Lyapunov instability of MHD equilibrium of a plasma with nonvanishing pressure

V. A. Gordin

Hydro-meteorological Research Institute of the USSR

V. I. Petviashvili

I. V. Kurchatov Atomic Energy Institute

(Submitted 16 December 1988)
Zh. Eksp. Teor. Fiz. **95**, 1711–1722 (May 1989)

We use a new functional series of first helicity integrals to obtain sufficient conditions for containing an ideal plasma in a magnetic trap. It is pointed out that in configurations satisfying these conditions almost the whole of the magnetic field is produced by currents in the plasma and that the toroidal field is of the order of the poloidal one. The pressure and the safety factor are increasing functions of the helicity density. The longitudinal current at the edge of the plasma is nonvanishing: the pressure and the helicity density, and also their gradients do vanish. We give a comparison with experiments.

1. INTRODUCTION

Stability in the ideal magnetohydrodynamics (MHD) approximation occupies an important place in the problem of the confinement of a plasma by a constant magnetic field. A definition of the stability of the stationary point of a system of ordinary differential equations with a clear physical meaning was proposed by Lyapunov.[1] He proposed a sufficient condition for such a stability based upon verification of the properties of the extremal points of a function, the so-called Lyapunov function, defined in the phase space of the system.

The ideas of Lyapunov's theory for partial differential evolution equations, i.e., in the case of an infinite-dimensional phase space, began to evolve in the Fifties, but they were extended to applications to the hydrodynamic equations only in the Seventies (see Refs. 2–4 and the citations therein). At that time linear MHD theory, which studies the necessary conditions for stability, had already been well developed (see, e.g., Refs. 5, 6). In the case ideal MHD the linear theory indicates a set of equilibrium states for which all frequencies of the linear oscillations are real. It is well known[1] that this is only a necessary, but not a sufficient condition for the stability of the original nonlinear system. For instance, it does not take into account the possibility of a thresholdless instability caused by the interaction of three or more linear waves in an inhomogeneous medium. Such a nonlinear instability is possibly observed in linearly stable flows of water. It is well known that according to the linear theory the instability of laminar flow must start with Reynolds numbers Re larger than 5800. Experimentally, the stability starts with Re ~ 2000. This is apparently connected with a nonlinear instability which is not found in the linear theory.

Interest attaches therefore to the task, undertaken in the present paper, of distinguishing, in the set of linearly stable plasma states, a narrower class satisfying the sufficient conditions for stability. The focal point of Lyapunov's theory is finding the appropriate Lyapunov function (functional). We find here that for this purpose a functional series of first integrals, connected with helicity,[7–10] turns out to be useful. The stationary solutions of the MHD equations which are checked for Lyapunov stability are obtained as extremals, i.e., as solutions of the Euler equations for the appropriate Lyapunov functionals. The class of equilibrium solutions is thereby restricted—we cannot select the necessary functional for any arbitrary solutuion of the Grad-Shafranov equation.[6]

The next step consists in checking the strict positive-definiteness of the second variation of the Lyapunov functional in the vicinity of the extremal. There appear here difficulties connected with the infinite-dimensionality of the phase space. These are the possibility of a continuous spectrum for the Jacobi equations with coefficients which have poles on resonance surfaces, and the diverse definitions of positive definiteness and of strict positive definiteness.

Earlier, Taylor[11] proposed to find stable states, basing himself on intuitive considerations, by minimizing the functional

$$\int (B^2+k\mathbf{AB})\,d^3x, \qquad \mathbf{B}=\operatorname{rot}A, \qquad k=\text{const} \tag{1}$$

where **AB** is the helicity density and **A** the vector potential of the magnetic field. Minimization leads in this case to stable force-free (pressure $p=\text{const}$) configurations. Here variations in the pressure are neglected, which is valid provided there exists in the plasma a mechanism for a fast equilization of the pressure.

At the extremals of (1), in a cylindrical or toroidal geometry, the longitudinal magnetic field can change sign ("turn").

Using the more general helicity integral[7–10] makes it possible to construct a Lyapunov functional which has an extremal with non-vanishing pressure gradients.

2. ENLARGING THE SET OF MHD EQUATIONS AND HELICITY INTEGRALS

The ideal MHD equations:

$$\begin{gathered}\partial_t\mathbf{B}=\operatorname{rot}[\mathbf{v},\mathbf{B}], \qquad \operatorname{div}\mathbf{B}=0,\\ d_tp+\gamma p\operatorname{div}\mathbf{v}=0, \qquad \partial_t\rho+\operatorname{div}\rho\mathbf{v}=0,\\ \rho d_t\mathbf{v}=-\nabla p+[\operatorname{rot}\mathbf{B},\mathbf{B}]/4\pi\end{gathered} \tag{2}$$

(γ is the adiabatic exponent, the remaining notation is the normal one) possess the following first integrals (respectively, the components of the momentum, the energy, the cross helicity, and the "entropy series"):

 0038-5646/89/050988-07$04.00

$$\int \rho\mathbf{v}\, d^3x, \quad W=\int\left(\frac{B^2}{8\pi}+\frac{p}{\gamma-1}+\frac{\rho v^2}{2}\right)d^3x,$$

$$\int \rho\mathbf{v}\mathbf{B}\, d^3x, \quad \int_D \rho f\left(\frac{p}{\rho^\gamma}\right) d^3x, \tag{3}$$

where f is an arbitrary function, D an arbitrary fluid volume; we assume that if the integration domain is not specified, integration is carried out over the whole volume V inside the conducting sheath.

On the boundary ∂V we assume here that the conditions of impenetrability and infinte conductivity are satisfied:

$$\mathbf{nv}\,|_{\partial V}=0, \quad \mathbf{nB}\,|_{\partial V}=0, \tag{4}$$

where $\mathbf{n}$ is the normal vector to the boundary.

From the first equation of the set (2) it follows that

$$\partial_t\mathbf{A}=[\mathbf{v},\mathbf{B}]-\nabla\varphi(\mathbf{x},t), \tag{5}$$

where φ is an arbitrary function.

It is shown in Ref. 12 that for any gauge or, which is approximately the same, any choice of the function φ the functional $\int \mathbf{A}\cdot\mathbf{B}\, d^3x$ is a first integral of the set (2).

To generalize this integral we supplement the set (2) by an equation for a new unknown function χ:

$$\partial_t\chi+\mathbf{v}\nabla\chi=\varphi-\mathbf{Av}. \tag{6}$$

We define the helicity density by $h=(\mathbf{A}+\mathbf{v}\cdot\nabla\chi,\mathbf{B})$. One shows easily that it satisfies a continuity equation

$$\partial_t h+\operatorname{div} h\mathbf{v}=0. \tag{7}$$

Hence it follows that the quantity $\mu=p^{1/\gamma}/h$ is conserved in particles:

$$\partial_t\mu+\mathbf{v}\nabla\mu=0.$$

For the enlarged set of Eqs. (2) and (6) there exists a functional series of first integrals

$$K_{F,D}=\int_D hF\left(\frac{\rho}{h},\mu\right)d^3x, \tag{8}$$

where F is an arbitrary function of two variables. We note that this conclusion differs from the conclusion of Refs. 7–10 where the gauge $\chi=0$, $\varphi=\mathbf{A}\cdot\mathbf{v}$ was assumed in order to satisfy the continuity of h. Simultaneously with this, they assumed in Ref. 13 a "Lagrangian" gauge which also guarantees that the continuity equation is satisfied for the helicity. It follows from Ref. 13 that for the gauge $\varphi=\mathbf{A}\cdot\mathbf{v}$ Eq. (5) with a given $\mathbf{v}$ can be integrated in Lagrangian coordinates.

3. LYAPUNOV FUNCTIONAL

As the Lyapunov functional we use a first integral in the form of the sum of the energy and the helicity

$$L=W+K_F. \tag{9}$$

We can put here $h=\mathbf{A}\cdot\mathbf{B}$ and minimize both with respect to the potential and with respect to the rotational part of $\mathbf{A}$. We choose the function F in the form

$$F=g(\mu)+c_1\rho^2/h^2-c_2\rho/h,$$

where c_1, $c_2=$ const, and g will be defined later. For the sake of simplicity we assume that there is no vacuum region, $D=V$. The boundary condition (4) is assumed also to be satisfied by the variations. Minimizing (9) with respect to $\mathbf{v}$ we find that equilibrium is realized, if there is no motion: $\mathbf{v}=0$. To obtain a minimum with $\mathbf{v}\neq 0$ we can add to (9) the first integrals (3) in which the velocity enters linearly.

Further minimizing the functional with respect to the variables ρ for $\mathbf{v}=0$, we get

$$\rho=\frac{c_2}{2c_1}h.$$

The condition that (9) is positive definite with respect to the variables ρ and $\mathbf{v}$ for $c_1h>0$ is satisfied; the helicity density h has a constant sign for the extremal of (9).

Minimizing the functional with respect to the pressure p we get the Euler equation:

$$\gamma p+(\gamma-1)g'(\mu)=0, \quad \mu=p^{1/\gamma}/h, \tag{10}$$

where the prime indicates differentiation with respect to the argument. Hence it is clear that in equilibrium the pressure also is a function solely of the helicity: $p=p(h)$.

Finally, varying with respect to $\mathbf{A}$ we get the Euler equation:

$$\mathbf{j}+2G\mathbf{B}+[\nabla G,\mathbf{A}]=0, \quad \mathbf{j}=\operatorname{rot}\mathbf{B}/4\pi,$$
$$G=g-\mu g'-c_2^2/4c_1. \tag{11}$$

Acting on Eq. (10) with the gradient operator and on Eq. (11) with the divergence operator we get:

$$\nabla p=h\nabla G, \quad \mathbf{B}\nabla p=0. \tag{12}$$

Taking the vector product of (11) with $\mathbf{A}$ and using (12) we get the equilibrium condition

$$[\mathbf{j},\mathbf{B}]=\nabla p, \tag{13}$$

which we might have obtained directly from (2). However, the set of Euler equations (10), (11), together with the condition (13), imposes on the current:

$$\mathbf{Aj}+2Gh=0, \tag{14}$$

an additional constraint that is obtained by taking the scalar product of (11) with $\mathbf{A}$. Hence, not any arbitrary equilibrium plasma configuration is an extremal of any functional L.

Turning to a study of the second variation of L we see that one can easily eliminate algebraically by minimization the density and velocity variations $\delta\rho$ and $\delta\mathbf{v}$ from

$$\delta^2L=\int\Bigg\{\frac{(\delta\mathbf{B})^2}{8\pi}+\frac{(\delta p)^2}{2\gamma p}+\frac{\rho(\delta\mathbf{v})^2}{2}+Gh_2$$
$$-G'p^{1/\gamma}\left(\frac{\delta p}{\gamma p}-\frac{h_1}{h}\right)^2\Big/\,2$$
$$+c_1\left[\frac{(\delta\rho)^2}{h}-\rho\frac{\delta\rho h_1}{h^2}-\frac{\rho^2}{h^2}h_2+\frac{\rho^2}{h^3}h_1{}^2\right]\Bigg\}d^3x,$$

where $h_1=\mathbf{A}\cdot\delta\mathbf{B}+\mathbf{B}\cdot\delta\mathbf{A}$, $h_2=\delta\mathbf{A}\cdot\delta\mathbf{B}$. One can after that eliminate the variation δp only under some restrictions. Moreover, we note that one can have such $\delta\mathbf{A}$ that $\delta\mathbf{B}=0$, $h_2=0$, $h_1\neq 0$. From the condition that δ^2L be positive there also follows restrictions in this case. Combining these restrictions[8] we get

$$G'\leqslant 0\Leftrightarrow\mu g''\geqslant 0, \quad 0\leqslant hd_hp\leqslant\gamma p. \tag{15}$$

Here the surfaces on which equality is reached in (15) need further study. They are connected with the possibility of a continuous spectrum of the problem and the necessity to introduce norms accurately such that strict positive definiteness can be proved:

$$\delta^2 L(\delta X) \geqslant c_0 \|\delta X\|^2.$$

If equality is reached in (15) in an open region, the second variation is only non-negative definite, and Lyapunov's theorem is inapplicable. In the examples considered by us all inequalities in (15) are rigorous except at the boundaries ∂V of the region. In view of (10) $\mu = \mu(h)$ and we can introduce a new funtion $U = U(h)$ such that

$$U'(h) = 4\pi G(\mu(h)).$$

It then follows from (10) and (12) that

$$4\pi p = \int hU''\,dh. \qquad (16)$$

After we have eliminated δp by minimization we get a quadratic functional which depends solely on $\delta\mathbf{A}$. We introduce a contracted functional

$$Y = \int \left[\frac{B^2}{2} + U(h) \right] d^3x, \qquad (17)$$

such that when (15) is satisfied the conditions of strict positive definiteness for $\delta^2 L$ and $\delta^2 Y$ are equivalent.

Further, we shall choose the function U such that the contracted functional be strictly positive definite and that at the extremals inequality (15) is satisfied.

Varying the functional Y with the boundary condition in $\mathbf{A}$ that $\mathbf{nB}|_{\partial V} = 0$, we get an Euler equation which is equivalent to (11):

$$\operatorname{rot} \mathbf{B} + 2U'\mathbf{B} + [\nabla U', \mathbf{A}] = 0. \qquad (18)$$

Through variations in the vicinity of the boundary one might obtain additional conditions of transversality for $\mathbf{A}$. However, in view of (4), this does not occur, as $\mathbf{n}\cdot\delta\mathbf{B}|_{\partial V} = 0$ and the integral over the surface ∂V which appears in δY vanishes identically.

We have already indicated that the ρ component of the extremal is proportional to the helicity h corresponding to the extremal, while the pressure component is given by Eq. (16). The extremal, i.e., the plasma configuration corresponding to it, is thus given by the choice of U and the form of the casing. The equilibrium state must satisfy the thermal insulation condition—the condition that the plasma temperature be zero at the boundary with the casing, whence

$$p|_{\partial V} = 0, \qquad (19)$$

Using (15) and (16) it thus also follows that

$$h|_{\partial V} = 0, \quad \mathbf{n}\nabla p|_{\partial V} = 0, \quad \mathbf{n}\nabla h|_{\partial V} = 0,$$

$$4\pi p = \int_{\bullet}^{h} hU''\,dh. \qquad (20)$$

To check the strict positive definiteness,

$$\delta^2 Y = \int \left[\frac{\delta B^2}{2} + U'h_2 + \frac{U''h_1^2}{2} \right] d^3x$$

we must make the choice of the norm in the space of the perturbations $\delta\mathbf{A}$ more precise. We choose the norm to be explicitly dependent on the extremal, in the vicinity of which we study the strict positive definiteness,

$$\delta^2 Y \geqslant c\|\delta A\|^2, \text{ где } c > 0.$$

In that case

$$\|\delta\mathbf{A}\|^2 = \|\delta\mathbf{B}\|^2_{L^2} + \|\alpha h_1\|^2_{L^2}; \qquad (21)$$

$\alpha = \alpha(h)$ is chosen from convenience considerations and we factorize the space of $L^2(\delta\mathbf{A})$ with respect to the kernel of this functional.

The Jacobi equation has the form

$$\operatorname{rot} \delta\mathbf{B} + 2U'\delta\mathbf{B} + [\nabla U', \delta\mathbf{A}] + U''h_1\mathbf{B} + \operatorname{curl}(U''h_1\mathbf{A}) = 0. \qquad (22)$$

The study of the strict positive definiteness of $\delta^2 Y$ can easily be completed if the casing and the extremal of $\mathbf{A}$ have a symmetry. We shall consider below cylindrical and axial symmetries.

4. STABILITY OF A CYLINDRICAL CONFIGURATION

We give here briefly the results of Ref. 10. As the extremal depends only on r, it is convenient to specify the function $\theta(r) = U'(h)$ rather than U', where $\{r,\varphi,z\}$ are cylindrical coordinates; we assume that the helicity h of the extremal is a monotonic function of the radius.

We can consider a more complicated case on non-monotonic functions $\theta(r)$ and $h(r)$, i.e., non-single-valued function $\theta(h)$. In that case the variation of L must be carried out also on the boundaries of the regions in which different branches of $g(\mu)$ are used.

The radial component of the extremal is $A_r = 0$. For $r = 0$ we put $A_\varphi = 0$ and $d_r A_z = 0$.

The Euler equations (18) have the form

$$\begin{aligned} &d_r B_z = 2\theta B_\varphi - A_z d_r\theta, \\ &r^{-1} d_r(rB_\varphi) = -2\theta B_z - A_\varphi d_r\theta, \\ &d_r A_z = -B_\varphi, \ r^{-1} d_r r A_\varphi = B_z. \end{aligned} \qquad (23)$$

In view of the cylindrical symmetry of the extremal we can expand the variation (all the components) in a twofold Fourier series with coefficients depending on the radius:

$$\delta A_\alpha = \sum_{k,m} a_{\alpha,k,m}(r) \exp[i(m\varphi + kz)], \quad \alpha = r, \varphi, z.$$

We can then obtain for $b = b(k,m) = \delta B_r$ from (22) with $k^2 + m^2 > 0$ one second-order differential equation that depends on the parameter k and m.

$$d_r^2 b + \frac{1}{r}\Gamma_1 d_r b = (\Gamma_2 + \Gamma_3/\sigma_1 + \Gamma_4/\sigma_1^2)\, b, \qquad (24)$$

where

$$\Gamma_1 = 1 + \frac{2m^2}{\varkappa^2 r^2}, \quad \Gamma_2 = r^{-2} + \varkappa^2 - 4\theta^2 - \frac{2m^2}{r^4\varkappa^2} + \frac{4mk\theta}{r^2\varkappa^2},$$

$$\Gamma_3 = -\alpha_1 d_r^2\theta + \left(4\alpha_2\theta - 3\sigma_2 + \frac{\alpha_1}{r} - \frac{2km\alpha_2}{r^2\varkappa^2}\right) d_r\theta,$$

$$\Gamma_4 = [\alpha_1 d_r\sigma_1 - (\varkappa^2 + U''\alpha_2^2) d_r h] d_r\theta, \quad \varkappa^2 = k^2 + m^2/r^2,$$

$$\alpha_1 = \frac{m}{r}A_\varphi + kA_z, \quad \alpha_2 = \frac{m}{r}A_z - kA_\varphi,$$

$$\sigma_1 = (m - nq) B_\varphi / r, \quad \sigma_2 = \frac{m}{r}B_z - kB_\varphi$$

(l is the length of the cylinder, $n = kl/2\pi$, and $q = 2\pi r B_z / lB_\varphi$ is the safety factor). When $k = m = 0$ the equation has a different form. From the factorization condition (see Sec. 3) it follows that we can restrict ourselves here to the case $\delta A_r = 0$.

After we have solved Eq. (24) the components $a_{\alpha,k,m}$ can be expressed uniquely in terms of the $b(k,m)$.

The condition for local stability (the absence of a continous spectrum for the Jacobi equation (24)) at the resonance point $\hat{r}$ (i.e., when $\sigma_1(\hat{r}) = 0$) are obtained for large $k^2 + m^2$ in a similar way as the Suydam stability criterion[6] and consists in requiring the absence of oscillations of the solution b in the vicinity of the point $r = \hat{r}$. One can show that this condition is the same as the Suydam criterion obtained from the energy principle,[6] which in our notation has the form

$$(d_r q/q)^2 \geqslant -8hd_r\theta/rB_z^2. \qquad (25)$$

We recall that we consider a narrower class of equilibrium configurations than usual. We note that l does not occur in either the Euler equations (23) or in the resonance-stability criterion (25). The right-hand side of (25) is, in view of (16), proportional to $d_r p$. Hence and from (15) and (16) it follows that if the plasma pressure decreases monotonically from the center to the edge, we also have $|d_r q| \neq 0$. In order to estimate the sign of this derivative we consider the asymptotic behavior of the extremal in the vicinity of $r = 0$:

$$A_\varphi \sim B_0 r/2,\ A_z \sim \psi_c - B_0 r^2/2,\ \theta \sim 1 - c_2 r^4/4,$$
$$lq \sim 1 - r^2,\ p \sim B_0 c_2 (1 - r^4), \qquad (26)$$

where B_0, ψ_c, and c_2 are arbitrary constants. Using (25) we find that $|q(r)|$ is a strictly decreasing function.

Because of the boundary condition we have $b(r_b) = 0$, where r_b is the radius of the cylinder. If in some point $r < r_b$ the solution of Eq. (24) vanishes, the points r and r_b will be conjugate and, hence, the second variation $\delta^2 Y$ is not positive definite.[14]

If we specify the function $\theta(r)$, we can use (23) to determine the equilibrium state corresponding to it. It must satisfy (this must be monitored during the calculation) the necessary conditions (15) and (25). After this we use (24) to determine whether the extremal studied satisfies the sufficient stability conditions.

From the condition that the solution of (23) and the inequality (25) be smooth it follows $d_r\theta = O(r^3)$ that in the vicinity of the axis of the cylinder. The smoothness conditions are not trivial: solutions of (23) are possible, which, moreover, are stable, but have as $r \to +\infty$ a power-law asymptotic form, different from (26), with a non-integer exponent.

From the condition $p = 0$ at the boundary of the plasma and the first inequality of (15) it follows that as $r \to r_b$

$$h = O((r - r_b)^\alpha),\ \alpha \geqslant (\gamma - 1)^{-1}.$$

When solving the Cauchy problem for the system (23) from the center, we get as usual, $\alpha = 2$. The quanitity $d_r\theta$ on the boundary can then be finite.

Numerically solving (23) under those conditions from the center to the edge, we could not get the function $B_z(r)$ to vanish before $h(r)$. This possibly means that for a monotonic pressure gradient there is no turning of the magnetic field. It is in this connection of interest to analyze the Taylor case[11] where $U' = \text{const}$. This case is exceptional and does not allow us to take the limit as $U'' \to 0$. Indeed, in the Taylor case the extremals do not need to satisfy the boundary condition $d_r h = 0$, since $\nabla p \equiv 0$.

We show in Fig. 1 a typical computer solution of Eqs. (23) for $\theta(r) = 1 - 0.004r^4 - 0.0013r^6$, satifying the Lyapunov stability conditions, as checked, apart from (15) and (25), by the solution of Eq. (24) with $|m| = 1$ and 2. The parameter k was chosen such that in the points $r \in [0, r_b]$ there was resonance, $k = mq(r)$. The resonance points were chosen particularly close together near the edges $r = 0$ and $r = r_b$.

In numerical experiments for stable configurations it was possible to obtain a maximum ratio of the plasma pressure to the magnetic pressure on the axis of the column up to 7.5% (see Fig. 1). It is clear that the azimuthal field is of the order of the longitudinal one and that the current at the boundary of the plasma, though small, does not vanish.

5. AXISYMMETRIC TOROIDAL CONFIGURATIONS

We study now more general extremals which depend not only on r but also on z. Since the magnitudes of the longitudinal and azimuthal field components are of the same order in the cylindrical configuration, it is possible that here too the toriodal and poloidal fields will be of the same order. In that case it is impossible to expect a qualitative difference from the cylindrical case, as occurs in tokamak type systems. In the latter the poloidal field is always much smaller than the toriodal field which is almost completely sustained by the external current.

A numerical experiment showed that the magnetic field in configurations constructed by the method described above is produced mainly by the currents in the plasma.

We express the magnetic field and the current in the cylindrical system of coordinates $\{r,\varphi,z\}$ in terms of Stokes potential. We write $\psi = -r\mathbf{A}\cdot\mathbf{e}$. Hence

$$r\mathbf{B} = [\mathbf{e},\ \nabla\psi] + \mathbf{e}I,\ \mathbf{B}\nabla\psi = 0, \qquad (27)$$

where $\mathbf{e}$ is the unit vector in the azimuthal direction, and ψ and I are arbitrary functions.

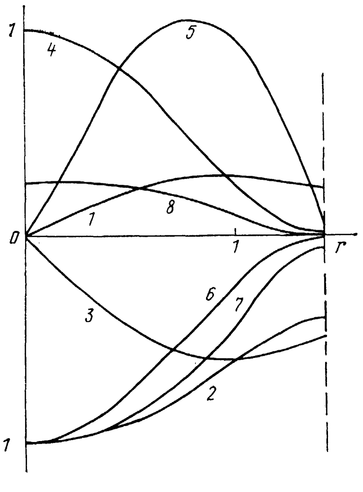

FIG. 1. Components of the extremal for a cylindrical plasma in a system of units where $\theta(0) = 1$ and $B_z(0) = -1$; 1) A_φ; 2) $A_z - 7$; 3) B_φ; 4) B_z; 5) $\text{curl}_\varphi \mathbf{B}$; 6) $\text{curl}_z \mathbf{B}/2$; 7) q; 8) $p \cdot 10^2$.

It follows from (27) that

$$r\,\mathrm{curl}\,\mathbf{B}=-[\mathbf{e}, \nabla I]+\mathbf{e}\tilde{\Delta}\psi,$$
$$\tilde{\Delta}=r\partial_r r^{-1}\partial_r+\partial_z^2. \qquad (28)$$

Substituting (27) and (28) into (14) and (13) we find that I, h, and p are locally single-valued functions of ψ. This means that if in the points (r_1,z_1) and (r_2,z_2) the function ψ takes on the same values, but these points are separated by closed iso-ψ lines, the functions I, h, and p can take on different values. In the numerical examples considered below the function ψ has no saddle points and the concepts of local single-valuedness and single-valuedness are the same.

Moreover, it follows from (18) that the function ψ satisfies the Grad-Shafranov equation which in our case has the form

$$\tilde{\Delta}\psi=-r^2h\theta-I\dot{I}, \qquad (29)$$

and the easily integrable equation

$$\dot{I}=2\theta+\psi\dot{\theta}. \qquad (30)$$

also follows. Here and henceforth a dot indicates differentiation with respect to ψ, $\dot{U}=\theta(\psi)$. There is thus in Eq. (29) one arbitrary function, $\theta(\psi)$, rather than two as usual.[6]

In view of the axial symmetry of the problem, the boundary condition (4) is equivalent to the condition

$$\psi|_{\partial V}=\mathrm{const}. \qquad (31)$$

We determine the vector potential $\mathbf{A}$. This is a single-valued vector field, continuous in the region V, such that the current $\mathbf{j}$ is finite everywhere in V.

We change to an orthogonal set of coordinates (ψ,ϑ,ζ), where ϑ and ζ are, respectively, the poloidal and toroidal angles. The components of the metric tensor are

$$l_1=1/|\nabla\psi|,\ l_2=1/|\nabla\vartheta|,\ l_3=r$$

and

$$B_1=0,\ B_2=\nabla_1\psi/r,\ B_3=I/r,\ A_3=-\psi/r.$$

It follows from (12) and (27) that $h=h(\psi)$ and, by definition, we have

$$h=A_2\nabla_1\psi/r+A_3I/r, \qquad (32)$$

where $\nabla_i=l_i^{-1}\partial_i$ is the ith component of the gradient in the (ψ,ϑ,ζ) coordinate system. The third component of the magnetic field is

$$B_3=I/r=(\partial_1l_2A_2-\partial_2l_1A_1)/l_1l_2. \qquad (33)$$

As $\mathbf{A}$ is single-valued we have

$$\oint\partial_2(l_2A_1)\,\partial\vartheta=0$$

and, in view of (33), we find

$$\partial_1(hM_1+\psi IM_2)=IM_2, \qquad (34)$$

where

$$M_1=\oint Hd\vartheta,\quad M_2=\oint Hr^{-2}\,d\vartheta,\quad H=l_1l_2l_3.$$

We note that $2\pi\int M_1d\psi=V$ is the volume inside the $\psi=\mathrm{const}$ surface, $\Phi=\int IM_2d\psi$ the toriodal magnetic field flux, $\chi=2\pi\psi$ the poloidal flux, and ψ_c the value of the function ψ on the magnetic axis; $q=IM_2/2\pi=d_\chi\Phi$. Equation (32) now gives a relation between h and q:

$$h=[\Phi(\psi)-\psi2\pi q]M_1^{-1}=\Phi d_V\chi-\chi d_V\Phi. \qquad (35)$$

From (32), (33), and (35) we get a unique expression for the components of the vector potential

$$A_1=l_1^{-1}\int_0^{\vartheta}(\partial_1l_2A_2-l_1l_2I/r)\,d\vartheta,$$

$$A_2=\psi I(1-r^2M_2/M_1)l_1+\Phi l_1r/M_1.$$

As we assume the current on the magnetic axis to be finite, it follows that the component A_2 must be a differentiable function. As $l_1=1/|\nabla\psi|$ has a pole on the magnetic axis, while M_1 is finite, we must put on the axis, where $\psi=\psi_c$

$$\Phi(\psi_c)=0. \qquad (36)$$

We can rewrite[15] the Grad-Shafranov equation in the variables (ψ,ϑ), assuming r and z to be unknown functions, if we introduce yet another function $\mu=\mu(\psi,\vartheta)$:

$$\dot{r}=\partial_2z/\mu r,\quad \dot{z}=-\partial_2r/\mu r,\quad \dot{\mu}=-DJ/\mu, \qquad (37)$$

where

$$D=(\partial_2r)^2+(\partial_2z)^2,\quad J=h\dot{\theta}+I\dot{I}/r^2,$$

$$|\nabla\psi|=r\mu D^{-1/2},\quad |\nabla\vartheta|=D^{-1/2}.$$

In the vicinity of the magnetic axis the following asymptotic expressions hold:

$$h=-\psi_cI(\psi_c)+O(\xi^2),\ q=-1/(\theta(\psi_c)R)+O(\xi^2),$$
$$r=R+k\xi\cos\vartheta+O(\xi^2),\ z=k\xi\sin\vartheta+O(\xi^2),$$
$$\mu=\xi^2/R+O(\xi^3),\ \text{где}\ \xi=[2(\psi-\psi_c)]^{1/2},$$
$$k^2=-1/[\theta(\psi_c)I(\psi_c)]>0. \qquad (38)$$

We chose the function θ in the form $\theta(\psi)=c_0-c_3(\psi-\psi_c)^3$ with the normalizing condition $c_0=1$. In view of (15) we have at the boundary

$$h(\psi_b)=0,\ \dot{h}(\psi_b)=0. \qquad (39)$$

The system (30), (34), (37) is of the elliptic type. The set (37), which is equivalent to the Grad-Shafranov equation, was solved in Ref. 15 for a given shape of the casing. In our case are imposed "additional" conditions (39), and this raises difficulties in the solution of problem (30), (34), (37)–(39). We solved the Cauchy problem for (30), (34), (37) under the initial conditions (38), with smoothing with respect to the variable ϑ. The shape of the casing is then obtained in the course of the solution.

The parameter c_3 was chosen for $R=3.5$ in the range $1\leqslant\psi_c\leqslant256$ for $I(\psi_c)=-1$ by a hit-and-miss method: integration of the set (30), (34), (37) was stopped either when $h(\psi)=0$ or when $\dot{h}(\psi)=0$.

We show in Fig. 2 the functions I, h, and p corresponding to $\psi_c=16$. It is clear that, just as in the case of the cylinder, we obtain solutions of the type of a diffusion pinch with a longitudinal field, and not of the tokamak type.

We show in Fig. 3 the distributions of the toroidal and poloidal components of the field and the current. Moreover,

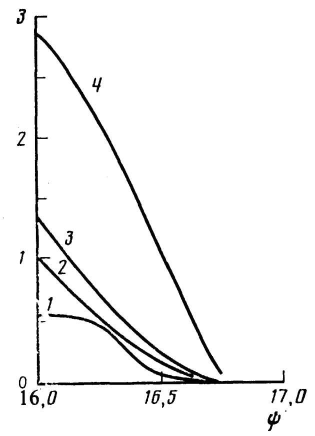

FIG. 2. Profiles of surface functions for a torus with $\psi_c = 16$, $R = 3.5$ in a system of units where $\theta(\psi_c) = 1$, $I(\psi_c) = -1$: 1) $p\cdot 10^2$; 2) $-I$; 3) h; 4) $-q\cdot 10$.

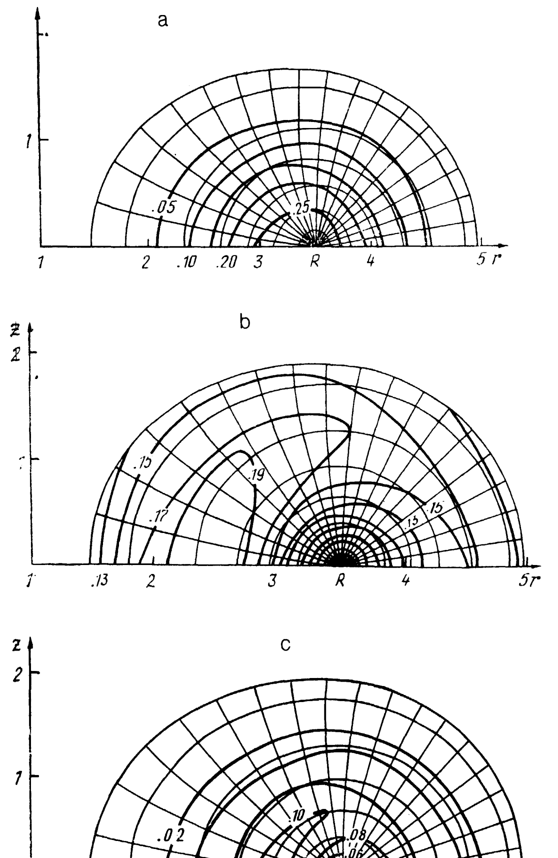

FIG. 3. Isolines in the (r,z) plane: a) poloidal magnetic field; b) toroidal magnetic field; c) poloidal current; d) toroidal current. The magnitudes of the corresponding parameters are shown by the fractional numbers on the isolines.

we performed experiments with other values of ψ_c. In particular, we found that $\beta = p(\psi_c)/2R^2$ increases with increasing ψ_c and is stabilized at a level of $\sim 14\%$. The graphs obtained are rather well approximated by the functions

$$c_3 \approx 1.1\psi_c^{-1}, \quad \beta \approx 0.14 - 0.14\psi_c^{-1}, \quad \delta\psi \equiv \psi_b - \psi_c \approx \approx 0.73 - 0.11\psi_c^{-1},$$

$$I(\psi_b) \approx -0.055\psi_c^{-1}, \quad q(\psi_b) \approx -0.11\psi_c^{-1}.$$

To check the strict positive definiteness of $\delta^2 Y$ we must study the analog of Eq. (24); in the case of a toroid this is already a partial differential equation that depends on a single parameter—a Fourier expansion is possible only with respect to the variable ζ. In the experiments described above we restricted ourselves to less—to Mercier's criterion[16]:

$$I(2\pi\dot{q})^2/4\dot{p} > \left[\dot{p}\oint\frac{H\,d\vartheta}{B_2^2} - \dot{M}_1\right]\oint\frac{\nu B^2}{B_2^2}\,d\vartheta + 2\pi\dot{q}I\oint\frac{\nu\,d\vartheta}{B_2^2} - \dot{p}I\left(\oint\frac{\nu\,d\vartheta}{B_2^2}\right)^2,$$

where $\nu = IHr^{-2}$.

Of course, this is not a complete check which can, for instance, be carried out by minimizing $\delta^2 Y$ under the condition $\|\delta A\| = 1$.

When we take dissipative effects into account, in first instance the finite conductivity, we must restrict ourselves to those extremals of the functional L on which $d_t L[X(t)] \leqslant 0$, where the derivative $\partial_t X$ is calculated already by means of the dissipative set of equations.

6. COMPARISON WITH EXPERIMENTS

The main object of the present paper is to show that the stability conditions which we obtained are not too rigid and that they can be satisfied. The configuration examples obtained here (i.e., stabilized pinches in which $B_2 \sim B_3$, $B_3 \neq 0$, and $q < 1$ and decreases montonically towards the edge) had been already been considered in Ref. 17. Amongst them is the well known ZETA installation. At the present time the installation closest of all to it with a "supersmall" q is Repute-1 at the University of Tokyo.[18] The aspect ratio in it is 82 cm/22 cm, the electrical current 350 kA, the temperature 350 eV, and the relative pressure 5–15%; q decreases from a value of 0.3 in the center and starts to grow near the boundary. The presence of a minimum in q inside the plasma leads to an instability connected with the finite pressure. It is assumed[18] that the level of magnetic fluctuations is lower than in installations with a turning field. This confirms the numerical experiment[19] which shows that in the case when B_3 turns there arises a nonlinear instability.

The ratio of the longitudinal current at the boundary to the current at the center is in Fig. 3 less than 1/300. Near the boundary the pressure $p \propto x^3$, $\partial_x q \propto x$, and $\rho \propto x^2 \sim h$, where x is the distance to the boundary. This guarantees stability and thermal insulation, but worsens the conductivity. One therefore need additional measures to sustain the longitudinal current which guarantees that q is monotonic up to the boundary. In spheromaks[20] q also decreases up to the boundary, but the stability condition (15) cannot be satisfied.

The authors express their gratitude to V. D. Shafranov for advice and discussions.

[1]A. M. Lyapunov, Obschchaya zadacha ob ustoĭchivosti dvizheniya (*The general problem of the stability of motion*), Gostekhizdat, Moscow. 1950.
[2]V. I. Arnol'd, *Matematicheskie metody klassicheskoĭ mekhaniki* (*Mathematical methods of classical mechanics*), Nauka, Moscow, 1974 [English translation published by Springer Verlag, Berlin].
[3]L. A. Dikiĭ, *Gidrodinamicheskaya ustoĭchivost' i dinamika atmosfery* (*Hydrodynamical stability and the dynamics of the atmosphere*), Gidrometeoizdat, Leningrad, 1976.
[4]V. A. Gordin, *Matematicheskaya zadachi gidrodinamicheskogo prognoza pogody* (*Mathematical problems of hydrodynamical weather forcasting*), Gidrometeoizdat, Leningrad, 1987.
[5]B. B. Kadomtsev, Voprosy teorii plazmy, Vol. 2, Atomizdat, Moscow, 1962 [Rev. Plasma Phys. **2**, 153 (1966)].
[6]G. Bateman, *MHD Instabilities*, MIT Press, Cambridge, Mass., 1978.
[7]V. A. Gordin and V. I. Petviashvili, Pis'ma Zh. Eksp. Teor. Fiz. **45**, 221 (1987) [JETP Lett. **45**, 276 (1987)].
[8]V. A. Gordin and V. I. Petviashvili, Fiz. Plazmy **13**, 880 (1987) [Sov. J. Plasma Phys. **13**, 505 (1987)].
[9]V. A. Gordin and V. I. Petviashvili, Proc. ICPP (Kiev) **3**, 52 (1987).
[10]V. A. Gordin and V. I. Petviashvili, Fiz. Plazmy **15**, 240 (1989) [Sov. J. Plasma Phys. **15**, 140 (1989)].
[11]J. B. Taylor, Rev. Mod. Phys. **58**, 741 (1986).
[12]L. Woltjer, Proc. Nat. Acad. Sc. **44**, 489 (1958).
[13]M. I. Monastryskiĭ and P. V. Sasorov, Zh. Eksp. Teor. Fiz. **93**, 1210 (1987) [Sov. Phys. JETP **66**, 683 (1987)].
[14]I. M. Gel'fand and S. V. Fomin, *Variatsionnoe ischislenie* (*Variational calculus*), Fizmatgiz, Moscow, 1961 [English translation published by Prentice-Hall, Englewood Cliffs].
[15]L. N. Vabishchevich, L. M. Degtyarev, and A. P. Faborskiĭ, Fiz. Plazmy **4**, 995 (1978) [Sov. J. Plasma Phys. **4**, 554 (1978)].
[16]J. W. Connor, R. J. Hastie, and J. B. Taylor, Proc. Roy. Soc. (London) **A365**, 1 (1979).
[17]L. A. Artsimovich, *Upravlyaemye termoyadernye reaktsii* (*Controlled thermonuclear reactions*), Fizmatgiz, Moscow, 1961 [English translation published by Gordon and Breach, New York].
[18]N. Inoue, Z. Ioshida, and Y. Kamada, XIIth Int. Conf. Plasma Phys. Contr. Nucl. Fusion, IAEA-CN-50/C-5-8, Nice, 1988.
[19]Y. Murakami, Z. Ioshida, and N. Inoue, Nucl. Fusion **28**, 449 (1988).
[20]M. N. Rosenbluth and M. N. Bussac, Nucl. Fusion **19**, 489 (1979).

Translated by D. ter Haar