Fractional regenerations of wave packets in the course of long-term evolution of highly excited quantum systems

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The long-term evolution of wave packets consisting of highly excited states of quantum systems executing regular periodic motion in the classical limit is considered. It is shown that after a certain stage of the dynamics, in accordance with the correspondence principle, the wave function of the packet evolves according to a universal scenario (independent of the actual nature of the system) involving the formation of symmetric structures which replace each other in a definite sequence. Each structure is a set of correlated packets, i.e., fractions which repeat the shape of the initial packet and move along a classical trajectory with a time shift (equal to a fractional part of the classical period) with respect to each other. The structures are coherent superpositions of macroscopically distinguishable quantum states resembling, for example, the generalized coherent states in quantum optics. The manifestation of the effect in the radiation from Rydberg atoms is discussed in connection with the experimental feasibility (which has only recently appeared) of the generation and detection of wave packets of highly excited atomic states by short laser pulses.

The question of the relation between the quantum and classical descriptions of the dynamics of physical systems, already posed at the dawn of quantum mechanics, continues to attract the attention of researchers. Recently, this question has been intensively studied in connection with the problem of the semiclassical quantization of highly excited multidimensional quantum systems,[1] the analysis of the quantum dynamics of systems which are chaotic in the classical limit,[2,3] and the recent emergence of the experimental possibility of creating electron wave packets from the Rydberg states of highly excited atoms by short laser pulses.[4–7] The question of the transition from the quantum description to the classical one as $\hbar \to 0$ requires very careful consideration in the case of systems executing bounded motion and whose spectra are discrete. As $\hbar \to 0$ (in the region of energies E corresponding to large quantum numbers n), the energy spectrum of systems executing regular periodic motion in the classical limit is quasi-equidistant, and the frequency distance $\omega_{n+1,n}$ between neighboring levels is equal to the inverse period of the classical motion $T_{\rm cl}$: $\omega_{n+1,n} \approx \omega_{\rm cl} = 2\pi/T_{\rm cl}$.[8] However, it is well known[9,10] that a large quantum number of a stationary state of the system does not in general imply its classicity. The transition to the classical description requires a consideration of the evolution of the wave packets (the superposition of quantum stationary states with different n). To ensure that the packet remain localized in space, the number Δn of states which form the packet must be quite large ($\Delta n \to \infty$). Note that packets which are composed of a moderate number of states exhibit nonclassical behavior even at large n.[11]

For times much shorter than the period of oscillations, the discreteness of the spectrum is unimportant and the packet moves along a classical trajectory, generally speaking, spreading as it propagates. However, such spreading is not irreversible as in the case of free motion, and the packet completely recovers its original shape after one period as a result of the equidistant character of the energy spectrum of the states of which it is composed. In this sense the dynamics of the packet can be interpreted in terms of quantum beats between a large number of states.[12] Such a correspondence between the quantum description and the classical description is maintained for as long as desired only in the case of a strictly equidistant spectrum, which obtains, for example, for a harmonic oscillator.

In general, during the long-term evolution of the packet, in the region of highly excited states, the levels are inevitably nonequidistant is a result of the dependence of $\omega_{\rm cl}$ on energy:

$$\omega_{n+1,n}-\omega_{n,n-1}\approx\hbar\omega_{cl}\frac{\partial\omega_{cl}}{\partial E}. \qquad (1)$$

The quantum dephasing of the contributions of the stationary states that arises as a result of this and leads to the disintegration of the packet after many periods limits the duration of the "classical" evolution of the packet (see, e.g., Refs. 1 and 3):

$$t\ll T_{rev},\qquad T_{rev}=2T_{cl}\left(\hbar\left|\frac{\partial\omega_{cl}}{\partial E}\right|\right)^{-1}. \qquad (2)$$

In a number of numerical investigations of the long-term evolution of Rydberg wave packets,[4,5] and of the evolution of coherent states in model nonlinear systems,[13–16] it has been discovered that the indicated dephasing is not completely irreversible and at $t \approx T_{\rm rev}$ the wave packet recovers its initial shape and again evolves according to "classical" laws. In Refs. 13–16 it was shown that also for $t < T_{\rm rev}$ complete localization in phase space does not always take place, and some regular, strongly localized structures have been discovered that occur already at intermediate times.

In the present paper we will show that the appearance of such structures in various nonlinear quantum systems is not random. It will be shown that the long-term evolution of quantum wave packets in systems executing regular periodic motion in the classical limit develops according to a single universal scenario, reversing itself after the dynamic stage of the packet according to the correspondence principle. In the course of this scenario the wave function of the system undergoes a determinate sequence of changes which correspond to the onset of regularly organized structures–concen-

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trations of the probability density with high degrees of localization. Since (as will be shown below) the shape of each such concentration is uniquely determined by the shape of the initial wave packet, we call this phenomenon "fractional regeneration." The structures discovered in Refs. 13–16 in concrete systems are particular episodes in the course of such a general scenario.

Let us consider a wave packet which consists of highly excited discrete states of a quantum system executing bounded motion in the energy region $E \approx E_{\bar{n}}$ $(\bar{n} \gg 1)$, in which the classical dynamics corresponds to regular periodic motion:

$$\psi(\mathbf{r},t)=\sum_n c_n\varphi_n(\mathbf{r})\exp\left(-i\frac{E_n}{\hbar}t\right); \qquad (3)$$

here $\varphi_n(\mathbf{r})$ are the wave functions corresponding to the stationary states with energies E_n and the quantities c_n are constants. We assume that at the initial instant of time $t=0$ the wave packet is strongly localized in space (its spatial dimension Δx is many times smaller than the characteristic dimension L of the classical orbit corresponding to $E\approx E_{\bar{n}}$). It follows from the uncertainty relation that the energetic width of the packet ΔE is of the order of

$$\Delta E \propto v\Delta p \sim \frac{\hbar\omega_{cl}L}{\Delta x}, \quad \Delta x \ll L, \qquad (4)$$

where v and Δp are characteristic values of the velocity and of the momentum uncertainty. This means that the $|c_n|^2$ distribution, which has a sharp maximum at $n\approx\bar{n}$, has a width $\Delta n \propto L/\Delta x$. For example, for systems which are close to a harmonic oscillator with mass M and frequency w, we have

$$L\propto (E/M\omega^2)^{1/2}, \quad \Delta n \propto \bar{n}^{1/2}\,[(\hbar/M\omega)^{1/2}/\Delta x].$$

For a coherent state of the oscillator $\Delta x \propto (\hbar/M\omega)^{1/2}$ and $\Delta n \propto n^{-1/2} \ll \bar{n}$.

Thus the condition $L/\Delta x \gg 1$, necessary for the transition to the classical limit, implies that $\Delta n \gg 1$. The values of E_n for n close to $\bar{n}$ (in the interval Δn) can be written in the form

$$E_n=E_{\bar{n}}+\hbar\omega_{cl}(n-\bar{n})+\frac{1}{2}\hbar^2\omega_{cl}\frac{\partial\omega_{cl}}{\partial E_{\bar{n}}}(n-\bar{n})^2+\dots \qquad (5)$$

In order that the packet evolution correspond to the motion of a classical particle for times $t \gg T_{cl}=2\pi/\omega_{cl}$ (i.e., in order that the correspondence principle be fulfilled), it is necessary to satisfy the inequality

$$\hbar\left|\frac{\partial\omega_{cl}}{\partial E_{\bar{n}}}\right|(\Delta n)^2 \ll 1, \qquad (6)$$

which imposes an upper bound on the energetic width of the packet. Here a necessary condition on the dynamics, according to the correspondence principle, is

$$\hbar\left|\frac{\partial\omega_{cl}}{\partial E_{\bar{n}}}\right| \ll 1. \qquad (7)$$

As has already been mentioned, with the passage of time the dephasing due to the terms in Eq. (5) which are quadratic in $(n-\bar{n})$ begins to play a major role. If

$$\frac{t}{T_{cl}} \ll \frac{1}{\hbar^2}\left|\frac{\partial}{\partial E}\left(\omega_{cl}\frac{\partial\omega_{cl}}{\partial E}\right)\right|^{-1}_{E=E_{\bar{n}}}, \qquad (8)$$

the influence of the subsequent terms in expansion (5) can be neglected. Below we will concentrate our attention on the long-term evolution in the indicated time interval.

The indicated dephasing for a wave packet with prescribed Δn leads to a distortion of its shape at

$$t\sim T_{rev}/(\Delta n)^2, \qquad (9)$$

i.e., when the additional phase shift between the different energetic components in Eq. (3) within the width of the packet is of the order of unity. However, at $T=T_{rev}$ the additional phases due to the terms in Eq. (5) which are nonlinear in $(n-\bar{n})$ are exact multiples of 2π, which means complete recovery of the shape of the initial packet. For $t \gtrsim T_{rev}$ the classical evolution of the packet recommences anew. This circumstance, which was mentioned in Ref. 4, was called the regeneration of the wave packet (see also Ref. 17).

Let us consider the evolution of the packet during times $0<t<T_{rev}$. We rewrite Eq. (3) in the form

$$\psi(\mathbf{r},t)=\sum_{k=-\infty}^{+\infty} c_k\varphi_k(\mathbf{r})\exp\left[-2\pi i\left(k\frac{t}{T_{cl}}+k^2\frac{t}{T_{rev}}\right)\right], \quad k=n-\bar{n}. \qquad (10)$$

Here and below, the energy is reckoned from $E_{\bar{n}}$. Let us consider the sum (10) at $t\approx mT_{rev}/n$, where m and n are mutually prime integers. The additional phase shifts due to the terms $\propto k^2$ in Eq. (10) are equal to $2\pi\theta_k$, where

$$\theta_k=\left\{\frac{m}{n}k^2\right\}, \qquad (11)$$

where the braces denote the fractional part of the argument. It is easy to see that the quantities θ_k form periodic sequences. Indeed,

$$\theta_{k+n}=\left\{\frac{m}{n}k^2+2km+mn\right\}=\left\{\frac{m}{n}k^2\right\}=\theta_k, \qquad (12)$$

i.e., the sequence θ_k is obviously periodic in k with period n. Let us see if there exists in this sequence a period l which is less than n. Let $\theta_k=\theta_{k+l}$ for arbitrary k. Then

$$\left\{\frac{m}{n}k^2\right\}=\left\{\frac{m}{n}(k+l)^2\right\}. \qquad (13)$$

Conditions which are necessary and sufficient for the satisfaction of Eq. (13) are

$$\{2ml/n\}=0, \qquad (14)$$

$$\{ml^2/n\}=0. \qquad (15)$$

For n odd, conditions (14) and (15) are satisfied for $l=n$. For even n (and, correspondingly, odd m) the first condition can be satisfied also for $l=n/2$. Substituting the value $l=n/2$ into Eq. (15), we obtain the condition

$$\{mn/4\}=0.$$

Thus, $l=n/2$ for n divisible by 4, and $l=n$ in all remaining cases.

Thus, near $t=mT_{rev}/n$ the terms in Eq. (10) that are quadratic in k lead to additional (in comparison with the situation at time $t=0$) phase factors which are l-periodic in k. Note that phase factors with the same periodicity in k arise in the spectral expansion of packets which execute motion according to the correspondence principle (neglecting nonlinear terms) with a time shift with respect to the initial packet which is a multiple of T_{cl}/l. This suggests that close to $t = mT_{\rm rev}/n$ the wave function of the system can be represented in the form

$$\psi(\mathbf{r},t)=\sum_{s=0}^{l-1} a_s\psi_{cl}\left(\mathbf{r},t+\frac{s}{l}T_{cl}\right), \quad (16)$$

where

$$\psi_{cl}(r,l)=\sum_{k=-\infty}^{+\infty} c_k\varphi_k(\mathbf{r})\exp\left(-2\pi ik\frac{t}{T_{cl}}\right) \quad (17)$$

describes the evolving packet in a "classical" way, and a_s are constants. Expression (16) follows directly from the possibility of representing the l-periodic sequence $\exp(-2\pi i\theta_k)$ in the form of an expansion in l fundamental sequences

$$\exp\left(-2\pi i\frac{S}{l}k\right),\ s=0,1,\ldots,l-1,$$

which have the same periodicity:

$$\exp(-2\pi i\theta_k)=\sum_{s=0}^{l-1} a_s\exp\left(-2\pi i\frac{s}{l}k\right). \quad (18)$$

Multiplying expansion (18) by $\exp(2\pi iqk/l)$, where q is an integer, and summing both parts of the equality over all k, we find

$$a_q=\frac{1}{l}\sum_{k=0}^{l-1}\exp\left(-2\pi i\theta_k+2\pi i\frac{kq}{l}\right). \quad (19)$$

Using the explicit form of θ_k and the l-periodicity of these quantities and shifting the summation index in Eq. (19) by 1, we obtain the relation

$$a_q=\exp\left(2\pi i\frac{m}{n}-2\pi i\frac{q}{l}\right)a_{q'}, \quad (20)$$

$$q'=(q+2ml/n)\ (\mathrm{mod}\ l).$$

If n is odd, then $l=n$ and an n-fold application of relation (20) shows that all of the quantities a_q are equal in modulus. For n divisible by 4, the quantity $l=n/2m$ is odd and relation (20) again interrelates quantities a_q, likewise causing them to be equal in modulus. For n even, but not divisible by 4, relation (20) relates differently the quantities a_q with even and odd q. All the coefficients with even q turn out in this case to be zero. Indeed, let us consider the quantity a_0, which belongs to this group. Using expression (19) with $q=0$, we shift the summation index by the whole number $n/2$:

$$a_0=\frac{1}{l}\sum_{k=0}^{l-1}\exp\left(-2\pi i\frac{m}{n}k^2\right)$$

$$=\frac{1}{l}\sum_{k'=0}^{l-1}\exp\left[-2\pi i\frac{m}{n}\left(k'+\frac{n}{2}\right)^2\right]$$

$$=\frac{1}{l}\exp\left(-i\pi\frac{mn}{2}\right)\sum_{k'=0}^{l-1}\exp\left(-2\pi i\frac{m}{n}k'^2\right)=-a_0. \quad (21)$$

Thus, a_0 equals zero, as do all the remaining a_q with even q. We find the modulus of the r nonzero coefficients a_q ($r=n/2$ for even n and $r=n$ for odd n) with the help of the relations

$$r|a_q|^2=\sum_{q=0}^{l-1}|a_q|^2$$

$$=\frac{1}{l^2}\sum_{qkk'}\exp\left[-2\pi i(\theta_k-\theta_{k'})+2\pi i\frac{q}{l}(k-k')\right]$$

$$=\frac{1}{l^2}\sum_{kk'}\exp[-2\pi i(\theta_k-\theta_{k'})]l\delta_{kk'}=1,\quad |a_q|=\frac{1}{r^{1/2}}. \quad (22)$$

Thus, for m/n an arbitrary irreducible rational number close to $t=mT_{\rm rev}/n$ the initial packet divides into r spatially separated packet-fractions undergoing a periodic evolution according to the correspondence principle and time-shifted by the r-th part of the classical period with respect to each other ($r=n/2$ for even n and $r=n$ for odd n). We call such a structure a fractional regeneration of the initial packet of order m/n. Naturally, it will be well pronounced if these fractions do not overlap, i.e., for $r<L/\Delta x\sim\Delta n$. The better the semiclassical conditions are satisfied, the higher the order of splitting that it will be possible to observe.

Let us consider some actual structures.

1. Let $t\approx(1/2)T_{\rm rev}$ (here $n=1$, $l=2$, $r=1$, and $m=1$), $a_0=1$, and $a_1=1$, i.e.,

$$\psi(\mathbf{r},t)=\psi_{cl}(\mathbf{r},t+{}^1/_2T_{cl}), \quad (23)$$

which is the initial packet shifted in time by half the classical period. Note that precisely this regeneration (of order 1/2 according to our classification) was in fact discovered in Refs. 4 and 5.

2. Let $t\approx(1/4)T_{\rm rev}$ (here $l=r=n/2$ and $m=1$). Then

$$\psi(\mathbf{r},t)=\frac{1}{2^{1/2}}\left[e^{-i\pi/4}\psi_{cl}(\mathbf{r},t)+e^{i\pi/4}\psi_{cl}\left(\mathbf{r},t+\frac{1}{2}T_{cl}\right)\right]. \quad (24)$$

Expression (24) describes an essentially nonclassical object that is a superposition of two correlated, localized packets, macroscopically separated by lengths of the order of the dimension of the classical orbit. In the case in which $\psi_{cl}(\mathbf{r},t)$ is a coherent state of the oscillator, such objects (which have acquired the name of generalized coherent states) have been studied in Ref. 18. The question of the generation of such states in nonlinear optical systems, with the goal of observing macroscopic quantum optical effects, was considered in Ref. 15.

Note that an analogous structure arises for $t=(3/4)T_{\rm rev}$ ($l=r=n/2=2$, $m=3$).

3. Let $t\approx(1/3)T_{\rm rev}$ (here $l=r=3$ and $m=1$). Then

$$\psi(\mathbf{r},t)={}^1/_3(1+2e^{-2\pi i/3})\{\psi_{cl}(\mathbf{r},t)$$
$$+e^{2\pi i/3}[\psi_{cl}(\mathbf{r},t+{}^1/_3T_{cl})+\psi_{cl}(\mathbf{r},t+{}^2/_3T_{cl})]\}. \quad (25)$$

Structures of such kind, consisting of three packets, also arise for $t/T_{\rm rev}\approx1/6$, 2/3, and 5/6.

4. Let $t\approx(1/8)T_{\rm rev}$ (here $l=r=n/2=4$ and $m=1$). Then

$$\psi(\mathbf{r},t)={}^1/_2e^{-i\pi/4}\{[\psi_{cl}(\mathbf{r},t)-\psi_{cl}(\mathbf{r},t+{}^1/_2T_{cl})]$$
$$+e^{i\pi/4}[\psi_{cl}(\mathbf{r},t+{}^1/_4T_{cl})+\psi_{cl}(\mathbf{r},t+{}^3/_4T_{cl})]\}. \quad (26)$$

Such structures, consisting of four packets, were discovered

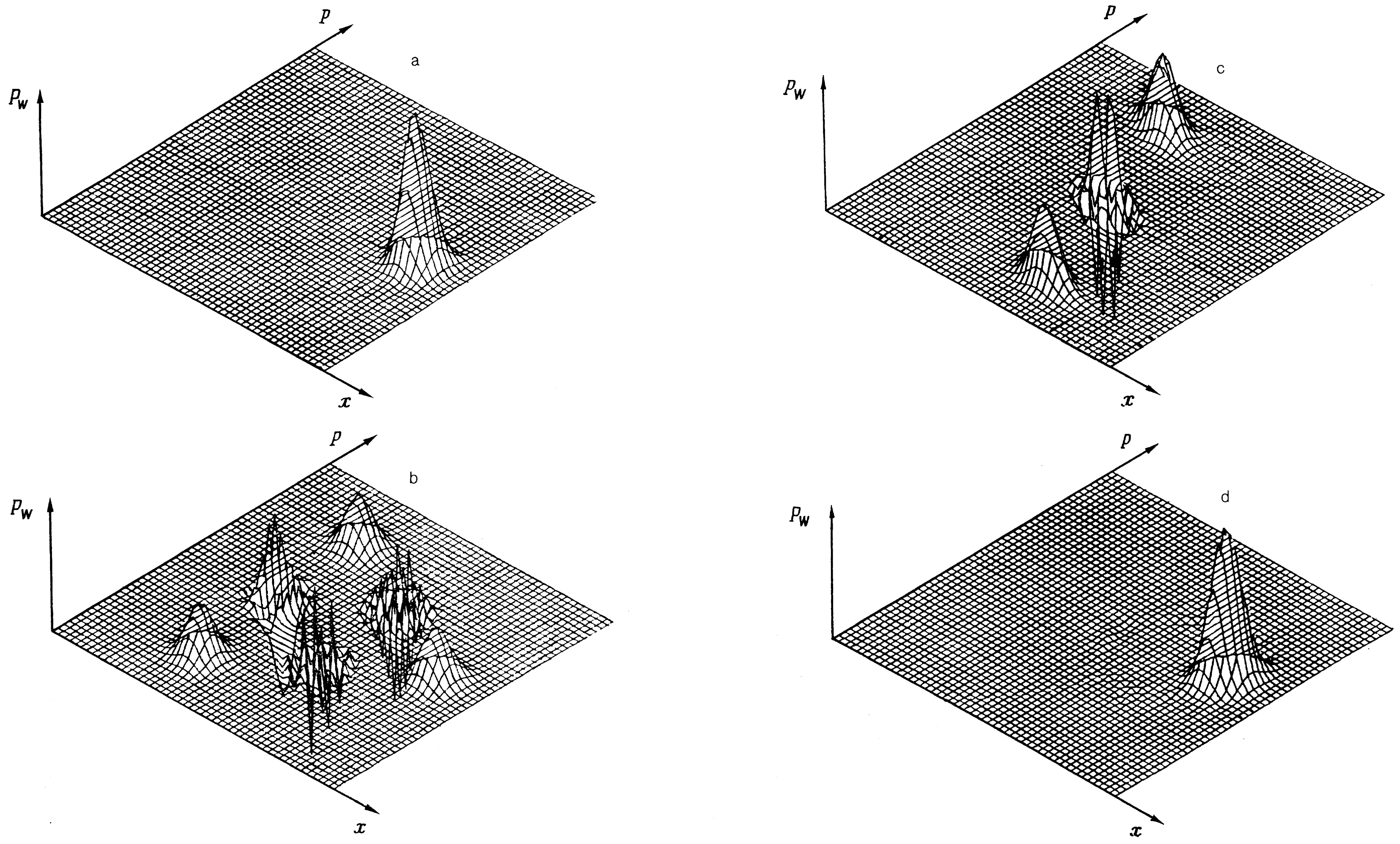

FIG. 1. Wigner distribution function of harmonic oscillator: $\varepsilon = 0\,001$, $T_{rev} = 667 \cdot 2\pi/\omega$, $\alpha = 5$; a—$\omega t/2\pi = 0$, b—$\omega t/2\pi = 111$, c—$\omega t/2\pi = 166$, d—$\omega t/2\pi = 333$.

in the course of the numerical investigation of nonlinear model problems in Ref. 13.

Let us illustrate the above-described general regularities by examples of concrete physical systems. We will first consider an anharmonic oscillator with potential energy of the form

$$V(x)={}^1/_2 M\omega^2x^2+\Gamma x^4. \qquad (27)$$

The eigenstates and the energy spectrum of such an oscillator have been studied in detail. We will limit ourselves to the case of weak anharmonicity ($\varepsilon\to 0$), in which case the following expression is correct[19]

$$E_n\approx\hbar\omega[(n+{}^1/_2)+{}^3/_2\varepsilon(n+{}^1/_2)^2+{}^3/_8\varepsilon],$$
$$\varepsilon=\Gamma\frac{\hbar}{M^2\omega^3}. \qquad (28)$$

The most lucid representation of the evolution of the wave packets under semiclassical conditions can be had with the help of the Wigner distribution function, which is defined in the phase space (x,p) by the expression[20]

$$P_W(x,p,t)=\frac{1}{\pi\hbar}\int_{-\infty}^{+\infty}dy\,\psi^*(x+y,t)\,\psi(x-y,t)\exp(2i\pi py/\hbar). \qquad (29)$$

Specifying the initial wave packet in the form of a coherent state of an unperturbed harmonic oscillator[10]

$$\psi(x,0)=\langle x|\alpha\rangle=\exp(-{}^1/_2|\alpha|^2)\sum_n\frac{\alpha^n}{(n!)^{1/2}}\langle x|n\rangle$$
$$=\left(\frac{\beta^2}{\pi}\right)^{1/4}\exp\left[-\frac{\beta^2}{2}\left(x-2^{1/2}\frac{\alpha}{\beta}\right)^2\right],\qquad \beta^{-1}=(\hbar/M\omega)^{1/2}, \qquad (30)$$

we obtain with the help of Eqs. (28)–(30), to leading order in ε

$$P_W(x,p,t)=\frac{2}{\pi}\sum_{n=1,m<n}(-1)^m e^{-|\alpha|^2}\frac{(\alpha^*)^n\alpha^m}{n!}2^{n-m}\cos[(n-m)\varphi$$
$$+(E_n-E_m)t/\hbar](E/\hbar\omega)^{(n-m)/2}e^{-2E/\hbar\omega}L_m^{n-m}(4E/\hbar\omega)$$
$$+\frac{1}{\pi}\sum_{n=0}^{\infty}e^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!}e^{-2E/\hbar\omega}L_n^0(4E/\hbar\omega),$$
$$\frac{\beta}{2^{1/2}}(x+ip/M\omega)=(E/\hbar\omega)^{1/2}e^{i\varphi}. \qquad (31)$$

Here $L_n^m(z)$ are associated Laguerre polynomials.

Figure 1 shows the results of numerical calculations which depict the distribution P_W at the initial time, and also at times corresponding to the fractional regeneration of various orders. The smooth humps of the distribution function correspond to the split initial packet, while the localized strongly oscillating spikes on the phase plane arise as a result of the interference between the various packet-fractions and do not have classical analogs.

Electron wave packets of highly excited Rydberg states, which arise in the excitation of the respective atoms by short laser pulses, can serve as an interesting object of study in the observation of the phenomenon of fractional regenerations. This object has been studied recently both experimentally and theoretically.[4–7] One means of observation of the dynamics of such packets is afforded by recording the radiation of such atoms. At the stage of "classical" motion of the packet about a Kepler orbit, the radiation consists of regularly repeating (with the period of the classical motion) spikes, which correspond to the passage of the pocket through its point of minimum distance from the nucleus, where its acceleration is at a maximum. The formation of the above-described macroscopic quantum states corresponding to the fractional regenerations of the packet should be manifested in the onset of regular radiation spikes, one following the other twice, three times, four times, etc. as often as in the classical motion. It is interesting to compare the asymptotic scenario described here with the results of a detailed numerical investigation[4] of the luminescence of a Rydberg atom excited by a laser pulse of duration ~ 10 psec in the region of energy states with principal quantum number close to $\bar{n}=85$, which corresponds to $T_{cl}\approx 94$ psec (see Fig. 2). The sharp peaks of radiation, repeating the shape of the exciting pulse at the initial stage of the evolution, correspond to motion according to the correspondence principle (see also Ref. 12). The authors of Ref. 4 also discovered and explained the regeneration of the initial structure of the time course of the radiation that is observed after ~ 35 periods of the classical motion (see Fig. 2) and is a regeneration of order 1/2 in our nomenclature ($T_{\rm rev}\approx 5.2$ nsec). The intermediate region in Fig. 2, described in Refs. 4 and 5 as a "complicated picture of quantum beats," possesses in fact a well-defined structure. The arrows in Fig. 2 indicate the times $(1/8)T_{\rm rev}$, $(1/6)T_{\rm rev}$, $(1/4)T_{\rm rev}$, and $(1/2)T_{\rm rev}$. The intensity spikes close to the indicated times correspond to fractional regenerations of order 1/8, 1/6, 1/4, and 1/2 with repetition periods $(1/4)T_{cl}$, $(1/3)T_{cl}$, $(1/2)T_{cl}$, and T_{cl}, respectively. The asymmetric character of Fig. 2 with respect to the time $(1/4)T_{\rm rev}$ can be attributed to the influence of higher terms of the expansion of E_n in powers of $(n-\bar{n})$ for the chosen values of the parameters.

In the present article we have described the universal behavior of wave packets consisting of highly excited states of quantum systems executing regular periodic motion in the classical limit. We have shown that during the course of the long-term evolution, following the well-known stage of the motion in accordance with the correspondence principle,

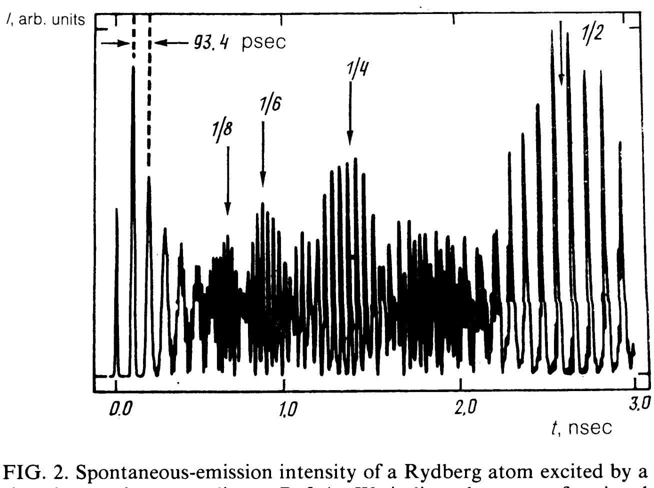

FIG. 2. Spontaneous-emission intensity of a Rydberg atom excited by a short laser pulse (according to Ref. 4). We indicate by arrows fractional regenerations of various order.

these superposition states undergo a universal sequence of fractional regenerations with formation of a correlated set of localized components distributed along a classical orbit. Such objects are in fact macroscopically distinguishable quantum states, whose properties have been extensively discussed over the course of the development of quantum mechanics. The preparation and detection of such states has at present become experimentally feasible.[7,21,22] As follows from the present article, states of this kind regularly arise in a wide class of quantum systems of varying physical nature, in the course of the evolution of an arbitrary "classical" initial state.

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