Spontaneous singularities in three-dimensional turbulence and the emission of sound during strong dynamical interaction between point vortex dipoles

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An “explosive” growth of the power of acoustic emission occurs after a finite time when two non-coaxial point vortex dipoles (infinitesimally small vortex rings) approach one another.

Onsager 1 was apparently the first to mention the fundamental problem of spontaneous singularities in three-dimensional turbulence; this problem has been studied intensively from various angles in present-day hydrodynamics. 2

In particular, the author 3 has obtained an exact solution of the dynamics of point vortex dipoles (infinitesimally small vortex rings) corresponding to an unbounded explosive growth of the localized vorticity in a finite time upon collapse (convergence into a single point) of two non-coaxial vortex dipoles. A particularly stimulating role is played here by experiments of the Stanford group of Klein 4 and by others, 5 (see also Ref. 3) who observed “bursts” of localized vorticity in turbulent boundary layers. The recorded finite (albeit relatively large) amplitude of the vorticity in Refs. 7, 8 during the time of the explosions is, apparently, caused by some dissipative mechanisms. For instance, the emission of acoustic waves by the turbulence 6-10 may be such a factor limiting the explosive growth of the local vortex field.

In the present paper we consider the possibility of an anomalously strong sound generation in a weakly compressible medium during the collapse of a pair of non-coaxial point vortex dipoles. We define more precisely the existing ideas (see Refs. 9, 10) about the weak efficiency of turbulence as a sound emitter in the limit of small Mach numbers.

We solve the problem of the generation of vortex sound by using the method of the joining of asymptotic expansions, 11,12 in which the Mach number $M = \frac{v}{c} < 1$ is the small parameter, where $v(t)$ is the velocity of approach (along a logarithmic spiral trajectory 13) of non-coaxial vortex dipoles, and $c$ is the velocity of sound in a weakly compressible medium.

Let the two non-coaxial vortex dipoles have Lamb momenta which are equal in absolute magnitude, but which have opposite directions, $\rho_1 \mathbf{y}(t) = - \rho_2 \mathbf{y}(t) = \rho \mathbf{y}(t)$, and let they be at time $t$ at a distance $r = |x_1 - x_2|$ from one another, where $x_1(t)$ and $x_2(t)$ are the Cartesian coordinates of the first and the second vortex dipole, satisfying [like $\mathbf{y}(t)$] the dynamic set of equations given in Ref. 6. If initially at $t = 0$ the vectors $\mathbf{y}$ and $\mathbf{r}$ lie in the same $(x, y)$ plane, it follows from the angular momentum conservation law $\mathbf{M} = \rho_0 |\mathbf{y}| = \text{const}$ ($\rho_0$ is the unperturbed density of the medium) that they remain in the same plane also for any other $t > 0$. We shall start from this assumption about the initial conditions and characterize the direction of the vectors $y$ and $r$ in the $(x, y)$ plane by the polar angles $\varphi_1(t)$ and $\varphi_2(t)$, respectively.

The motion of the fluid outside the vortex dipoles is potential and is described by the velocity potential $\Phi$.

Choosing the origin of the spherical coordinate system $(r, \theta, \varphi)$ at the point $\mathbf{B} = [x_1(t) + x_2(t)]/2 = \text{const}$ (into which the vortex dipoles collapse) we get $x_1 = t(t)/2$, $x_2 = -t(t)/2$, and for the potential $\Phi$ we have in the limit $r \rightarrow \infty$ the expression

$$
\Phi = \frac{\gamma(x - x_2)}{4\pi|x - x_2|^3}. 
$$

Under the influence of the non-stationary pressure field corresponding to (1) the point vortex dipoles can generate acoustic oscillations $\Psi$, the propagation of which in the wave zone $r \gg \lambda$ ($\lambda$ is the wavelength of $\Psi$) is described by the equation $c^2 \partial^2 \Psi/\partial t^2 - \Delta \Psi = 0$, where $\Psi$ is the sound potential and $\Delta$ the three-dimensional Laplace operator. We shall apply a standard technique, 14,15 which uses an expansion of $\Psi$ in a series in the spherical functions $Y_{lm}$ and the radial Hankel functions $H^+_l(r/\lambda)$, to look for a solution $\Psi$ of this equation which satisfies the emission conditions as $r \rightarrow \infty$ and which is the same as the potential (1) in the vortex zone $\lambda \ll r \ll 1$ when $M \ll 1$. We then get from the equation $p = -\rho \partial^2 \Psi/\partial t^2$ for the oscillations of the pressure in the acoustic wave which is emitted by the pair of vortex dipoles in the wave zone $r \gg \lambda$,

$$
p(t + \lambda, r, \varphi) = \frac{5p \sin \beta \Omega}{4\pi c^2 r^2} (A_0 \cos (\varphi - q_0(t))) + A_0 \sin (\varphi - q_0(t)),
$$

where

$$
\frac{\lambda}{\rho_0} A_0 = 4H 4H + \frac{8H}{10} y^2 - 2 \left(\frac{F}{\rho_0}\right) \frac{1}{c^2} (\mathbf{F}^2) + \frac{7H^2}{10} y^2, \quad F = \frac{\gamma}{\rho_0},
$$

and $H = \frac{\gamma}{\rho_0}$.

1. Onsager.
2. See Ref. 3.
3. Chefranov.
5. Refs. 7, 8.
6. Ref. 6.
7. Ref. 9.
8. Ref. 10.
9. Refs. 11, 12.
11. Refs. 15.
is the invariant interaction energy of the vortex dipoles. In agreement with Ref. 6
\begin{equation}
\gamma_1 = \frac{9}{2} \phi_{1/2} + \frac{9}{2} \phi_{1/2},
\end{equation}
where
\begin{equation}
\Psi(t) = \frac{5}{6} \left( \phi_{1/2} + \frac{5 \phi_{1/2}}{2} \right).
\end{equation}
\end{center}

There are therefore in this approximation with respect to the small parameter \( Ma \ll 1 \) no emission in the direction \( \theta = 0 \) [i.e., in the \((x, \gamma)\) plane] and the frequency of the \( \phi \) oscillations increases without bounds in the time of the collapse of the vortex dipoles, i.e., \( \omega(t) \to \infty \) as \( l(t) \to 0 \).

2. In particular, for almost coaxial merging vortex dipoles the energy flux of the acoustic emission
\begin{equation}
I = \int \frac{\phi_1}{\phi_2} \sin \theta \, d\theta.
\end{equation}
(see Ref. 10) through the surface of a sphere of radius \( r = \lambda \) has, in accordance with (2), the form (in the limit as \( t \to t_0 \))
\begin{equation}
I = \frac{\lambda}{\pi} \left[ \frac{M_0}{(1 - t_0 \lambda)^{3/2}} + 1 + O(\phi_1) \right],
\end{equation}
where \( \phi_1 \ll 1 \) but \( \phi_2 \neq 0 \) when
\begin{equation}
\cos \phi_1 = \frac{\lambda}{\phi_2}, \quad t_0 = 2(1 - \phi_2) > 0.
\end{equation}
\( M_0 = \lambda/\epsilon \ll 1 \), \( \lambda_0 = \lambda/\epsilon \), \( \epsilon = \rho_0 \lambda^2 |\phi_2| / 60 \phi_1 \) is the magnitude of the vortex energy flux. In this limit \( Ma = |\phi(t)| / \epsilon = M_0(1 - t / t_0)^{3/2} \) and the applicability of (3) is clearly justified under the condition
\begin{equation}
(1 - t_0 \lambda)^{3/2} < M_0(\lambda)(t) < 1
\end{equation}
(i.e., \( M_0(\lambda)(t) < 1 - t_0 \lambda < M_0(\lambda)(t) \)), when the acoustic efficiency
\begin{equation}
K = \frac{I}{\gamma_1} = \frac{M_0(\lambda)(t)}{(1 - t_0 \lambda)^{3/2}},
\end{equation}
corresponding to (3) becomes already close to unity. The situation is not changed quantitatively for larger \( \phi_2 \) since we have, for instance for \( \phi_2 = \pi \)
\begin{equation}
I = \int \frac{M_0(\lambda)(t)}{\pi} \left[ 1 - \frac{t}{(1 - t_0 \lambda)^{3/2}} \right]^{\gamma_1/2}.
\end{equation}

At the same time we have for coaxial vortex dipoles (\( \phi_2 = 0 \) or \( \phi_2 = \pi \)) already \( I = O(M_0^2) \). We note that the estimate \( I = O(M_0^2) \) in Ref. 11 (see also Ref. 9) for the sound emission intensity of two coaxial vortex rings of finite radius \( R(t) \) is obtained in the limit when \( l(t) \ll R(t) \)—of small distances \( l(t) \) between the centers of the rings—and is determined by the effect of the periodic time dependence of \( R(t) \) in the "vortex leap-frogging" process. In the present paper, however, we consider essentially the opposite limit \( l(t) \gg R(t) \), (e.g., when \( R = M_0 \lambda_x = R_0 \), \( 1 - t / t_0 \gg M_0(\lambda)(t) \) and \( l(t) \approx O((1 - t / t_0)^{3/2}) \) simulated by the dynamics of point vortex dipoles which do not change their structure even as \( l(t) \to 0 \).

Equation (3) thus shows that the emission of vortex sound at times close to \( t_0 \) can be very efficient notwithstanding that the magnitude of the acoustic efficiency \( K = O(M_0^2) \), as is usually the case for sound emission by turbulence in a weakly compressible medium. The possibility of similar, although appreciably weaker, effects for the magnification of \( K \) was obtained for point vortices in two-dimensional hydrodynamics, and also in Ref. 14 for vortex dynamics (vorton dynamics itself, however, does not satisfy all conservation laws of the three-dimensional equations of hydrodynamics, in contrast to the dynamics of point vortex dipoles). In connection with the results obtained above there is interest in developing experimental studies related to those described in Ref. 15: of acoustic radiation by small vortex rings (with \( R - R_0 \) which collide at a nonzero angle, and also the realization of acoustic time measurements of the vorticity bursts observed in a turbulent boundary layer.

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