Regularization of the self-energy of point vortex dipoles and the increase in the total vorticity during stretching of vortex lines

S. G. Chefranov

Institute of Atmospheric Physics, Academy of Sciences of the USSR
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The balance equation for the total vorticity (the enstrophy) corresponding to a "strong" singularity, i.e., the explosive increase in the enstrophy in a dynamic interaction of two regularized point vortex dipoles (infinitesimal vortex rings which have been "smeared out"), is derived in the three-dimensional case. The effect of viscous dissipation on the process is evaluated.

A comparison is made with the dynamics of two-dimensional vortex dipoles which leave the enstrophy invariant (in the absence of dissipative factors), despite the possible realization of a "weak" singularity for the local properties of the vortex field.

The regularization problem always arises in the study of point vortex entities which behave an indefinite self-energy. For point vortices in the two-dimensional case (i.e., for rectilinear vortex filaments) an infinite self-energy is simply discarded in the analysis of the interaction of such vortices. A more complicated question is whether it is valid to discard the self-energy in treating the interaction of point vortex dipoles in either two or three dimensions. The complexity stems from the fact that the vortices have an infinite self-induction velocity along the symmetry axis. The mathematical difficulties which arise (and which, incidentally, also arise for point vortices in a plane) have the same source as in a quantum field theory based on the concept of a point interaction described by the product of $\delta$-function operators taken at the same spatial point. In quantum field theory this problem of the regularization of expressions containing products of $\delta$-functions has been approached by various formal paths which nevertheless lead to excellent agreement with experimental data. Dyson regularization, for example, can be associated with a subtraction of the self-energy of point vortex entities from the total invariant kinetic energy of the system, as mentioned above. A somewhat less formal approach to quantum field theory was derived by Landau, who suggested treating the point interaction* as the limit of some "smeared" interaction of finite range as this range decreases to zero. In other words, the idea is to abandon the use of $\delta$-functions to describe a point interaction. This idea of Landau's underlies the procedure developed in Sec. 1 of the present paper for regularizing the self-energy of point vortex dipoles, $T_r$, in which $T_r$ does not affect the Hamiltonian dynamics of the relative motion in a system of such vortex entities. A system of this sort was studied in Sec. 2 in connection with an analysis of the problem of spontaneous singularities in three-dimensional turbulence.**

1. REGULARIZATION OF THE VORTEX SELF-ENERGY

1. In the three-dimensional case, the total kinetic energy corresponding to the vorticity distribution $\omega(x)$ in an unbounded space is

$$T = \frac{\Omega}{2\pi} \int \int d^3x \omega(x) \omega(x) / |x-x'|,$$  

where $\rho_v$ is the constant density of the liquid. The self-energy of a point vortex dipole, $T_r$, is found from (1) for the vorticity

$$\omega_{\nu} = \mu \gamma_{\nu} / \partial x_{\nu},$$  

which corresponds to a point vortex dipole at the origin of coordinates, with a Lamb momentum $\gamma$. In (2), $\epsilon_{\nu}$ is the Levi-Civita pseudotensor, and a repeated index means a summation from 1 to 3. The value of $T_r$ is undetermined because of the product of $\delta$-functions, taken at the same point, in (1) for as from (2).

We will regularize the energy $T_r$ by Landau's approach: We replace the $\delta$-function in (2) by a finite "smeared" modification $\delta(x)$, where the regular function $\delta(x)$ satisfies the parity and normalization conditions

$$\delta(x) = \delta(-x), \quad \int d^3x \delta(x) = 1.$$  

The distribution of the solenoidal velocity field corresponding to (2) takes the following form as a result of the replacement $\delta \to \delta^r$:

$$\delta^r(x) = \gamma_{\nu} \delta(x) / \partial x_{\nu},$$  

where

$$\Phi = \Phi_{\nu} \gamma_{\nu} f(x), \quad f(x) = \frac{1}{4\pi} \int d^3x' \delta(x') / |x'-x|.$$  

Let us consider the difference between the Laplacian of $\delta$ and $\delta^r$ as $|x| \to 0$, and we have div $\omega = 0$, regardless of the nature of $\delta(x)$, since we have $\Delta \delta = -\delta(x)$, where $\Delta$ is the Laplacian. For $\delta(x) \to 0$ as $|x| \to 0$, the velocity field (4) of the "smeared" vortex dipole no longer has a singularity at the point $x = 0$, since we have $\delta^r(x) \to 0$ as $|x| \to 0$.

In general, we can write a regular finite function $\delta^r(x)$ which satisfies conditions (3) as an infinite series in spherical harmonics. For simplicity we restrict the analysis to the representation (at $t = 0$)

$$\delta^r(x) = \phi_{\nu}(r) \{ 1 + \epsilon P_{\nu}(\cos \theta) \}.$$  

$$P_{\nu}(\cos \theta) = \frac{\Gamma(\nu + 1/2)}{\sqrt{2\pi} \Gamma(\nu + 1)} (3\cos^2 \theta - 1/2)$$

is the Legendre polynomial, $\phi$ is the spherical angle measured from the direction specified by the momentum vector $\gamma$, $r = |x|$, $\delta$ is the scale of the smearing of $\delta(x)$.
the $\delta$-function, and $y$ is an arbitrary constant factor.

Representation (5) corresponds to expression (A1) for
$f(r, \theta)$, while for $T_c$ we find the following results from (1) and (2), respectively, when we make the replacement $\delta \to \delta$:

$$T_c = -\frac{\alpha^2}{2} \int d^2x \delta(x) \left[ \delta(x) + \frac{\partial^2}{\partial x^2} \right] f(r,0).$$

Expression (6) can also be derived in an elementary way
from the definition

$$T_c = \frac{\alpha^2}{2} \int d^2x u^\alpha$$

for $u$ from (4), since in an unbounded space we have

$$\int d^2x \delta \theta \theta_{x=0} = 0,$$

by virtue of the condition $\text{div } u = 0$.

1. We first consider the possibility of separating from $T_c$
in (6) that part of the energy ($T_{a\alpha}$) which is most responsible
for the existence of a self-induced motion of the vortex
dipole in the direction specified by the vector $y$, i.e., along
the axis $\theta = 0$. Here we have $T_c = T_{a\alpha} + T_{c\alpha}$, where $T_{a\alpha}$
is given by (6) when we make the replacement
$f(r,\theta) - f(r,\theta = 0)$ [See (A1)]. Note also that the $\theta$
dependence of $f(r,\theta)$ is determined by the $\theta$ dependence of $\delta$
even in the limit $\delta \to 0$, in which we have, for $\delta \to \delta$, $f(r,\theta) \to -1/4\pi r$. In the Appendix we derive an expression for $T_{a\alpha}$, specifically, expression (A2), which corresponds to an initial (at $t = 0$) smearing:

$$T_{a\alpha}(r) = \alpha^2 (r-b)^2 \delta(y-b-y),$$

where $\alpha > 3/2$ and

$$\delta(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$$

is the unit step function. It follows from (A2) that for real
values of the parameter $y$ the quantity $T_{a\alpha}$ can vanish only
for large values of the structure parameter $\alpha \alpha = 137$ (we
are rounding to integers). For example, we have $T_{a\alpha} = 0$ for
$\alpha = 137$ and $y \geq 3$. A vortex structure of this sort, corre-
sponding to (2), (5), and (7), is topologically equivalent to
a vortex ring of radius $b$ which is smeared out toward the
center of the ring, since we have $\alpha = 0$ at $r = 0$ for any
$0 < r < b$ and $\alpha = 0$ at $r = 0$ and $r > b$. A large value of $\alpha$ here
is supposed to correspond to the greatest concentration of the vorticity
which is nevertheless near the periphery of the vortex region,
and $\alpha = 0$ at $r = b$. The total energy $T_c$ of course increases without
bound in the limit $b \to 0$, although $T_{a\alpha}$ no longer depends
on this limit, remaining zero for arbitrary $b$ (for the values of $y$
and $\alpha$ stated above). As a result, there is a cancellation of the energy

$$\frac{\alpha^2}{2} \int d^2x \delta(x),$$

associated with the anisotropic induced motion of the vortex
dipole.

3. To evaluate the possibility that the total self-energy
$T_c$ vanishes, we must also consider complex values of the
parameter $y$. We assume a factor $y = iy$, where $y = -1$
and $y$ is a real random quantity with a zero mean, $(y) = 0$.

The values of the vorticity field and the velocity which are aver-
aged over $y$, and which are real in this case, are physically
meaningful, of course. The same comment applies to the glo-
bal characteristics of the vortex field: the energy, the integral
vorticity (enstrophy), etc. Despite the formal nature of this
statistical approach to the problem of the regularization of
the self-energy $T_c$, it may be justified by the circumstance
that in practice only average properties of vortex fields, not
the fields themselves, are measurable and, correspondingly,
predictable. Furthermore, complex vortex structures also find applications in elementary particle theory.

For $(T_c)$ in this case we find the following expression from
(6) and (A1), after we average over $y$:

$$\langle T_c \rangle = -\frac{4\alpha}{3} \int \delta(x) (x \cdot \theta) \int d^2r \delta(y) f(r,0)^2 \left(1 - \frac{y^2}{4\alpha^2} \right).$$

where we should have $(y) = 7$ according to physical con-
siderations. From the Cauchy-Bunyakovskii inequality we
find the following estimate of the coefficient in front of the
parentheses in (7a):

$$\frac{4\alpha}{3} \left(\delta(x) \int \delta(y) f(r,0)^2 \right)^{\frac{1}{2}} \langle \delta(y) f(r,0)^2 \rangle,$$

This estimate does not depend on the form of $\delta(y)$ at
$0 < r < b$. In the case of $y^2 = 7$, however, we have $(T_c) = 0$, and
this energy no longer depends on the last limit $(b \to 0)$.

We wish to stress that the smearing in (5) and (7) does not by itself determine the steady-state solution of the Helm-
holtz equations (Ref. 13, for example), but it can serve as an
initial condition for a time-varying vortex structure (which is
localized in the limit $b \to 0$) for which the self-energy
should again be zero, $(T_c) = 0$, by virtue of energy conser-
vation. It is shown in Sec. 2 that a vortex structure of this
sort, corresponding to (5) and (7), is nevertheless a steady-
state structure on the average.

We also note that the results of the regularization found
above agree with the conclusion reached by Goman et al.1
that it is possible to eliminate the self-induction velocity for a
localized time-varying vortex region modeled by a set of thin
vortex rings. Furthermore, Finkel'shtein11 used arguments
similar to those presented by Goman et al.1 as a basis for
equating the self-energy for vortex filaments in type II super-
conductors to zero. In quantum field theory also, regulariza-
tion brings the self-energy of a photon to zero.

In the Appendix we present some procedures for regu-
larizing the self-energy of point vortex dipoles in the two-
dimensional case. Those procedures are similar to the ones
discussed in Subsections 2 and 3.

4. In the three-dimensional case, for a system of $N$ point
vortex dipoles with a vorticity distribution

$$\omega_v(x, t) = \sum_{m=1}^{N} \gamma_v(\omega_{x}^{(m)}(x) - \omega_{x}^{(m)}(t)),$$

a Hamiltonian formulation with a Hamiltonian

$$H = \sum_{m=1}^{N} \gamma_v(\omega_{x}^{(m)}),$$

and with canonical variables $\gamma_v$, $x_v^{(m)} (m = 1,2,\ldots,N)$, satisfying
the system of equations

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The total kinetic energy corresponding to the vorticity distribution in the plane, \(\rho(s)\), is
\[
T = \sum_{n=1}^{N} \frac{1}{2} \lambda_{n}^{2} \int d^{2} x \int d^{2} x' \rho(s) \ln |x-x'|^{2}.
\]
For a system of \(N\) point vortex dipoles we have
\[
u_{i} = \sum_{n=1}^{N} e_{i} e_{n} \partial_{x_{n}} \int d^{2} x' \rho(s) \ln |x-x'|^{2}.
\]
where \(e_{i}\) is the Levi-Civita density, and a repeated index means a summation from 1 to 2. From (10) and (11) we find the interaction energy of the dipoles, \(W_{2}\), and the corresponding Hamiltonian \(H_{2}\), which is given by (with \(W_{2} = \rho_{0} H_{2}/2\))
\[
H_{2} = \sum_{n=1}^{N} \gamma_{n}^{2} \varphi(s),
\]
where
\[
\varphi(s) = \frac{1}{2\pi} \int d^{2} x \int d^{2} x' \gamma_{n}^{2} \ln |x-x'|^{2}.
\]

The canonical variables \(\gamma, \varphi, \gamma_{n}, \gamma_{n}\) \(r = 1, 2, m = 1, 2, \ldots, N\) are described by the system of equations (which corresponds to \(H_{2}\))

\[
\begin{align*}
\frac{d\gamma_{n}}{dt} &= \gamma_{n} - \gamma_{n}, \\
\frac{d\varphi}{dt} &= \varphi - \varphi,
\end{align*}
\]
which can also be derived as a weak solution of the two-dimensional Helmholtz equations for the vorticity field (11) by eliminating the corresponding singular self-induced velocity.

Equations (12) leave invariant the Hamiltonian \(H_{2}\), the total momentum
\[
P_{\gamma} = \sum_{n=1}^{N} \gamma_{n}^{2},
\]
and the angular momentum
\[
M = \sum_{n=1}^{N} \lambda_{n} \gamma_{n}.
\]

As in the three-dimensional case, we can use these invariants to derive (without any difficulty) an exact solution of Eqs. (12) for two vortex dipoles. In particular, we consider the case \(P_{\gamma} = 0\), i.e., the case with \(\gamma = \gamma = \varphi\). We assume that \(x_{1} = x_{2} = 1\); then we have
\[
M = \rho_{0} \gamma_{1}, \quad H_{1} = \frac{\rho_{0}}{2\pi} \left(1 - \frac{2\rho_{0}}{\gamma_{1}^{2}} / \gamma_{1}^{2}\right),
\]
From (12) we then find the system of equations
\[
\begin{align*}
\frac{d\gamma_{1}}{dt} &= \gamma_{1} - \gamma_{1}, \\
\frac{d\varphi}{dt} &= \varphi - \varphi,
\end{align*}
\]
which agrees qualitatively with that found in Ref. 6 for three dimensions, and it corresponds to Eqs. (9), (9') with \(N = 2\). Furthermore, despite the difference between the exponents in (14) and (15), on the one hand, and those in the three-dimensional case, on the other [see the expressions for \(\gamma(t)\) and \(\varphi(t)\) in Ref. 6, and see also expressions (21) and (22) below], the exponent of the explosive increase in the square of the local gradient in a neighborhood \(|x - B| < r(t)\) of the invariant "center of gravity" \(B = (x_{1} + x_{2})/2\) [3 in the limit \(r(t) \to 0\) for \(t \to t_{*}^{-}\) in (14), i.e.,
\[
\Omega = \left(\partial_{x} / \partial_{x'}\right)^{2} \Omega(\gamma^{2}/\varphi) \mathcal{O}(1/|t^{-1} - t_{*}^{-}|)\]
\]
is exactly the same as the three-dimensional value \(\Omega = 2\) [in Ref. 6, \(\Omega \approx 0 \gamma(t)^{2} / 2 \gamma(t)^{2} - \gamma(t)^{2}\)]. Here \(\mathcal{O}(1)\) is the vorticity field which would be set up at point \(x\) by a pair of vortex dipoles which are collapsing (closing on each other).

On the other hand, a fundamental difference between three-dimensional vortex dynamics and two-dimensional dynamics, which stems from the circumstance that only
three-dimensional vortex lies can be stretched, is demonstrated by the analysis in Sec. 2 of the evolution of the enstrophy, for which there is a "strong" singularity at \( t \to -t^* \). This strong singularity is not found for the invariant enstrophy in the two-dimensional case, despite the existence of a "weak" singularity in the localization of the quantity \( \Omega \) for \( t \to -t^* \).

2. INTENSIFICATION OF THE INTEGRAL VORTICITY (ENSTROPHY)

1. The regularization of the vortex field \( \hat{\omega}(x) \) which was introduced in Sec. 1 through the replacement of a \( \delta \)-function by a regular finite function \( \hat{\delta}(x) \) (with infinitesimal support \( b \to 0 \)) also makes it possible to introduce, in a correct way, the square of the regularized vorticity, \( \omega^2(x) \), and corresponding integrals of the enstrophy, \( I = \int d^3\hat{\omega}^2 \) and \( I_2 = \int d^2\hat{\omega}^2 \), for point vortex dipoles in the three-dimensional case, (8), and the two-dimensional case, (11). In contrast with the integral \( \int d^3\hat{\omega}^2 \) and \( \int d^2\hat{\omega}^2 \), the enstrophy \( I \) and \( I_2 \) does not vanish for the vortex dipoles (8) [or (11)] and is thus a convenient measure of the integral vorticity throughout the volume of the liquid. In an unbounded three-dimensional space, the balance equation for the enstrophy is

\[
\frac{d}{dt} \left\{ \int d^3\hat{\omega}^2 \right\} = \int d^3\frac{\partial \hat{\omega}^2}{\partial t} + \int d^3\hat{\omega} \frac{\partial \hat{\omega}}{\partial t},
\]

(16)

where \( \hat{\omega} \) is the regularized vorticity (8), \( \hat{\omega} \) is the corresponding velocity field, and \( \omega \) is a kinematic viscosity coefficient. Everywhere below, we will consider finite smearing of the \( \delta \)-functions in the form of \( \hat{\delta}(x) \) [as in (5) and (7) and the effects of viscous dissipation only in the enstrophy balance equation (16)]; we will be assuming that we have already taken the inverse limit \( \delta - \delta \) (i.e., \( b \to 0 \)) in the Helmholtz equation (A6). That limit, in contrast with (16) in (A6), is permissible [and leads to a weak solution described by system (9), (9')] at large Reynolds numbers we can again use the system (9), (9'), which follows from (A6), to describe the dynamic interaction of vortex dipoles and the corresponding evolution in the enstrophy \( I \) in (16). Since the smearing of \( \delta(x) \) which we discussed in Sec. 1 corresponds to only the initial time, \( t = 0 \), we must introduce some new notation: \( \hat{\delta}(t) \), which determines some "average" scale of the smearing at \( t > 0 \). Specifically, it determines the scale for the enstrophy balance equation (16) which we are considering here, (16), from which we also find the quantity \( \hat{\delta}(t) \) and, correspondingly, \( \hat{\delta}(t) \) under the initial condition \( \hat{\delta}(t) = 0 = b \) [for \( b \) from (5) and (7), for example] and for the solutions of Eqs. (9), (9') which were found in Ref. 6.

In particular, for a single vortex dipole (in a cylindrical coordinate system with \( z \) axis in the \( z \)-plane), with \( \omega = \text{const} \), we have

\[
\omega_0 = 0, \quad \omega_0 = \frac{\hat{\delta}}{\omega}, \quad \omega_0 = 0,
\]

\[
\omega_0 = \frac{\hat{\delta}}{\omega}, \quad \omega_0 = \frac{\hat{\delta}}{\omega}, \quad \omega_0 = \frac{\hat{\delta}}{\omega}, \quad \omega_0 = \frac{\hat{\delta}}{\omega}.
\]

(17)

The corresponding enstrophy is

\[
I = \int d^3\omega_0^2 \frac{\partial \omega_0^2}{\partial t} + \frac{1}{2} \int d^3\omega_0 \frac{\partial \omega_0}{\partial t}
\]

(18)

Expression (18) incorporates the assumption that the smeared vortex dipoles are separated from each other by a distance large in comparison with the smearing \( b \) (i.e., \( \delta - b \to 0 \)).

When viscous forces are taken into account, the enstrophy balance equation (16) takes the following form, to within the omitted terms \( O(\hat{\delta}^2/t^2) \):

\[
\frac{d}{dt} \frac{1}{2} \int d^3\omega_0^2 \frac{\partial \omega_0^2}{\partial t} + \frac{1}{2} \int d^3\omega_0 \frac{\partial \omega_0}{\partial t} = \frac{3\gamma^2(y)}{2\pi} \left[ \int d^3x \left( \frac{5}{\gamma^2} (\nabla \omega_0)^2 + \frac{1}{2} (\nabla \omega_0)^2 \right) - 2\omega_0^2 \right],
\]

(19)

We ignore the viscous forces only in the balance equation (19); everywhere below we assume that we are dealing with large Reynolds number, \( Re = \epsilon(t)/\omega \cdot y \sim 1 \) [for two vortex dipoles, the relative velocity is \( \omega(t) \sim y(t)/4\pi t^2(t) \), in which case we can ignore the effect of viscous dissipation on the relative Hamiltonian dynamics of the vortex dipoles described by Eqs. (9), (9').]

In (19) we are using a rather smooth representation of the function \( \hat{\delta}(x) \), which corresponds to, for example, (5) and (7) for \( \omega > 137 \) and for real values of the parameter \( y \) [corresponding to \( T_{\omega} = 0 \) at \( t = 0 \); see (A2)]. From (19) we find, in the limit \( \omega \to 1 \),

\[
d\omega dt = A(\tau) = \beta(\tau) \omega^2,
\]

(20)

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where
\[ u(t) = \frac{I(t)}{I(0)}, \quad I(t) = \frac{\gamma(t)}{4\delta(t)} \left( 1 + \frac{\gamma}{5} + \frac{8}{3\gamma} \right) \gamma(t) + O \left( \frac{1}{\gamma^2} \right), \]

\[ \tau_{c} = t_{0} = \frac{\gamma(\tau_{c})}{\gamma(0)}. \]

\( \tau_{c} \) is the time at which the two vortex dipoles collapse (at which they come together at the same point).\n
According to Ref. 6, we have the following expressions for \( \ell(t) \) and \( \gamma(t) \):
\[ \ell(t) = \left( \frac{25H}{2\pi} \right) \gamma(t)^{-1}, \]
\[ \gamma(t) = \gamma(0) \left( 1 - \frac{\gamma}{5} \right) \gamma(t)^{-1} \gamma(t)^{-1}, \]
where
\[ r^{*} = 4bM_{d} \left( -\cos \varphi_{1} - \frac{1}{2} \sin \varphi_{1} \right) / 5(1 - 3 \cos \varphi_{1}), \]
\[ \gamma_{0} = \gamma(0), \quad \gamma_{1} = \gamma(1), \]
\( \varphi_{1} \) and \( \varphi_{2} \) are the polar angles of the vectors \( \mathbf{l} \) and \( \gamma \) in the \((x, y)\) plane,
\[ \gamma_{1} = \gamma_{0} \left( 1 - \frac{\gamma}{5} \right) \gamma_{0}^{1/2}, \]
\[ \gamma_{0} = \gamma(0), \quad \gamma_{1} = \gamma(1), \]
\( \gamma(0) \) is the Mach number, \( e \) is the velocity of sound in the slightly compressible medium, with \( M_{d} < 1 \). This limit must be associated with an initial restriction on the scale of the smearing, \( H(\tau) > b(\tau) \).

In the limit \( H(\tau) \rightarrow H_{0} \), we have a viscous effect that is able to balance the threedimensional stretching of vortex lines which leads to the unbounded increase in the enstrophy over a finite time as vortex dipoles close each other in the limit \( r \rightarrow 1 \). Because of the pronounced increase in the Reynolds number, \( \Re_{\gamma}(r) > a_{\gamma}^{-1} \), as \( r \rightarrow 1 \), the viscous forces cannot substantially influence the relative Hamiltonian dynamics of point vortex dipoles, described by Eqs. (9), (9') and the corresponding solutions in (21) and (22) in this regime. In this situation, the intense acoustic emission of the collapsing vortex dipoles can apparently act as a dissipative mechanism which is more effective than the viscosity in limiting the explosive growth of the enstrophy [and eliminating the corresponding singularity \( \omega(\tau) = 0 \) as \( r \rightarrow 1 \)]...
from each other under the initial condition \( \phi_0 = \pi \). In this case we have \( (\gamma) = -y \); at any time \( t \neq 0 \) this situation corresponds to a positive value \( A(t) \neq 0 \) and the possibility that the enstrophy will grow in time. In this case we indeed find from (23) \( \{t(\gamma)\} = (1 + t)^{\gamma/3} \), \( y(t) = y(0)(1 + t)^{\gamma/3} \), \( A(t) = (1 + t)^{-\gamma/3} \), \( \beta(t) = \beta(0)(1 + t)^{\gamma/3} \) 

\[
\psi(t) = (1 + t)^{\gamma/3} \left[ \left( \frac{1}{\beta(0)} \right)^{1/\gamma} + \frac{1}{\beta(0)^{1/\gamma}} \right]^{-1/4}, 
\]

(25) 

where

\[
\tau = \left( \frac{L}{l} \right) \left( \frac{\nu}{\nu_0} \right), \quad \beta_0 = 6n \pi \nu_0 / h^3, \quad \nu = \frac{n \pi \nu_0}{h^3} \nu_0^n. 
\]

With \( \gamma > 1 \) and \( \beta_0 < 1 \) it thus follows from (25) that there can be a significant power-law increase in the enstrophy, since we have \( \psi(t) \approx O(t^{\gamma/3}) \) on the time interval \( 1 < t < \infty \). On this time interval we have an expansion of the localization region, \( h(t) \approx O(t^{2/3}) \). Furthermore, at \( \tau > \tau_0, \beta_0, \nu_0 \) we find \( h(t) \approx O(t^{2/3}) \), since in this limit we have \( \psi(t) \approx O(t^{-\gamma/3}) \) and \( \gamma(t) \approx O(t^{-\gamma/3}) \). There thus exists an intermediate time interval \( t \) in which we have \( h(t) \approx O(t^{2/3}) \); i.e., the increase in the average radius of the smearing of the vortex dipole, \( h(t) \) in time is proportional to the change in the distance between the vortex dipoles, since we have \( I(t) \approx O(t^{2/3}) \) and the condition \( I(t) \approx O(t^{-1/3}) \) definitely holds for \( I > \beta_0 \), \( \beta_0 \). The same tendency toward an increase in the radius of a spherical vortex proportional to the displacement of the vortex, was noted in Ref. 18, where this process was driven by buoyancy effects. It thus obviously follows from (25) that the role of viscous forces in the case \( M = 0 \) (\( c = 0 \)), is a very important one, and it may differ qualitatively from that in the explosive regime (24) for \( H = 0 \) and \( M \neq 0 \), since even in the case of an arbitrarily small viscosity \( \nu \) the viscous forces will be capable in principle, after a sufficiently long time \( t \gg \beta_0^{-2/3} \), of cancelling the effect of the stretching of the vortex lines.1 On the other hand, it follows from (25) that at sufficiently large Reynolds numbers \( Re_0 \gg 1 \) the relative increase in the enstrophy can reach extremely large maximum values before the viscous damping comes into play. Specifically, the function \( \psi(t) \) in (25) reaches its maximum value

\[
u_{\text{max}}(t_0) = \beta_0 \nu_0 \left( \frac{1.14 - \beta_0}{1.47 - \beta_0} \right) \psi(t_0),
\]

at

\[
t = \tau_0 = \left[ \frac{L}{l} \right] \left[ \frac{\nu}{\nu_0} \right] \left[ \frac{1.47 - \beta_0}{1.14 - \beta_0} \right]^{-1} - 1,
\]

(26) 

where \( \tau_0 > 0 \) only for sufficiently small values \( \beta < \beta_0 \), \( \beta_0 = 1.41 \), i.e., only at Reynolds number \( Re_0 \) exceeding a critical value

\[
Re_0 > Re_{\text{crit}} = 3 \nu_0 / \beta_0 \nu_0^n.
\]

where \( \alpha = 1 \) for \( \alpha = 1 \) we have \( \text{max} / \text{min} = O(Re_0^{11/8}) \). Consequently, in this case, \( M = 0 \) (\( c = 0 \)), the enstrophy can be significantly enhanced, to a maximum value \( \text{max} / \text{min} = O(Re_0^{11/8}) \) at \( t = \tau_0 = \left[ \frac{\nu}{\nu_0} \right] \left[ \frac{1.47 - \beta_0}{1.14 - \beta_0} \right]^{-1} \) only for Reynolds numbers above the critical value, \( Re_0 > Re_{\text{crit}} \).

The voricity bursts of large (but finite) amplitude which are observed in turbulent boundary layers are indeed linked with an interaction of (coaxial) dipole vortices.20

Accordingly, solutions (23)-(25) (especially (25)), derived above, can be of some use in interpreting the corresponding experimental data.21 In fact, the separation of a horseshoe-shaped vortex having a dipole structure22 from the wall which was observed in Refs. 21 and 22 may be thought of as a result of the interaction of this vortex with its coaxial mirror image (with respect to the plane of the wall) (or with a dipole vortex induced near the wall)22.23

The conclusions reached in Sec. 2 can also be used to model statistically uniform turbulence, if the integral in the definition of \( I \) (and \( I_1 \)) is replaced by a statistical averaging.24 Furthermore, it would be interesting to study the intensification of voricity when the effects of buoyancy and a temperature stratification of a medium are taken into account.

We also note that the conclusion, reached in Sec. 2, that a strong singularity of the enstrophy \( I \) can be reached in the three-dimensional case (in the limit \( t \rightarrow 1 \)) is itself independent of the nature of the regularization of the point vortex dipole (cf. the case with \( \gamma = 0 \) and 3). On the other hand, the index \( A_1 \) of the corresponding explosive increase of \( I \) nevertheless depends on the nature of the smearing of the \( \delta \)-func tion. This circumstance leads in turn to different tendencies in the time evolution of the smearing radius \( b \) (in contrast with the case in which there is only an expanding vortex region, studied in Ref. 18). Consequently, and in contrast with the two-dimensional case (in which the enstrophy is invariant, and only a weak singularity of the local characteristics of the velocity field is possible; see Subsection 1.5), in three-dimensional turbulence there can be a special mechanism by which energy is drained off to point vortex singularities. This mechanism would not depend on the presence of viscous dissipation, because of the strong singularity of the enstrophy which was mentioned in Sec. 2. The possibility that a mechanism of this sort would operate to drain off the energy of turbulence (and lead to the establishment of a corresponding universal Kolmogorov-Obukhov regime) because the solutions of the three-dimensional hydrodynamic equations lose their smoothness over a finite time (there is no need to introduce viscous dissipation in the region of small scales) was apparently studied first in the well-known paper by Onsager25 (see also Refs. 6 and 9 and the bibliographies there).

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APPENDIX

1. To derive the expression

\[
J(x) = \int \frac{dx}{(x^2 - 1) x^2},
\]

which corresponds to \( \delta(x) \) from (5), we use the representation

\[
\int \frac{dx}{(x^2 - 1) x^2} = \sum \frac{a_n \cos \phi}{x^2 - 1} \left( \frac{\delta(x) - \delta(-x)}{x} \right) - \sum \frac{a_n \sin \phi}{x^2 - 1} \left( \frac{\delta(x) + \delta(-x)}{x} \right),
\]

where

\[
\cos \phi = \cos \theta \cos \phi \sin \theta \sin \phi \cos \phi \sin \theta \sin \phi.
\]

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\( \theta \) and \( \varphi \) are the polar angles of the vector \( x \), and \( \theta' \) and \( \varphi' \) are those of the vector \( x' \), over which the integration is carried out. Making use of the orthogonality of the spherical harmonics,

\[
\int d\Omega' \sin \theta' \int d\varphi' P_n(\cos \theta') P_n(\cos \theta') = \begin{cases} 
 0, & m' \neq m \\
 4\pi \delta \left( \frac{m}{2\pi + 1} \right) P_n(\cos \theta), & m = m' 
\end{cases}
\]

we find the following expression for \( f(x) \):

\[
f(r, \theta, \varphi) = \frac{1}{r} \int \frac{1}{r} dx' \delta_0(x') + \int dx \delta_0(x) + \frac{\delta}{\delta \theta} \left( \frac{1}{\sin \theta} \int \frac{1}{r} dx' \delta_0(x') + \int dx \delta_0(x) \right).
\]

To find \( T_{\phi} \), we substitute (A1) into (6) and set \( \theta = 0 \) in \( I(r, \theta) \) \( [P_n(1) = 1] \). For \( \delta_0(r) \) \( (r) \), we then find

\[
T_{\phi} = \frac{np^2(\alpha + 3)^3(\alpha + 4)^3(\alpha + 5)^4}{90 \delta_0(\alpha + 2)(\alpha + 3)(2\alpha + 3)(2\alpha + 5)(2\alpha + 7)} \times \left[ g' \left( 1 + \frac{6p^3 + 143p^2 + 227}{(\alpha + 3)(\alpha + 4)(\alpha + 5)} \right) \right].
\]

In particular, with \( a_{\delta}(r) = a_{\delta}(\delta_0) \) \( (k = 0, 1) \) \( (k = 0, 1) \), \( a_{\delta} = 0 \) \( (k = 1) \), \( a_{\delta}/a_0 = r \) we have the following expression for \( T_{\phi} \) \( T_\phi = T_{\phi 0} + T_{\phi 1}, \) where \( T_{\phi 0} \) is found from \( T_\phi \), as \( f_2(r, \varphi) - f_2[r, \varphi = 0] \), in the case \( \delta_0(r) \)

\[
T_\phi = \frac{2p^2}{\delta_0(\alpha + 2)(\alpha + 3)} \left[ g' \left( 1 + \frac{4(\alpha + 3)}{(\alpha + 4)(\alpha + 5)} \right) \right] - \frac{2p^2}{\delta_0(\alpha + 2)(\alpha + 3)} \left[ \frac{4(\alpha + 3)}{(\alpha + 4)(\alpha + 5)} - \frac{8}{\alpha + 3} \right] + 2.
\]

(A4)

It follows from (A4) that we have \( a_{\phi} = 98 \) (we are rounding to integers), while in the case \( \alpha = a_{\phi} = 0 \) we have \( T_{\phi 0} = 0 \) for \( y = 2 \) and 1 in (A4) \( (\alpha = 98, \text{we have} \ T_{\phi 0} = 0 \text{for} \ y = 1.388 \pm 0.009) \).

For the case of complex \( y, \)

\[
\delta(x) = a_{\delta}(r) \left( 1 + \cos \varphi \cos \varphi' \right),
\]

where \( \langle \varphi \rangle = 0 \) \( (\varphi \text{is a random quantity}) \), we have, correspondingly, the following expression for the total self-energy, averaged over \( y; \)

\[
\langle T_\phi \rangle = -p^2 \delta_0(\alpha + 2) \left[ \frac{1}{2} \left( \frac{\delta_0(\alpha + 2)}{2} \right) + \frac{1}{2} \right] \delta(x) \delta(x'),
\]

(A1)

where

\[
a_{\delta} \delta_0 \int dr r \delta_0(r) = 1.
\]

In (A5) we have \( \langle y' \rangle \leq 2, \text{and at} \langle y \rangle \leq 2 \text{we have} \ T_{\phi 0} = 0. \text{3. In the three-dimensional case, the Helmholtz equation for the vortex field} a(x, t) \text{is given for the case of an ideal, incompressible fluid} \)

\[
- \Delta \vec{a}(x, t) = \vec{b}(x, t),
\]

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Substituting the expression for $s$ from (8) into (A6), and multiplying both sides of (A6) by an arbitrary smooth function $p(x)$, we can carry out the integration over the entire space, including the region in which point vortex dipoles (8) are concentrated. From (A6) we find

$$
\frac{\partial \psi_m}{\partial t} + \nabla \psi_m \cdot \mathbf{u} = \frac{\partial \psi_m}{\partial x} \cdot \mathbf{u} + \frac{\partial \psi_m}{\partial x^2} \cdot \mathbf{u} = 0.
$$

From (A8) and (A10) we find system (9), (9'). We also note that system (A8), (A10) [and, correspondingly, (9), (9')] was derived in Ref. 15 for the particular case of a weight function by carrying out a vector multiplication of (A6) [for $s$ from (8)] by $x$, followed by integration over the entire volume. Furthermore, the procedure used above is analogous to the derivation of a weak solution for two-dimensional point vortex dipoles which has been carried out by V. M. Gryazik (private communication) for the two-dimensional Helmholtz equation $i \Delta \psi = s(x, t)$.

Translated by Dave Parsons