 - \boldsymbol{\gamma}\mathbf{n} \cdot \boldsymbol{\gamma}\mathbf{n}') \Big[\pm \frac{i}{2} y I_{2\nu}'(y) + Z\alpha I_{2\nu}(y) \operatorname{cth} s \Big] B(x)$$
$$\pm (1 + \boldsymbol{\gamma}\mathbf{n} \cdot \boldsymbol{\gamma}\mathbf{n}') \gamma_0 i l A(x) I_{2\nu}(y) - i \Big[\frac{|E|(r - r')}{2 \operatorname{sh}^2 s} (\boldsymbol{\gamma}, \mathbf{n} + \mathbf{n}') B(x)$$
$$\pm l A(x) (\boldsymbol{\gamma}, \mathbf{n} - \mathbf{n}') \operatorname{cth} s \Big] I_{2\nu}(y) \Big\}. \tag{20}$$

The notation is the same as in Eq. (6). Similarly to what was done for g we extract the large logarithm and represent the sought-for quantity in the form

$$q = \frac{\alpha}{30\pi} \ln\left(\frac{\lambda_e}{R}\right) f_Q(Z\alpha), \tag{21}$$

where the function f_Q reduces to the following after taking the trace and integrating over angles:

$$f_Q = 24 \int_0^\infty \frac{dr}{r^3} \int_0^\infty (r')^2 dr' \int_0^\infty ds \int_0^\infty ds' \sum_{\sigma=\pm 1} \exp[2i\sigma Z\alpha T - (r + r') \operatorname{cth} s + \operatorname{cth} s')]$$
$$\times \sum_{l,l'=1}^{\infty} \Big\{ - \Big[\frac{y}{2} I_{2\nu}'(y) - i\sigma Z\alpha I_{2\nu}(y) \operatorname{cth} s \Big] \cdot \begin{pmatrix} s \to s' \\ l \to l' \end{pmatrix}$$
$$+ \Big[\frac{(r - r')^2}{4 \operatorname{sh}^2 s \operatorname{sh}^2 s'} - ll'(-1)^{l+l'} \times (1 - \operatorname{cth} s \operatorname{cth} s') \Big] I_{2\nu}(y) I_{2\nu'}(y') \Big\} \Delta_{ll'}, \tag{22}$$

with

$$y = 2(rr')^{1/2}/\operatorname{sh} s, \quad y' = 2(rr')^{1/2}/\operatorname{sh} s', \quad T = s + s',$$
$$\Delta_{ll'} = \frac{l^3 - l}{4l^2 - 1} \delta_{l,l'} + \frac{6l(l+1)}{(4l^2 - 1)(2l + 3)} \delta_{l',l+1} + \frac{3l(l+1)(l+2)}{(2l+1)(2l+3)} \delta_{l',l+2}. \tag{23}$$

Thus, in the case under consideration we have along with "diagonal" ($l = l'$) also "off-diagonal" transitions of two types: with $l' = l + 1$ and $l' = l + 2$. Further integration leads to the result

$$f_Q(Z\alpha) = 12 \int_0^1 \frac{dx}{x^4} (1 - x^2) \left(\frac{4}{x} - 1 \right) \sum_{l,l'=1}^{\infty} [\Phi(x, Z\alpha) - \Phi(x, 0)] \Delta_{ll'},$$

$$\Phi(x, Z\alpha) = [1 + {}^1/_3 ll'(-1)^{l+l'}] B^2_{2\nu,2iZ\alpha}(x) B^2_{2\nu',2iZ\alpha}(x)$$
$$+ {}^1/_6 ll'(-1)^{l+l'} \operatorname{Re}[B^1_{2\nu,1+2iZ\alpha}(x) B^3_{2\nu',1+2iZ\alpha}(x) + B^3_{2\nu,1+2iZ\alpha}(x) B^1_{2\nu',1+2iZ\alpha}(x)], \tag{24}$$

where

$$B^\beta_{\nu,\mu}(x) = 2^{\beta-2} x^{(\nu+\beta)/2} \frac{\Gamma((\nu+\beta+\mu)/2)\Gamma((\nu+\beta-\mu)/2)}{\Gamma(\nu+1)} \times F\left(\frac{\nu+\beta+\mu}{2}, \frac{\nu+\beta-\mu}{2}; \nu+1; x \right)$$

A plot of the function $f_Q(Z\alpha)/(Z\alpha)^2$ is shown in Fig. 2. In this case strong cancellation occurs between contributions corresponding to the transitions with $\Delta l = 0$ and $\Delta l = 2$—their sum is less than 3% of f_Q for all values of the parameter $Z\alpha$. Similarly to the case of the magnetic dipole moment rapid growth is observed in the induced electric quadrupole moment near the critical value of the parameter $Z\alpha$, which now equals $\sqrt{15}/4$. The main source of this growth is the pole in the term in the partial-wave expansion

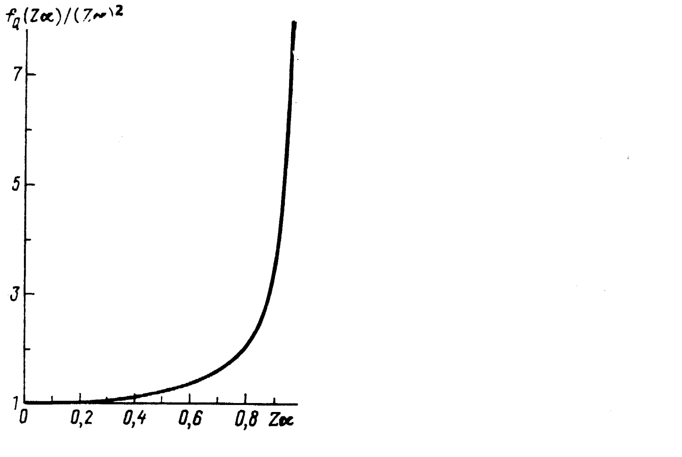

FIG. 2. Dependence of the ratio $f_Q(Z\alpha)/(Z\alpha)^2$ on $Z\alpha$.

with $l=1, l'=2$ for $\nu+\nu'=2$. Upon accurate inclusion of finite-nuclear-size effects this pole singularity is transformed into an additional logarithmic factor $1/2 \ln(\lambda_e/R)$, just as in the case of the magnetic moment.

Evidently the appearance of the logarithm $\ln(\lambda_e/R)$ in the expressions for the induced moments is a general feature, which can be easily understood from dimensional considerations. Indeed, we are evaluating a dimensionless quantity (g or q) as an integral over a product of homogeneous functions (potentials and massless propagators). Such an integral is necessarily logarithmically divergent. In view of the previously mentioned arguments this divergence is cut off at R in the ultraviolet region and at λ_e in the infrared region.

4. INDUCED POTENTIALS

To calculate atomic energy shifts we need to know not only the total induced moments but also the corresponding charge and current distributions and the potentials due to them. For distances in the region $R \ll r \ll \lambda_e$ it easily follows from simple dimensional considerations that

$$\rho(\mathbf{r}) = \frac{\alpha}{48\pi^2} f_Q(Z\alpha) \frac{Q_{ij}n_in_j}{r^5}, \quad \mathbf{J}(\mathbf{r}) = \frac{\alpha}{2\pi^2} f_M(Z\alpha) \frac{[\boldsymbol{\mu}\mathbf{r}]}{r^5}. \tag{25}$$

Precisely such a distribution of charge and current densities gives the logarithmic contribution to the total induced moments found above. At distances considerably in excess of λ_e the induced densities are found most easily with the help of the Euler–Heisenberg Lagrangian (see Ref. 15). We have

$$\rho(\mathbf{r}) = \frac{4\alpha(Z\alpha)^2}{15\pi^2} \lambda_e^4 \frac{Q_{ij}n_in_j}{r^9}, \quad \mathbf{J}(\mathbf{r}) = \frac{11\alpha(Z\alpha)^2}{90\pi^2} \lambda_e^4 \frac{[\boldsymbol{\mu}\mathbf{r}]}{r^9}. \tag{26}$$

The electrostatic and vector potentials due to the distributions (25) and (26) are (to within logarithms)

$$\begin{aligned}
\varphi(\mathbf{r}) &= \frac{\alpha}{30\pi} f_Q \frac{Q_{ij}n_in_j}{2r^3} \ln\frac{r}{R}, \\
\mathbf{A}(\mathbf{r}) &= \frac{2\alpha}{3\pi} f_M \frac{[\boldsymbol{\mu}\mathbf{r}]}{r^3} \ln\frac{r}{R}, \quad R \ll r \ll \lambda_e, \\
\varphi(\mathbf{r}) &= \frac{\alpha}{30\pi} f_Q \frac{Q_{ij}n_in_j}{2r^3} \ln\frac{\lambda_e}{R}, \\
\mathbf{A}(\mathbf{r}) &= \frac{2\alpha}{3\pi} f_M \frac{[\boldsymbol{\mu}\mathbf{r}]}{r^3} \ln\frac{\lambda_e}{R}, \quad r \gg \lambda_e.
\end{aligned} \tag{27}$$

The zeroth-order perturbation-theory contribution to φ is found by means similar to those used for the induced vector potential [see (15) and (16)]:

$$\varphi(\mathbf{r}) = \frac{1}{6} Q_{ij} \nabla_i \nabla_j \frac{\phi(r)}{Z|e|}. \tag{28}$$

5. FEASIBILITY OF EXPERIMENTAL OBSERVATION OF THE EFFECT

The question of feasibility of experimental observation of the phenomenon under discussion should be given a separate study. Nonetheless we should like to make several comments on the subject. It follows from our analysis that the main contribution to the induced magnetic moment is from distances between the nuclear radius and the electron Compton wavelength. It therefore seems to us that the best candidate for experimental observation of the phenomenon is a mu-mesic atom. The characteristic size of the muon wave function for low-lying levels in such atoms is approximately $1/Z\alpha m_\mu \ll \lambda_e$. Measuring the ratio of hyperfine intervals for strongly and weakly excited levels we can exclude the magnitude of the "bare" magnetic moment of the nucleus μ, whose calculation contains large theoretical uncertainties, and compare the ratio obtained that way with the results of our calculations. We consider an example, which permits judging the size of the effect and the possibility of its experimental observation. Using the standard method for the calculation of the hyperfine splitting (see, for example, Ref. 17) and Eqs. (16) and (25) we find for states with quantum numbers $n-l\sim 1$, $n \gg 1$

$$\Delta E = \frac{\mu\mu_0 F(F+1)}{ij(j+1)(l+^1/_2)} \left(\frac{m_\mu Z\alpha}{n}\right)^3 \left[1 + \frac{2\alpha}{3\pi} f_M(Z\alpha) \ln\frac{n^2}{m_\mu Z\alpha R} + \frac{8\alpha}{3\pi} \ln\frac{m_\mu Z\alpha}{m_e n^2}\right]. \tag{29}$$

Here μ and μ_0 are respectively the magnetic moments of the nucleus and the muon, $F=i+j$, $j=l\pm\frac{1}{2}$ is the total angular momentum of the muon, i is the spin of the nucleon. We note that precisely such states are of greatest interest from the experimental point of view.[18] In the experiment of Ref. 18 hyperfine splitting in mu-mesic atoms with nuclear charges $63<Z<77$ was measured and accuracy of $<1\%$ was achieved. As is easily estimated with the help of Eq. (29), the correction to the ratio of hyperfine intervals due to the effect discussed here could reach 0.5%. We note that the ratio has contributions also from nonlogarithmic corrections and such effects as the influence of vacuum polarization on the muon wave function, Coulomb corrections to the magnetic form factor of the muon,[1)] and so forth. We anticipate, however, that the effect discussed here dominates numerically since it contains a large logarithm, and that the precision of the predictions for the hyperfine intervals will be sufficient for comparison with experiment. Such an experiment would serve as an essentially new test of the predictions of nonlinear effects of quantum electrodynamics in a strong Coulomb field.

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[1)] We have included in (29) the logarithmic part of this effect.

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Antiferromagnetic axion detector

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Antiferromagnets with easy-plane anisotropy are proposed as axion detectors. It is shown that the response of the detector is proportional to the ratio of the Dzyaloshinskiĭ field to the external magnetic field; this makes it possible to strengthen the bounds on the axion-electron interaction constant.

1. INTRODUCTION

The discovery of long-range action transferred by massless or almost massless pseudoscalar particles (arions, axions) would be of great significance both for constructing an adequate cosmological model and for studying physics at very small distances. The experimental status of the search for such "exotic" long-range actions at the beginning of 1989 is reviewed in Ref. 1.

The interaction of an axion field a with fermions ψ in matter is described by the Lagrangian

$$\mathscr{L}_{int} = -qa\bar{\psi}i\gamma^5\psi, \qquad (1.1)$$

where q is the axion charge of the fermion.

We shall study the interaction of the field a with magnetically ordered dielectrics. At sufficiently low energies an axion can excite only the spin degrees of freedom of the electrons localized at the sites of the crystal lattice. In this limit the effect Lagrangian, following from Eq. (1.1), coupling the axion field with the medium is equal to

$$\mathscr{L}_{ma} = \varkappa \nabla a \mathbf{m}(\mathbf{r}). \qquad (1.2)$$

Here $\varkappa = \mu_a/\mu_B$ is the ratio of the axionic magneton of the electron to the Bohr magneton, $\mu_a = q/2m_e$, m_e is the electron mass, and $\mathbf{m}(\mathbf{r})$ is the magnetization density of the medium. [The relation $m_e\bar{\psi}i\gamma^5\psi = \partial_\mu \bar{\psi}\gamma^5\gamma^\mu\psi +$ the contribution of the axial anomaly, which we ignored and which leads to direct axion-photon conversion in an external field[2,3] (see also Ref. 1), was used in the derivation of the Lagrangian (1.2).]

This "quasimagnetic" character of the field ∇a implies, in particular, that the static axion field generated by a ferromagnet induces a constant magnetization in paramagnetic samples separated from this ferromagnet by a superconducting screen.[4] The experiment of Ref. 4, performed according to this scheme, gave a limit on the constant $\varkappa$:

$$\varkappa < 2\cdot 10^{-7}. \qquad (1.3)$$

In Refs. 5 and 6, on the other hand, an experiment on the generation and detection of a dynamic axion field was studied. This experiment employed the existence of a point of intersection of the dispersion relations of axions and spin waves in a ferromagnet. Detailed calculations of the coefficients of coherent axion-magnon and double magnon-axion-magnon conversion in a ferromagnetic medium were presented in Ref. 6. (In Ref. 7, which appeared at the same time as Ref. 5, it is suggested that galactic axions with wavelength 10–100 m be detected based on their resonance interaction with the homogeneous (ferromagnetic precession.) Magnons excited in the ferromagnetic detector[5,6] are detected based on the electromagnetic oscillations which are coupled with them. Ultimately it is precisely the number of these coupled photons, which is proportional to $\varkappa^4$, that is recorded.

The number of excited magnons can be determined independently by measuring the variation of the macroscopic magnetic field (which is proportional to the measurement of the macroscopic magnetization) with the help of a SQUID magnetometer. It is known, however, that antiferromagnets (more accurately, weak ferromagnets) with anisotropy of the "easy plane" type and large Dzyaloshinskiĭ field H_D are more suitable for such measurements (see, for example, Ref. 8). In this case the decrease in the weak ferromagnetic moment accompanying excitation of one (quasi) goldstone magnon can exceed μ_B by more than an order of magnitude (this phenomenon was used in Ref. 8).

In this paper we shall study the possibility of such an antiferromagnetic detector of axions. We shall show that the response to the axion signal measured by a SQUID contains the enhancement factor H_D/H_0, where H_0 is the external magnetic field. An experiment of the type described in Ref. 5 on the generation and detection of axions can also strengthen the limit on the constant $\varkappa$ by two orders of magnitude.

2. AXION-MAGNON CONVERSION IN AN ANTIFERROMAGNETIC MEDIUM

Let $\mathbf{M}_1$ and $\mathbf{M}_2$ be the magnetizations of the sublattices of the antiferromagnet,

$$\mathbf{M}_1 = \mathbf{M}_0 + \mathbf{m}_1, \quad \mathbf{M}_2 = -\mathbf{M}_0 + \mathbf{m}_2,$$

where $2\mathbf{M}_0$ is the (almost) equilibrium value of the antiferromagnetic moment and $\mathbf{m}_{1,2}$ are the dynamic variables of the medium. Let the z-axis be oriented along $\mathbf{M}_0$ and $|\mathbf{m}_{1,2}| \ll |\mathbf{M}_0|$. Then the transformation to canonical variables c_1, c_2 (see, for example, Ref. 9), neglecting higher-order nonlinear terms, is made by means of the following substitution:

$$\begin{gathered} m_{1x} + im_{1y} = c_1(2\omega_m)^{1/2}\left(1 - \frac{g}{4M_0}c_1^*c_1\right), \\ m_{2x} - im_{2y} = c_2(2\omega_m)^{1/2}\left(1 - \frac{g}{4M_0}c_2^*c_2\right), \\ \omega_m = gM_0, \quad g = 2\mu_B/\hbar \approx 2\pi\cdot 2{,}8 \text{ MHz/Oe}, \\ m_{1z} = -gc_1^*c_1, \quad m_{2z} = gc_2^*c_2 \end{gathered} \qquad (2.1)$$

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