A black hole in a magnetic universe

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The general properties of a rotating electrically charged black hole in an external magnetic field are examined, using the Ernst-Wild metric. We find the electromagnetic potential corresponding to a uniform magnetic field oriented parallel to the rotation axis of the black hole. It is shown that in the presence of an external magnetic field, a rapidly rotating black hole may have angular momentum and electric charge exceeding those permitted for a Kerr-Newman black hole. Such a highly extreme black hole nevertheless possesses an event horizon, and does not evolve into a naked singularity. We find the parameters of a black hole in electrical equilibrium in a uniform magnetic field. The magnetic flux through a hemisphere of arbitrary radius which is symmetric with respect to the equatorial plane is calculated in the Ernst-Wild metric. The magnetic flux through the upper hemisphere of the horizon of an equilibrium black hole in a uniform magnetic field is found not to depend on the angular momentum of the black hole. We determine how the parameters of a highly extreme black hole transform when the uniform magnetic field is turned off.

1. INTRODUCTION

The physics of electromagnetic phenomena in the vicinity of accreting black holes has undergone active development in recent years.[1–7] This field, known as black hole electrodynamics, holds out the prospect of a quantitative explanation for quasars and active galactic nuclei. The electric and magnetic fields engendered by plasma accretion onto massive black holes make it possible, in principle, to accelerate particles to the highest energies observed in cosmic rays.[1,6,7] The schemes proposed for electrodynamic accretion assume that a magnetosphere with a regular distribution of electric and magnetic fields, induced by the plasma flow onto the black hole, is formed around the latter. Although there is as yet no self-consistent picture of such a flow, qualitative arguments and simple model calculations indicate that it is possible for a quasiuniform magnetic field distribution to be produced dynamically near the black hole horizon, due to rapid smoothing of fine-scale irregularities and loops in the accreting plasma.[6,8–10] A model with a uniform external field is therefore a fairly good approximation with which to explain the qualitative features of black hole behavior in a magnetic field. The external field is usually considered to be a probe of the black hole metric background. Wald[11] has shown that a stationary axisymmetric black hole in equilibrium, with angular momentum J, possesses an electric charge $q_0 = 2B_0J$, where B_0 is the external magnetic field strength. In particular, for a Kerr black hole of mass m, the angular momentum is $J = ma$, and the angular momentum per unit mass can take on values $\hat{a}^2 \leqslant m^2$. The existence of a finite electric charge on a rotating black hole in equilibrium, if Wild's result holds qualitatively in the presence of plasma, will have a significant effect on charged particle trajectories, and can result in rearrangement of the whole magnetosphere. Another question which arises is how a black hole in extreme rotation can acquire a charge $q_0 = 2B_0m^2$ without becoming a naked singularity. One can only answer this question by considering the reciprocal influence of the external magnetic field on the black hole.

In the present paper, we will examine some of the properties of a black hole in an external magnetic field using the Ernst-Wild metric, which is an exact solution of the Einstein-Maxwell equations. The Ernst-Wild solution is a stationary, axisymmetric magnetic universe having a magnetic field of arbitrary strength, containing a rotating, electrically charged black hole. In the limit of a gravitationally weak magnetic field in a finite region around the black hole, this metric describes a Kerr–Newman black hole immersed in a uniform magnetic field. Consideration of the more general case, which allows for the effect of the magnetic field back on the black hole, makes it possible to detect qualitatively new features of the behavior of a black hole in an external field, as well as clarifying the physical meaning of previous results obtained via perturbation methods. In particular we will show that a magnetized, rapidly rotating extremal black hole can have a specific angular momentum and electric charge which exceed the allowable values for a Kerr-Newman black hole. The excess specific angular momentum and electric charge will not, however, transform such a highly extreme black hole into a naked singularity. We will find a potential to specify an external magnetic field which is uniform at large distances from an arbitrarily charged, rotating black hole. In equilibrium, as it turns out, such a black hole has the same electric charge as that given by Wald[11], even when the rotation parameter takes its limiting value $a = m$. The presence of an electric charge has a considerable effect on the magnitude of the magnetic flux passing through the upper and lower hemispheres of the black hole horizon, which in turn may influence the efficiency of extraction of black hole rotational energy. Specifically, the magnetic flux through the upper hemisphere of the horizon of a maximally rotating uncharged black hole whose rotation axis is parallel to an external probe magnetic field is zero.[12,13] It will be shown, however, that when a black hole has an equilibrium electric charge, the magnetic flux does not depend on the angular momentum. We will also consider the possible transformation undergone by the parameters of a highly extreme black hole when a uniform external magnetic field is turned off.

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2. THE ERNST-WILD METRIC

In the stationary axially symmetric case, the Einstein-Maxwell equations reduce the equations for the complex potentials $\mathscr{E}$ and Φ, which depend on two variables such as the radius r and polar angle θ.[14,15] The metric can then be represented in the form[16]

$$ds^2=f^{-1}(-P^2d\zeta d\zeta^*+\rho^2dt^2)-f(d\varphi-\omega dt)^2, \quad (1)$$

where the real functions $f<0,P,\rho,\omega$, and the complex conjugate functions ζ and ζ^* depend on R and θ and φ and t are the azimuthal angle and the time, respectively. For the Kerr-Newman metric,

$$d\zeta=\Delta^{-1/2}dr+id\theta, \quad P=(A^{1/2}\sin\theta)^{-1}, \quad \rho=\Delta^{1/2}\sin\theta, \quad f=f_0=-A\sin^2\theta/\Sigma, \quad \omega=\omega_0=(2mr-e^2)a/A; \quad (2)$$

$$\Delta=r^2-2mr+a^2+e^2, \quad \Sigma=r^2+a^2\cos^2\theta, \quad A=(r^2+a^2)^2-a^2\Delta\sin^2\theta. \quad (3)$$

In these expressions m, a, and e are the mass, specific angular momentum, and charge of the metric as measured by an observer at infinity. For the Kerr-Newman metric, the complex potentials $\mathscr{E}$ and Φ are given by

$$\Phi=\Phi_0=-ie\cos\theta+ea\sin^2\theta(r+ia\cos\theta)^{-1}, \quad (4)$$

$$\mathscr{E}=\mathscr{E}_0=-[(r^2+a^2)\sin^2\theta+e^2\cos^2\theta]+i2ma\cos\theta(3-\cos^2\theta) -2a\sin^2\theta(ma\sin^2\theta+ie^2\cos\theta)(r+ia\cos\theta)^{-1}. \quad (5)$$

The orthonormal components of the electromagnetic field in a locally nonrotating frame of reference (LNFR), in which the observer moves along world lines with $r=\text{const}$, $\theta=\text{const}$, and $\varphi=\omega t+\text{const}$, may be expressed in terms of the potential Φ as follows:

$$H^{(r)}+iE^{(r)}=P(\partial\Phi/\partial\theta), \quad H^{(\theta)}+iE^{(\theta)}=-P\Delta^{1/2}(\partial\Phi/\partial r). \quad (6)$$

The Ernst-Wild metric,[17] which generalizes the Kerr-Newman solution to the case in which there is a magnetic field of strength B_0 parallel to the rotation axis of a black hole, can be obtained by replacing the functions f_0 and ω_0 in (2) by new functions f and ω with

$$f=|\Lambda|^{-2}f_0, \quad (7)$$

$$\Lambda=1+B_0\Phi_0-{}^1/_4B_0{}^2\mathscr{E}_0, \quad (8)$$

and deriving the potentials Φ_0 and $\mathscr{E}_0$ from (4) and (5). The angular rotation speed ω of the metric can be represented in the form

$$\omega=(\alpha-\beta\Delta)(r^2+a^2)^{-1} \quad (9)$$

when $e=0$, with

$$\alpha=a(1-B_0{}^4m^2a^2),$$
$$\beta=a\Sigma A^{-1}+{}^1/_8B_0{}^4ma\{-r[3\sin^4\theta+4\cos^2\theta(3-\cos^2\theta)] +a^2A^{-1}\sin^6\theta[(r^2+a^2)r+2ma^2]+2ma^2A^{-1}\cos^2\theta \times[(r^2+a^2)(3-\cos^2\theta)^2-4a^2\sin^2\theta]\}, \quad (10)$$

and when $e\neq0$, to first order in B_0, with

$$\alpha=a-2B_0er, \quad \beta=(a/A)(\Sigma+2B_0era\sin^2\theta). \quad (11)$$

The electromagnetic field components in a LNFR in the Ernst-Wild metric can be found using (6), in which

$$\Phi=\Lambda^{-1}(\Phi_0-B_0\mathscr{E}_0/2), \quad (12)$$

where Φ_0, $\mathscr{E}_0$, and Λ are given by (4), (5), and (8) respectively. As is the case in the Kerr-Newman metric, the black hole horizon in the Ernst-Wild metric can be obtained by setting $\Delta=0$ in (3). The radius of the black hole horizon, which is equal to the larger of the two roots of the equation $\Delta=0$, is

$$r_H=m+(m^2-a^2-e^2)^{1/2} \quad (13)$$

and is independent of B_0. Just as for the Kerr-Newman metric, it is necessary to have $a^2+e^2\leqslant m^2$ for a black hole horizon to exist, and this determines the possible values of the parameters a and e. For $m=0$, the Ernst-Wild solution is transformed into a solution known as the Melvin magnetic universe.[18–20] The latter is a stable self-gravitating magnetic field configuration with a single nonvanishing component along the symmetry axis; it varies according to

$$H_z=B_0[1+({}^1/_4)B_0{}^2l^2]^{-2}, \quad (14)$$

where l is the coordinate distance along the symmetry axis. With $m\neq0$ and a magnetic field gravitationally weak compared with the black hole, and characterized by the dimensionless parameter

$$B_0m=4{,}2\cdot10^{-20}(B_0/1\text{ G})(m/M_\odot\ll1), \quad (15)$$

there exists a region defined by the condition $B_0{}^2r^3\ll$ in a magnetic Ernst-Wild universe within which test particles do not "feel" the gravitational influence of the magnetic field B_0, by virtue of the small contribution it makes to the curvature compared with that of the black hole.[21–23] The outermost part of this region, specified by $(B_0r)^2\ll m/r\ll1$, is almost flat (Newtonian). At the boundaries of the Newtonian region, the magnetic field can be considered uniform. All realistic astrophysical objects apparently satisfy condition (15). The "curtailed" part of the Ernst-Wild metric $B_0{}^2r^3\ll m$, with $B_0m\ll1$, can be treated as a physical model of a black hole in a uniform magnetic field, assuming that the sources of the magnetic field are confined to within this region. Outside the magnetic field region occupied by the sources, the "curtailed" Ernst-Wild metric then goes into an asymptotically flat metric with a black hole "weighed down" by the matter and magnetic field which surround it.

The quantities which characterize a black hole in a magnetic universe, such as mass, angular momentum, and electric charge, will depend on B_0. In a magnetic universe, the parameters m, a, and e therefore lose the physical significance they had in the case $B_0=0$. Below, we find the B_0-dependence of the fundamental characteristics of a black hole in an Ernst-Wild universe.

3. ELECTRICAL CHARGE OF A BLACK HOLE IN A MAGNETIC UNIVERSE

The total electric charge q on the three-dimensional hypersurface Σ_μ containing a black hole and the currents j^μ external to the black hole horizon is

$$q=\int j^\mu d^3\Sigma_\mu+q_H, \quad (16)$$

where the electric charge q_H of the black hole is given by Gauss' Law expressed as an integral of the electromagnetic field $F^{\mu\nu}$ over the surface of the black hole horizon:

$$q_H=\frac{1}{8\pi}\oint_H F^{\mu\nu}d^2\Sigma_{\mu\nu}. \quad (17)$$

Here we employ the following parametrization of the surface element in the coordinate system of the X^α

$$d^2\Sigma_{\mu\nu}=\frac{1}{2!}\varepsilon_{\mu\nu\alpha\beta}\frac{\partial(x^\alpha,x^\beta)}{\partial(\theta,\varphi)}d\theta\,d\varphi, \qquad (18)$$

where $\varepsilon_{\mu\nu\alpha\beta}$ is the antisymmetric unit tensor. We may write the Maxwell equations $F^{\mu\nu}{}_{;\nu}=4\pi j^\mu$ in the form

$$j^\mu=\frac{1}{4\pi(-g)^{1/2}}\frac{\partial}{\partial x^\nu}(F^{\mu\nu}(-g)^{1/2}), \qquad (19)$$

where the determinant of the metric tensor $g_{\mu\nu}$ for the Ernst-Wild metric is

$$g=-|\Lambda|^4\Sigma^2\sin^2\theta. \qquad (20)$$

Using (6) for the components of the electromagnetic field in a locally nonrotating coordinate system, and expressing the field components $F^{\mu\nu}$ in terms of these in the (t,r,θ,φ) system, we obtain for the electric charge density outside the black hole

$$j^t=\frac{1}{4\pi(-g)^{1/2}}\left[\frac{\partial}{\partial r}(P^{-1}E^{(r)})+\frac{\partial}{\partial\theta}(P^{-1}\Delta^{-1/2}E^{(\theta)})\right]$$
$$=\frac{1}{4\pi(-g)^{1/2}}\mathrm{Im}\left(\frac{\partial^2\Phi}{\partial r\,\partial\theta}-\frac{\partial^2\Phi}{\partial\theta\,\partial r}\right)=0. \qquad (21)$$

Taking the time-independence of the metric into account, and choosing as the hypersurface Σ_μ a volume bounded by the radius $r\to\infty$, we obtain for the charge due to external currents in (16)

$$q_{ext}=\int j^\mu d^3\Sigma_\mu=\int j^t d^3\Sigma_t=0. \qquad (22)$$

Thus, the entire electric charge of the Ernst-Wild metric is contained in the black hole.[1)] Taking Gauss' Law and Eq. (22) into account, we can write the electric charge on the black hole in the form

$$q_H=q=\frac{1}{8\pi}\oint F^{\mu\nu}d^2\Sigma_{\mu\nu}$$
$$=\frac{1}{4\pi}\oint\left[E^{(r)}-\frac{\partial r(\theta,\varphi)}{\partial\theta}\Delta^{-1/2}E^{(\theta)}\right]A^{1/2}\sin\theta\,d\theta\,d\varphi$$
$$=\frac{1}{2}\mathrm{Im}[\Phi(r_2,\pi)-\Phi(r_1,0)], \qquad (23)$$

where r_2 and r_1 are the points at which the rotation axis of the black hole intersects an arbitrary closed surface of integration $r(\theta,\varphi)$ enclosing the singularity. The last equality in (23) makes use of (6), which relates the components of the electromagnetic field to the potential Φ. It can be seen from (4), (5), (8), and (12) that when $\theta=0$ or $\theta=\pi$, the potential Φ is independent of r, and consequently, as might be expected the integral in (23) does not depend on the shape of the surface of integration. Substituting Φ from (12) and (23), we obtain

$$q=\frac{e[1-{}^1/_4B_0^2e^2]+2B_0ma}{[1+{}^1/_4B_0^2e^2]^2+B_0^2(e+B_0ma)^2}. \qquad (24)$$

All of this charge is concentrated at the singularity. For $B_0m\ll1$ (uniform magnetic field approximation), the electric charge on the black hole is

$$q=e+2B_0ma. \qquad (25)$$

When $a=0$, this charge can exceed in absolute value the allowable electric charge on a Kerr-Newman black hole with the same parameters m and a.

The fact that the electric charge on a black hole depends on the quantity B_0 permits us to suggest that an equilibrium charge may be induced on a magnetized black hole by an external magnetic field. To determine this charge, we find the electromagnetic potential A^μ around a magnetized black hole. The time-independence axisymmetry of the Ernst-Wild metric enable us to use the gauge $A_r=A_\theta=0$. In the formalism of Ernst,[14,15] the potential Φ giving the electromagnetic field is

$$\Phi=A_\varphi+iA_t',$$

the component A_t and the auxiliary field A_t' by the differential equation

$$\nabla A_t=if^{-1}\rho\nabla A_t'-\omega\nabla A_\varphi, \qquad (26)$$

where

$$\nabla=\Delta^{1/2}\frac{\partial}{\partial r}+i\frac{\partial}{\partial\theta}. \qquad (27)$$

Here ρ, Δ, f, and ω are defined by Eqs. (2), (3), (7), and (9) respectively. To first order in B_0 (or to second order when e 0), it is easy enough to solve for the real and imaginary parts of Eq. (26), giving

$$A_t=-\frac{er}{\Sigma}-\frac{aB_0}{\Sigma}\left[mr\sin^2\theta-e^2\left(1-4\cos^2\theta-2\frac{a^2}{\Sigma}\sin^2\theta\cos^2\theta\right)\right], \qquad (28)$$

$$A_\varphi=e\frac{ar}{\Sigma}\sin^2\theta+\frac{1}{2}B_0\left(r^2+a^2+2mr\frac{a^2}{\Sigma}\sin^2\theta\right)\sin^2\theta$$
$$+B_0e^2\left\{\left[\left(\frac{r^2+a^2}{\Sigma}\right)^2+\frac{1}{2}\right]\cos^2\theta\right.$$
$$\left.+\frac{a^2}{\Sigma}\left(\cos^2\theta-\frac{r^2}{\Sigma}\sin^2\theta\right)\sin^2\theta\right\}. \qquad (29)$$

The potential found in this way specifies the electromagnetic field distribution about a rotating charged black hole in a magnetic field of strength $B_0\ll m^{-1}$ which is oriented parallel to its rotation axis and is uniform at large distances from the black hole. This is a generalization of the potential given by Wald[11] to the case of a black hole with arbitrary electric charge, and is consistent with the results of Wald to terms linear in e, taking into account the relationship (25) between B_0, a, e, and the charge on the black hole.

In order to find the equilibrium charge on a black hole in a uniform magnetic field, we make use of the concept of the comoving potential[24]

$$V=-(\eta^\mu+\omega\psi^\mu)A_\mu=-(A_t+\omega A_\varphi) \qquad (30)$$

in the stationary and axially symmetric gravitational field (1), which is characterized by the Killing vectors $\eta^\mu=(1,0,0,0)$ and $\psi^\mu=(0,0,0,1)$. The comoving potential V is a generalization of the electrostatic potential to the case in which an angular rotation rate ω is present in the metric. In point of fact, when moving in the gravitational and electromagnetic fields in question, a test particle of mass m and charge e conserves its total energy $E = -p_\mu \eta^\mu = \mu U_t - \varepsilon A_t$, where p^μ is the generalized momentum of the particle, and u_t and A_t, are the covariant time component of the particle four-velocity and electromagnetic potential in the (t, r, θ, φ) coordinate system. If the particle is stationary, the electromagnetic contribution to E consists of its electrostatic energy. For $\omega \neq 0$, however, the concept of immobility loses its global sense, and becomes local.[2] Particles at rest in a LNFR will in fact be stationary with respect to the metric. Such particles rotate with the geometry along trajectories given by $\varphi = \omega t + \text{const}$, $r = \text{const}$, $\theta = \text{const}$. In the coordinate system $(t,r,\theta\tilde{\varphi}; d\tilde{\varphi} = d\varphi - \omega dt)$, which is rotating with respect to a distant observer, the total particle energy may be written in the format $E = -\mu U_t + \varepsilon V + \epsilon L$, where V is given by Eq. (30), and $L = p_\mu \psi^\mu = p_\phi$ is the conserved generalized angular momentum of the particle. For a particle which is stationary in the $(t, r, \theta, \tilde{\varphi})$ coordinate system, and is therefore stationary with respect to the LNFR, the energy

$$E = \mu\rho(-f)^{-1/2} + \varepsilon V + \omega L = \mu|\Lambda|(\Sigma\Delta/A)^{1/2} + \varepsilon V + \omega L$$

is the sum of the mechanical potential energy, the electrostatic energy εV, and the energy of rotation ωL relative to a distant observer. It can be seen from these expressions for E that the quantity $V = -(A_t + \omega A_\phi)$ physically represents the electrostatic potential in a metric with $\omega \neq 0$.

The electrostatic energy of a test particle injected slowly with respect to the LNFR from infinity into a black hole is

$$E_\varepsilon = \varepsilon[V(r_H) - V(\infty)]. \tag{31}$$

By "infinity" in the present case, we mean a Newtonian region $(B_0 r)^2 \ll m/r \ll 1$ in the truncated Ernst-Wild metric. Then $V(\infty) = 0$. On the other hand, according to a theorem of Carter (Ref. 24, p. 173), the comoving potential at the horizon, $V_H = V(r_H)$ is a constant. The electrostatic injection energy (31) therefore turns out not to depend on the path taken by the injected particle. This energy is essentially the energy difference between charged and uncharged particles of differing mass injected into the black hole slowly relative to the metric, or the energy of slow injection of a charged particle of negligible mass. When $E_\varepsilon < 0$, it is energetically favorable to have accretion of particles with the same sign of ε; conversely, when $E_\varepsilon > 0$, charges with sign opposite to ε are accreted. Making use of (9), (28),a nd (29), we obtain

$$E_\varepsilon = \varepsilon V_H = (1/2)\varepsilon e(2r_H + 3B_0 ea)(r_H^2 + a^2)^{-1}. \tag{32}$$

Hence, we find that the condition for electrostatic equilibrium $E_\varepsilon = 0$ is satisfied when $e = 0$. In equilibrium, according to (25), a black hole has a charge $q_0 = 2B_0 ma$. The equilibrium charge q_0 is the same as that found by Wald[11] in considering the equilibrium of a weakly charged black hole in a uniform test magnetic field. In the present case, however, the equilibrium condition has been derived with the influence of the magnetic field back on the black hole taken into account, and it is also valid for the extreme value of the rotation parameter $a = m$. It is significant that a black hole rotating at its extremal rate, with equilibrium charge $q_0 = 2B_0 m^2$, has a horizon at $r_H = m$, according to (13), and does not evolve into a naked singularity.

In the case of an equilibrium black hole, the comoving potential V is identically zero everywhere outside the black hole, as follows from (9), (28), and (29) with $e = 0$. For particle injection along the black hole rotation axis, where immobility in the LNFR corresponds to absolute immobility, the electrostatic injection energy then vanishes by virtue of the absence (to second order in B_0) of an electric field anywhere on the rotation axis, with $e = 0$.[17] In particular, in the Newtonian region, the electric field in the LNFR takes the form

$$E^{(r)} = (e + 3B_0 ma \sin^2\theta)/r^2, \quad E^{(\theta)} = -2B_0 ma^3 \sin^3\theta \cos\theta/r^4. \tag{33}$$

In section 6, we will drive the equilibrium condition $e = 0$ for black hole in a uniform magnetic field, starting with the first law of black hole in a uniform magnetic field, starting with the first law of black hole thermodynamics. For a gravitationally strong magnetic field $B_0 \gtrsim m^{-1}$, there is no Newtonian region, and the electric equilibrium state of the black hole becomes indeterminate. We will nevertheless refer to a black hole in a strong magnetic field with $e = 0$ as being in equilibrium, meaning that it is actually in the transition to a week field $B_0 \ll m^{-1}$.

4. MASS AND ANGULAR MOMENTUM OF A BLACK HOLE IN A MAGNETIC UNIVERSE

When the metric is both stationary and axisymmetric, the existence of temporal and axial Killing vectors η^μ and ψ^μ enables one to write out integral relations for the mass and angular momentum contained within an isolated region:

$$M = -\int (2T^\mu{}_\nu - \delta^\mu{}_\nu T)\eta^\nu d^3\Sigma_\mu = -\frac{1}{4\pi}\int R^\mu{}_\nu \eta^\nu d^3\Sigma_\mu = \frac{1}{4\pi}\int \eta^{\mu;\nu}{}_{;\nu} d^3\Sigma_\mu = \frac{1}{8\pi}\oint \eta^{\mu;\nu} d^2\Sigma_{\mu\nu}, \tag{34}$$

$$J = \int T^\mu{}_\nu \psi^\nu d^3\Sigma_\mu = \frac{1}{8\pi}\int \left(R^\mu{}_\nu\psi^\nu - \frac{1}{2}R\psi^\mu\right) d^3\Sigma_\mu = -\frac{1}{8\pi}\int \psi^{\mu;\nu}{}_{;\nu} d^3\Sigma_\mu = -\frac{1}{16\pi}\oint \psi^{\mu;\nu} d^2\Sigma_{\mu\nu}. \tag{35}$$

The manipulations involved in (34) and (35) employed the Einstein equations

$$R_{\mu\nu} - {}^1/_2 g_{\mu\nu} R = 8\pi T_{\mu\nu}, \tag{36}$$

where $R_{\mu\nu}$ is the Ricci tensor, and $T_{\mu\nu}$ is the energy-momentum tensor of the matter; the equation for the commutator of the second derivatives of two arbitrary vectors,

$$\xi_{\mu;\nu\alpha} - \xi_{\mu;\alpha\nu} = R_{\mu\beta\nu\alpha}\xi^\beta, \tag{37}$$

where $R_{\mu\beta\nu\alpha}$ is the Riemann tensor; the equation for the Killing vectors,

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0 \tag{38}$$

and finally Stokes' theorem. The final expressions in Eqs. (34) and (35) can be considered to define the mass and angular momentum inside some closed two-dimensional surface, for metrics which contain black holes. Integration to spatial infinity then gives the total mass and total angular momentum for an asymptotically flat metric.[24,26] The Ernst-Wild metric is not asymptotically flat, and the total mass given by (34) is formally infinite. However, the total angular momentum for this metric is finite, at least when the angular rotation rate of the metric given by (9)–(11) is known. As already noted in section 2, in a real universe, only the truncated part of the Ernst-Wild metric is physically realizable, containing the black hole and a regular magnetic field with sources restricted to a finite volume. In that event, the total metric is asymptotically flat, and the integral relations (34) and (35) retain their usual meaning of mass and angular momentum for an observer located in the Newtonian part of the metric. Even in the case of a total Ernst-Wild metric, however, we will use the designations "mass' and "angular momentum" for the M and J derived from (34) and (35), considering these to be characteristics which generalize the corresponding concepts to the case of a gravitationally strong magnetic field. For the mass and angular momentum contained in a sphere of radius r, we then obtain

$$M(r)=(r-m)-\frac{1}{4}\int_0^\pi\left[|\Lambda|^2\frac{\Delta\Sigma}{A}\frac{\partial}{\partial r}\left(\frac{A}{\Sigma|\Lambda|^2}\right)+\frac{A^2\sin^2\theta}{\Sigma^2|\Lambda|^4}\omega\frac{\partial\omega}{\partial r}\right]\sin\theta\,d\theta, \quad (39)$$

$$J(r)=-\frac{1}{8}\int_0^\pi\frac{A^2\sin^3\theta}{\Sigma^2|\Lambda|^4}\frac{\partial\omega}{\partial r}d\theta, \quad (40)$$

where the integrands are given by Eqs. (3), (8), and (9). Specifically, for the case $B_0=0$ (Kerr-Newman metric), we have

$$J(r)=ma-\frac{1}{2}e^2\frac{a}{r}\left[1-\frac{r^2+a^2}{2a^2}\left(1-\frac{r^2+a^2}{ra}\operatorname{arctg}\frac{a}{r}\right)\right], \quad (41)$$

and the total angular momentum is $J(\infty)=ma$. In the Kerr-Newman metric, the expression for $M(r)$ is quite complicated, but it simplifies considerably at $r=r_H$:

$$M(r_H)=m-\frac{1}{2}\frac{e^2}{r_H}\left(1+\frac{r^2+a^2}{ar_H}\operatorname{arctg}\frac{a}{r_H}\right). \quad (42)$$

The total mass in the Kerr-Newman metric is $M(\infty)=m$. We will call the quantity $M_H=M(r_H)$ the mass of the black hole, and $J_H=J(r_H)$ its angular momentum. In the Kerr-Newman metric with $e\neq 0$, a fraction of the angular momentum and mass are included in the electromagnetic field outside the black hole horizon (this follows from (41) and (42)), and the specific angular momentum of the metric, $a=J(\infty)/M(\infty)$, is not the same as that of the black hole, $a_H=J_H/M_H$. For a black hole in a magnetic universe, with the limiting value of the rotation parameter $a=\pm m$ and the equilibrium electric charge (24), carrying out the integration in (39) and (40) with $r=r_H$ gives

$$M_H=m\left(1-\frac{4B_0^2m^2}{1-B_0^4m^4}\operatorname{arctg}\frac{1-B_0^2m^2}{1+B_0^2m^2}\right), \quad (43)$$

$$J_H=\pm m\frac{M_H}{1-B_0^4m^4}. \quad (44)$$

Similarly, in the uniform magnetic field limit for a rapidly rotating extremal black hole with parameters $m\ll B_0^{-1}$, $a^2+e^2=m^2$, $e^2\ll m^2$ we obtain (to terms bilinear in e and B_0).

$$M_H=m\left[1-\frac{1}{2}\left(1\dot{+}\frac{\pi}{2}\right)\frac{q^2}{m^2}+\frac{1}{2}\frac{q_0^2}{m^2}\right], \quad (45)$$

$$J_H=ma\left(1-\frac{\pi}{4}\frac{q^2}{m^2}\right), \quad (46)$$

where the electric charge on the black hole is $q=e+2B_0ma$, and $q_0=2B_0ma$. The specific angular momentum of such a black hole is $a_H=\pm[1+q_0(q-q_0)m^{-2}]m$, and the sign of a_H is determined by the sign of a. For a black hole rotating extremally, with equilibrium charge $q=q_0$, we see from (44) that $a_h=\pm(1-B_0^4m^4)m$. Clearly, in a uniform magnetic field, the absolute value of the specific angular momentum of a rapidly rotating extremal black hole with electric charge $q\geqslant q_0$ (if $q_0>0$) or $q\leqslant q_0$ (if $q_0<0$) is larger than for an extremal Kerr-Newman black hole with the same value of m. Such a highly extreme black hole nevertheless possesses an event horizon with $r_H=m$.

The condition for the absence of rotation of a black hole in an external magnetic field is $J_H=0$. The integration in (40), to first order in B_0, and with $a\ll m$ and $r\ll B_0^{-1}$, gives

$$J(r)=\left(m-\frac{2}{3}\frac{e^2}{r}\right)a-\frac{1}{3}B_0er^2. \quad (47)$$

Hence, for the rotation parameter a_0 of the corresponding black hole in a uniform magnetic field, we obtain

$$a_0=B_0er_H^2/(2r_H-m). \quad (48)$$

Bearing in mind the approximations being used, the ra dius of the event horizon in this expression is $r_H=m+(m^2-e^2)^{1/2}$.

5. THE MAGNETIC FLUX THROUGH A HEMISPHERE OF THE BLACK HOLE HORIZON

One important quantity which characterizes the interaction of a black hole with an external magnetic field is the magnetic flux through one of the two hemispheres of the horizon which are separated by the equatorial plane of the black hole. The effectiveness with which the black hole acts as a unipolar inductor may possibly depend on the magnitude of this magnetic flux. The case of rapid black hole rotation, with $(m^2-a^2)/m^2\ll 1$, is of great interest here, as it provides the opportunity, in principle, for energy to be extracted from the black hole at the Eddington luminosity,[3,7] which is a feature of quasars and active galactic nuclei.[8]

Let a closed curve in the equatorial plane, $l=l(\varphi)$, encircle the singular point $r=0$. Then the magnetic flux in an Ernst-Wild universe through an arbitrary surface $r(\theta,\varphi)$ lying above the equatorial plane and bounded by the curve $l(\varphi)$ will be

$$F=\frac{1}{2}\int {}^*F^{\mu\nu}d^2\Sigma_{\mu\nu}$$
$$=\int_0^{2\pi}d\varphi\int_0^{\pi/2}d\theta\left[H^{(r)}-\Delta^{-1/2}\frac{\partial r(\theta,\varphi)}{\partial\theta}H^{(\theta)}\right]A^{1/2}\sin\theta$$
$$=\operatorname{Re}\left\{\int_0^{2\pi}d\varphi\Phi\left[l(\varphi),\frac{\pi}{2}\right]-2\pi\Phi(r_1,0)\right\}, \quad (49)$$

where ${}^*F^{\mu\nu}=1/2\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$. The last equality in (49) has been derived using Eq. (6). The quantity $\Phi(r,0)$ does not depend on r, so the magnetic flux (49) is entirely determined by the shape of the equatorial curve $l(\varphi)$. When the curve is a circle, $l(\varphi) = r$, we obtain

$$F(r)=2\pi\left\{\frac{ea/r+{}^1/_2B_0(r^2+a^2+2a^2m/r)}{1+B_0ea/r+{}^1/_4B_0^2(r^2+a^2+2a^2m/r)}\right.$$

$$\left.-\frac{({}^1/_2)B_0e^2[1+{}^1/_4B_0^2e^2]+B_0(e+B_0ma)(e+2B_0ma)}{[1+{}^1/_4B_0^2e^2]^2+B_0^2(e+B_0ma)^2}\right\}. \quad (50)$$

Hence, we obtain for the magnetic flux $F_H = F(r_H)$ through the upper half of black hole horizon, with $e = 0$[3]

$$F_H=4\pi B_0m^2\frac{1-B_0^2a^2}{(1+B_0^2m^2)(1+B_0^4m^2a^2)}. \quad (51)$$

Equation (50) also implies that the magnetic flux $F(\infty)$ in an Ernst-Wild universe is finite. One case of practical interest is that of a black hole in a uniform magnetic field, corresponding to the approximation for Eq. (50) which is linear in B_0. The magnetic flux through the upper half of the horizon is then

$$F_H=2\pi\left\{e\frac{a}{r_H}+B_0\left[2m^2-e^2\left(1+3\frac{m}{r_H}-\frac{e^2}{r_H^2}\right)\right]\right\}. \quad (52)$$

It can be seen that for a black hole with equilibrium charge $q_0 = 2B_0ma$, corresponding to a parameter $e = 0$, the magnetic flux through the upper (lower) hemisphere of the black hole horizon does not depend on the rotation parameter a, and it is equal to its Schwarzschild value $F_H = 4\pi B_0m^2$. This in fact means that it is independent of the black hole angular momentum, since as can be seen from (39) and (40), with $e = 0$ in the truncated Ernst-Wild metric, to first order in B_0, the mass satisfies $M_H = M(\infty) = m$, and the angular momentum satisfies $J_H = J(\infty'ma$. Figure 1 shows qualitatively the regions with $F_H > 0$ and $F_H < 0$ for a black hole in a uniform magnetic field with $0 < B_0 \ll m^{-1}$ as a function of the parameters a and e the allowed values of which are bounded by the circle $a^2 + e^2 \leqslant m^2$. The heavy curves correspond to $q = 0$ and $J_H = 0$ in (25) and (47). The dashed curves show the characteristic level lines $F_H = \text{const}$, and the lighter lines correspond to F_H in (52). These intersect the line $a = 0$ at the points $e = \pm(3^{1/2}/2)m$. Two separatrices, on which F_H takes on its Schwarzschild value, pass through the coordinate origin. One of these coincides with the $e = 0$ coordinate axis, corresponding to a black hole with equilibrium charge $q = q_0$. The second separatrix (dashed curve) tends to $a = 0$ as $B_0 \to 0$. It can be assumed that when the magnetic field B_0 is turned on, the curves on the axes $a = 0$ and $e = 0$ are distorted and transformed into curves $a = a_0$ and $e = q_0$, corresponding to a black hole with no angular momentum or charge, respectively.

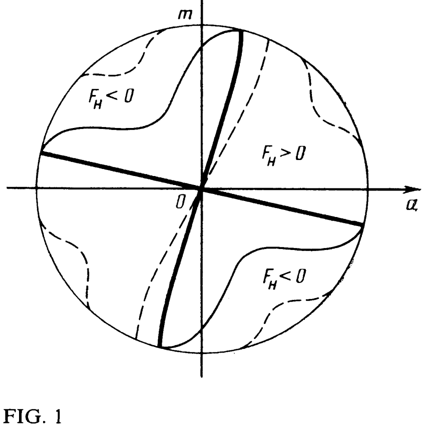

FIG. 1

Using (25) and (52) for a black hole with electric charge $q = 0$ in the linear approximation in B_0 and e, we find

$$F_H=\pi B_0r_H^2(1-a^4/r_H^4)=4\pi B_0m(m^2-a^2)^{1/2}. \quad (53)$$

In this expression, the radius of the horizon is $r_H = m + (m^2 - a^2)^{1/2}$. Equation (53) was derived in Refs. 12 and 13 by calculating the flux of a uniform test magnetic field through a hemisphere of the horizon of a Kerr black hole. The electromagnetic field distribution around the black hole can then be found by taking into consideration the requirement that it carry no electric charge. The need for this requirement is dictated by the fact that the electromagnetic characteristics of a black hole, such as charge q and magnetic flux F_H, depend linearly on B_0 and e. At the same time, the influence of small values of B_0 and e on the metric can be neglected, since the components of the metric depend on these at least bilinearly, which justifies the use of the Kerr metric in Refs. 12 and 13. Taking the reciprocal influence of a uniform magnetic field on a black hole with magnetic flux (53) into account is equivalent to the black hole having $e = -2B_0ma$, whereupon it will have no electric charge. In Fig. 1, varying F_H according to (53) corresponds to moving along the heavy line $e = -2B_0ma$. Similarly, making use of (48) and (52) for a black hole with angular momentum $JH = 0$, we find

$$F_H=6\pi B_0r_H\frac{m^2-e^2}{2r_H-m}. \quad (54)$$

This expression is consistent with the result obtained in Ref. 27 for the magnetic flux through the upper hemisphere of the horizon of a Reissner- Nordstrom black hole in a uniform magnetic field. The reciprocal effect of the magnetic field in this case is equivalent to the black hole having a rotation parameter $a = a_0$ (the second heavy line in Fig. 1), as given by Eq. (48). In the approximate expressions (53) and (54), $F_H = 0$ for $a = m$ and $e = m$ respectively. In the figure, this is equivalent to the intersection points of the circle $a^2 + e^2 = m^2$ with the lighter curves, which correspond to the condition $F_H = 0$ in Eq. (52), the exact expression in a and e.

6. HIGHLY EXTREME BLACK HOLES AND THE PRINCIPLE OF COSMIC CENSORSHIP

It was shown in section 4 that the specific angular momentum and electric charge of a black hole in the presence of an external magnetic field depend on the magnitude of that field, and can exceed the allowed values for a Kerr-Newman black hole. One question which arises is what happens to such a highly extreme black hole when the external magnetic field is turned off. Because of the interaction of the black hole with the magnetic field, its parameters must change when the field strength changes. We can identify the possible change in parameters of a highly extreme black hole using the first law of black hole thermodynamics.[24,28] This law is similar to the first law of thermodynamics, and governs the variation of total mass (energy) in a black hole and the surrounding material in two nearby asymptotically flat axisymmetric stationary configurations. In the uniform magnetic field approximation with $B_0 \ll m^{-1}$, the truncated part of the Ernst-Wild metric refers to such configurations. We can write the first law of black hole thermodynamics in a form which permits a familiar physical interpretation.[24,29]

$$\delta M=\Omega_H\delta\left(J_H+\oint A_\varphi\sigma_H\,dS_H\right)+V_H\delta q_H+(8\pi)^{-1}k\delta S_H+\delta U, \tag{55}$$

$$\delta U=\int\Omega\delta(d^3J_m+A_\varphi\,d^3q)+\int V_m\delta\,d^3q+\int\mu\delta\,d^3N+\int T\delta\,d^3S_m, \tag{56}$$

where the subscript H refers to values of the corresponding quantities at the black hole horizon. The variations δU constitute the contribution to the change in total energy δM from matter outside the black hole. It is assumed that the external material is distributed around the black hole as a ring of charged perfect fluid made up of different types of particles (summations over the various types of particles have been omitted), revolving the angular velocity Ω, and having a temperature T and chemical potential μ relative to an observer at infinity. The quantities $d^3q=j^\mu d^3\Sigma_\mu$, d^3J_m, d^3S_m, and d^3N are the volume elements of electric charge, angular momentum, entropy, and particle number in the ring, respectively. The electromagnetic field is characterized by the azimuthal component of the potential A_ϕ and the comoving electric potential $V_m = -(A_t+\Omega A_\phi)$. The angular rate of rotation of the black hole Ω_H equals the angular rotation rate of the metric (9) at the horizon, $\sigma_H = E^{(r)}/4\pi$ is the effective electric surface charge on the black hole, and J_H, Q_H, S_H, and k are the black hole angular momentum, electric charge, area of the event horizon, and surface gravity respectively. The area of the black hole event horizon in a magnetic universe is

$$S_H=\oint dS_H=\oint_H A^{1/2}\sin\theta\,d\theta\,d\varphi=4\pi(r_H^2+a^2). \tag{57}$$

It is independent of the magnetic field strength B_0, and is identical to the horizon area for a Kerr-Newman black hole with the same parameters m, a, and e. The quantity

$$\chi_H=\oint A_\varphi\sigma_H\,dS_H,$$

which appears as an angular momentum of the effective surface charge of the black hole, is the angular momentum of the stationary electromagnetic field external to the black hole event horizon, with no contributions from external sources.[29] The angular momentum contributed by external sources is contained in δU. The sum of the black hole and external electromagnetic field angular momenta, $J=J+\chi_H$, is therefore the total angular momentum of the metric, in which matter can be treated as a collection of test particles. In particular, in the Kerr-Newman metric, $J_H+\chi_H=ma$.

Note that $e=0$, the condition for electrostatic equilibrium of a black hole in a uniform magnetic field derived in section 3, follows directly from the first law of black hole thermodynamics. In fact, according to (55), when the total angular momentum of the metric $J=J_H+\chi_H$, the area of the event horizon (black hole entropy) S_H, and invariant energy of external matter U (if there is any) are fixed, a change δq_H in the charge on the black hole leads to a change $\delta M = V_H\delta q_H$ in the total mass energy. In uniform magnetic field approximation, making use of (25) and (32), we obtain $V_H\delta q_H \sim (q_H-q_0)\delta q_H$, and consequently

$$V_H(q_0)=0,\qquad \left.\frac{\partial V_H}{\partial q_H}\right|_{q_H=q_0}>0.$$

It can be seen that a black hole achieves electrical equilibrium (conditional minimum of the functional M) when its charge is $q_H=q_0=2B_0ma$, i.e., when $e=0$.

Turning off the magnetic field, which is equivalent to removing the source of this field to infinite distance from the black hole or dissipating the source currents, is only feasible for the truncated Ernst-Wild metric discussed above, where the sources of the magnetic field are located within a finite volume about the black hole, and the magnetic field itself is gravitationally weak. Suppose that a highly extreme black hole, is situated in such a uniform magnetic field of strength B_0, with parameters $m\ll B_0^{-1}$, $a^2+e=m^2$, and $e^2\ll m^2$. The electric charge is then $|q_H|=|e+2B_0ma|\ll m$. For definiteness, we assume that $a>0$, thereby fixing the direction of the black hole angular momentum in space. For subsequent calculations, it will suffice to know the expression for the surface gravity of a Kerr-Newman black hole,

$$k=(r_H-m)(r_H^2+a^2)^{-1}. \tag{58}$$

Carrying out the calculations in (55) with $\delta M=\delta q_H=0$ and using Eq. (57) for S_H for the parameters of the Kerr-Newman black hole produced from a highly extreme black hole when the magnetic field B_0 is turned off, we obtain to bilinear accuracy in B_0 and $q=q_H$

$$\begin{gathered}e'=q,\quad a'=m\left(1-\frac{1}{2}q^2\right),\quad r_H'=m'(1+\eta),\\ m'=m\left(1+\frac{15\pi-32}{24}B_0q-B_0^2m^2-m^{-1}\delta U\right),\end{gathered} \tag{59}$$

where the quantity $\eta\ll 1$ characterizes how extreme the resulting black hole is, and is equal to

$$\eta=\frac{1}{8\pi m^2}\delta S_H+\frac{56-15\pi}{12}B_0q+\frac{2}{m}\delta U. \tag{60}$$

The second term in this expression comes from the contributions to δM of the variation of the angular momentum of the black hole surface charge. In the present case, this angular momentum is equal to

$$\chi_H=\oint A_\varphi\sigma_H\,dS_H=\left(\frac{\pi}{4}q-\frac{56-15\pi}{12}B_0m^2\right)q. \tag{61}$$

The principle of cosmic censorship,[25] which prohibits a black hole from becoming a naked singularity, requires that $\eta\geqslant 0$. Furthermore, Hawking has proven[26] that whenever the principle of cosmic censorship holds, $\delta S_H\geqslant 0$ for any classical process. This theorem constitutes the second law of black hole thermodynamics, in which the area of the horizon S_H plays the role of entropy.[28] We see from (60) that there are possible ways of turning off the field B_0 such that, for certain values of δU which depends on $\delta\chi_H$, there is no increase in the area of the horizon of a highly extreme black hole ($\delta S_H=0$), and it turns into an extreme Kerr-Newman black hole ($\eta=0$). Let us consider now one way to turn off the field B_0, assuming that the energy in the current sources in the ring dissipates into heat sufficiently slowly that $\delta U = 0$. Equation (60) then implies that when $B_0 \geqslant 0$, the field can be turned off adiabatically with respect to the black hole, so that $\delta S_H = 0$. The transformation of a highly extreme black hole into an extreme Kerr-Newman black hole is in that case only possible when $q = 0$. When $\delta U < 0$ and $B_0 q < 0$, the principle of cosmic censorship unavoidably requires that the area of the horizon increase as the magnetic field is turned off. The minimum allowable increase in area under these conditions occurs with $\eta = 0$ in (60), corresponding to the transformation of a highly extreme black hole into an extreme Kerr-Newman black hole. Note that a highly extreme black hole with equilibrium electric charge $q_0 = 2B_0 m^2$ is transformed, when $\delta U = 0$ and the magnetic field is turned off adiabatically with respect to the black hole, into a black hole with

$$\eta = [(56-15\pi)/6] B_0^2 m^2. \quad (62)$$

7. CONCLUSION

The accretion of plasma can result in the formation of a magnetosphere with a regular magnetic field distribution in the vicinity of a black hole. The simplest configuration which sheds light on some of the qualitative features of black hole behavior in an external magnetic field is a uniform field oriented parallel or antiparallel to the angular momentum vector of the black hole. This situation is described by the Ernst-Wild metric in the limit of a gravitationally weak magnetic field. The presence of an external magnetic field results in the black hole parameters being dependent on the magnitude of that field.

Even for the limiting value $a = m$ of the rotation parameter, a rotating black hole in a uniform magnetic field may possess a nonvanishing equilibrium electric charge which is consistent with the charge found by Wald.[11] A black hole achieves electrical equilibrium when the comoving potential vanishes; the later determines the electrostatic energy of test particles. A nonvanishing equilibrium charge on a black hole ensures a conditional minimum in the total energy (mass) of the metric in the presence of an external magnetic field. It is possible that a rotating black hole with a magnetosphere and nonvanishing electric charge can also come to equilibrium. In that event, the entire system will be electrically neutral because of the compensating charge contained in the magnetosphere. In a uniform magnetic field, the magnetic flux through the upper hemisphere of the event horizon of an equilibrium black hole turns out not to depend on its angular momentum. The angular momentum and electric charge of a black hole in a magnetic field can exceed the allowed values for a Kerr-Newman black hole. Because they acquire an equilibrium electric charge in their highly extreme state, massive black holes may be found in quasars and galactic nuclei, which rotate rapidly due to their accretion disk. When the magnetic field is turned off, in accordance with the principle of cosmic censorship, a highly extreme black hole must in general be transformed into a nonextreme Kerr-Newman black hole. Under certain circumstances, such a transformation leads to an extreme Kerr-Newman black hole

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[1)] The Ernst-Wild universe is not bounded in radius as $r \to \infty$ for all values of the angles θ and φ. In this universe, the electric lines of force from the black hole go off to spatial infinity, so in contrast with the situation in a closed universe, no problems of closure of the lines of force from the charge arise.

[2)] The ultimate manifestation of the absence of global immobility of a particle when $\omega \neq 0$ is the impossibility of bringing particles within the ergosphere of a black hole to rest with respect to a distant observer.[25]

[3)] The expression for F_H in Eq. (41) of Ref. 13, which was obtained for the case $a = e = 0$, is incorrect.

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