Theory of polarizational optical bistability

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A symmetric approach to the analysis of optical bistability (OB) is proposed. It is shown that the propagation of radiation along the optic axis of a nongyrotropic nonlinear system generally gives rise to two types of optical bistability, namely, intensity OB and polarizational OB. The associated physical mechanisms and the criteria for the appearance of these types of OB are indicated. The dependence of the bifurcation value of the incident intensity on its polarization parameters is analyzed in terms of symmetry properties. It is shown that, for polarizational OB, this dependence is nonanalytic (it has a spinodal edge). The criteria obtained in this way are used to investigate the onset of OB in special cases, including ring cavities filled with nonabsorbing crystals exhibiting arbitrary optical nonlinearity.

Optical bistability (OB), i.e., the presence of two stable stationary states of radiation in a system for given incident radiation, is now known to occur in many nonlinear optical systems. As a rule, optical bistability (see Ref. 1) is associated with the dependence of the refractive index and adsorption coefficient of a nonlinear medium on the incident intensity (most calculations and experiments have been carried out for different media in resonant cavities). However, in general, the variation in the polarization of radiation in a medium is also found to depend on intensity. The propagation of radiation polarized in a symmetric direction (e.g., in the plane of symmetry) and propagating along the optic axis is often found to give rise to polarizational instability, i.e., lateral increments in the field **E** become amplified in the course of propagation. This type of instability occurs, in particular, in an isotropic medium exhibiting the self-induced rotation of the polarization ellipse observed in Ref. 2 (see below). It was also reported in Ref. 3 for nonlinearly absorbing cubic crystals.

Fluctuations in polarization may grow in time when a nonlinear medium is placed in a resonant cavity. As a result, the radiation transmitted by the system in the stationary state is found to be in one of two symmetric states differing by the sign of the angle of rotation of the polarization ellipse relative to the original direction of **E**, and by the direction of rotation of $\mathbf{E}(t)$ in time.

The possibility of this type of polarizational OB will obviously primarily depend on the symmetry properties of the system. Examples of polarizational OB were discussed in Refs. 4 and 5 in two special cases. The "dichroic" OB, discussed in Refs. 6 and 7, can also be described as polarizational OB. Below, we shall use the symmetry approach to derive the general conditions for the appearance of polarizational OB, and will analyze its observable properties.

The complete analysis of OB consists of establishing the dependence of the intensity I_t and of the polarization parameters ψ_t, χ_t of transmitted radiation on the corresponding parameters I_i, ψ_i, χ_i of the incident radiation [ψ is the angle between the semimajor axis of the polarization ellipse and the chosen symmetric direction in the system, i.e., the x-axis, and the parameter χ determines the degree of ellipticity [through the formula $\cos 2\chi = (I_\parallel - I_\perp)/(I_\parallel + I_\perp)$] and the direction of rotation of the vector $\mathbf{E}(t)$; see Ref. 8]. In general, these functions are very complicated. The significant point is, however, that they have a number of important properties that are determined by the symmetry of the system and the type of OB, and are common to different mechanisms of optical nonlinearity.

We shall show in this paper that these properties can be found, and that the basic types of OB can be established, by investigating the transmission of the system when the parameters I_i, ψ_i, χ_i of the incident radiation lie near points on the bifurcation surface $I_B(\psi_i, \chi_i)$ that are determined by symmetry properties, where $I_B(\psi_i, \chi_i)$ is the value of I_i for which new stationary states of radiation in the system occur for given ψ_i, χ_i. In Section 1, we shall obtain the criteria for the appearance of two general types of OB, namely, intensity OB and polarizational OB. Absorptive and refractive OB, discussed in Ref. 1, are special cases of intensity OB. The shape of the $I_B(\psi_i, \chi_i)$ surface near its symmetry axes is analyzed in Section 2 in the case of intensity and polarizational OB. Section 3 investigates the transmission of nonlinear systems near the POB threshold. In Section 4, we find general criteria for the appearance of intensity and polarizational OB in special cases, and derive explicit conditions for the onset of OB in ring cavities filled with transparent media with arbitrary optical nonlinearity. The main results are discussed in Section 5.

1. CRITERIA FOR INTENSITY AND POLARIZATIONAL OB

Let us consider the case—frequently investigated experimentally—where the radiation propagates in a system along its optic axis, where, in a nonlinear medium, this may be a third- or higher-order symmetry axis (z-axis), the system being nongyrotropic so that there is no change in polarization when the intensities are low. Examples of such systems include cavities filled with nongyrotropic isotropic media or suitably oriented cubic, tetragonal, or hexagonal crystals or heterostructures.

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Under the conditions of optical bistability or multistability, and for given incident parameters I_i, ψ_i, χ_i, the parameters I_t, ψ_t, χ_t of the transmitted radiation can assume a number of stationary values (in general, not all will correspond to stable states). The number of stationary states of transmitted radiation is different in different ranges of I_i, ψ_i, χ_i, e.g., for low enough I_i, for which optical nonlinearity is unimportant, there is only one stationary state. The relationship between I_i, ψ_i, and χ_i, for which there is a change in the number of states (two or more merge) is determined by the standard equation

$$D(I_B, \psi_i, \chi_i)=0, \quad D(I_i, \psi_i, \chi_i)=\partial(I_i, \psi_i, \chi_i)/\partial(I_t, \psi_t, \chi_t). \tag{1}$$

This condition describes the situation where, at the point of merging of different states of transmitted radiation, which is a singular point, there is a nonlinear relationship between small increments in the parameters of transmitted and incident radiation. We note that, clearly, different $D(I_i, \psi_i, \chi_i)$ correspond to different states of transmitted radiation; the values of I_t, ψ_t, χ_t and, correspondingly, of D in (1) refer to any of the merging states.

When the x axis, from which the angles ψ_i and ψ_t are measured, lies along one of the symmetry axes C_{2n} of the medium (spatial dispersion is assumed absent), or when the xz plane is a plane of symmetry, the bifurcation value of the intensity is unaffected by a change in the signs of ψ_i and χ_i:

$$I_B(\psi_i, \chi_i)=I_B(-\psi_i, -\chi_i), \tag{2}$$

i.e., a symmetry axis of the surface I_B (ψ_i, χ_i) passes through the point $\psi_i=\chi_i=0$. The property described by (2) follows directly from (1) and from the symmetry properties. Actually, the field component E_x is unaffected by reflection in a plane or by rotation around the x-axis through 180°, whereas E_y changes to $-E_y$, i.e., both ψ and χ change sign for both transmitted and incident radiation, and the relationship between the intensities I_i, I_t (and, consequently, the value of I_B) remains unaltered by virtue of symmetry.

We shall assume that, for a chosen x direction, the function I_B (ψ_i, χ_i) is finite, continuous, and nonself intersecting near $\psi_i=\chi_i=0$.[1)] In this general and most important case, (2) shows that the point $\psi_i=\chi_i=0$ is an extremum of sections of the surface I_B (ψ_i, χ_i) that pass through this point.

For $\psi_i^2+\chi_i^2 \ll 1$, we have $I_i \simeq I_B(0,0)$ and the incident parameters I_i, ψ_i, χ_i are uniquely determined by I_t, ψ_t, χ_t (although there is no one-to-one correspondence between the OB conditions). If the transmitted radiation is then polarized in the symmetric direction ($\psi_t=\chi_t=0$), the incident radiation is polarized in the same way ($\psi_i=\chi_i=0$); conversely, for given I_t, ψ_t, χ_t, there are two sets of values of the parameters I_i, ψ_i, χ_i which differ by the signs of ψ_i and χ_i. Hence, it follows that, when $\psi_t=\chi_t=0$, we have $\partial\psi_i/\partial I_t=\partial\chi_i/\partial I_t=0$, so that the Jacobian D in (1) can be factored, and (1) reduces to one of the two equations

$$\partial I_i/\partial I_t=0, \quad \psi_{i,t}=\chi_{i,t}=0, \tag{3}$$

$$\partial(\psi_i, \chi_i)/\partial(\psi_t, \chi_t)=0, \quad \psi_{i,t}\to 0, \quad \chi_{i,t}\to 0, \tag{4}$$

which define the characteristic values (usually, limiting thresholds) of the incident intensity I_B $(0,0)$ for the two types of OB.

Equation (3) describes the onset of intensity OB: an increase in the incident radiation intensity I_i leads to the "break" I_t ($|\partial I_t/\partial I_i|\to\infty$, $I_i\to I_B(0,0)$. Equation (4) describes the appearance of polarizational OB. This becomes clearer if we express ψ, χ in terms of the ratio of the field components E_x, E_y:

$$E_yE_x^{-1}\equiv|E_y/E_x|\exp(i\delta)=\psi+i\chi, \quad |\psi|,\ |\chi|\ll 1 \tag{5}$$

and write (4) in the form

$$\frac{I_t}{I_i}\frac{\partial\delta_i}{\partial\delta_t}\frac{\partial|E_{yi}|^2}{\partial|E_{yt}|^2}=0, \quad |E_{yi}|\to 0, \quad |E_{yt}|\to 0 \tag{4a}$$

(we have taken into account the fact that, as $|E_y|\to 0$, the change in the phase difference δ between the components in a nonlinear medium is independent of $|E_y|$). It is clear from (4a) that the polarizational OB threshold corresponds to the threshold of polarizational instability in the system: $|E_{yt}/E_{yi}|\to\infty$ as $|E_{yi}|\to 0$.

2. SHAPE OF THE BIFURCATION SURFACE NEAR A SYMMETRY AXIS

The behavior of the function I_B (ψ_i, χ_i) for $\psi_i^2+\chi_i^2\ll 1$ is different depending on the type of OB, i.e., it depends formally on which of the conditions (3) or (4) determines $I_B(0,0)$. To analyze the shape of I_B (ψ_i, χ_i), we use an expansion for the incident parameters δI_i, ψ_i, χ_i $[\delta I_i=I_i-I_B(0,0)]$ in terms of δI_t, ψ_t, χ_t $[\delta I_t=I_t-I_{tB}(0,0)]$:

$$\delta I_i=A_I\delta I_t+{}^1/{}_2A_{II}(\delta I_t)^2+L_I^{(2)}[\psi_t;\ \chi_t], \tag{6}$$

$$\begin{pmatrix}\psi_i\\ \chi_i\end{pmatrix}=\hat{B}\begin{pmatrix}\psi_t\\ \chi_t\end{pmatrix}+\delta I_t\hat{C}\begin{pmatrix}\psi_t\\ \chi_t\end{pmatrix}+\begin{pmatrix}L_\psi^{(3)}[\psi_t;\chi_t]\\ L_\chi^{(3)}[\psi_t;\chi_t]\end{pmatrix},$$

where, for brevity, we use the matrix form, $\tilde{B}$, $\tilde{C}$ are 2×2 matrices, and $L^{\{2\}}[x;y]$, $L^{\{3\}}[x;y]$ are homogeneous polynomials of degree two and three, respectively [by symmetry, the above expansion for δI_i does not contain odd powers of ψ_t, χ_t, whereas the expansions for ψ_i and χ_i do not contain even powers].

In view of (6), the condition for the onset of intensity OB (3) assumes the form

$$A_I\equiv A_I(I_i)=0, \quad I_i=I_0'\equiv I_B(0,0) \tag{7}$$

(this equation actually determines I_0'). Substituting (6) in (1), we obtain the shape of the bifurcation surface:

$$\begin{aligned}\delta I_B'(\psi_i,\chi_i)=L_I^{(2)}[(\hat{B}^{-1})_{11}\psi_i+(\hat{B}^{-1})_{12}\chi_i;\quad (\hat{B}^{-1})_{21}\psi_i\\ +(\hat{B}^{-1})_{22}\chi_i]\equiv g_{11}\psi_i^2+2g_{12}\psi_i\chi_i+g_{22}\chi_i^2,\\ \delta I_B'(\psi_i,\chi_i)=I_B(\psi_i,\chi_i)-I_0'.\end{aligned} \tag{8}$$

When $g=g_{12}^2-g_{11}g_{22}<0$, (8) shows that the surface $\delta I_B'$ (ψ_i, χ_i) is an elliptic paraboloid, whereas, for $g>0$, it is a hyperbolic paraboloid. Accordingly, the point $\psi_i=\chi_i=0$ is an extremum or a saddle point of $\delta I_B'$. Optical bistability occurs for

$A_{II}[\delta I_i - \delta I'_B(\psi_i, \chi_i)] > 0.$

When $g > 0$, and whatever the sign of δI_i, it is possible to enter the OB region or leave it by varying the polarization of the incident radiation. When, however, $g < 0$, this is possible only for $(g_{11} + g_{22})\delta I_i > 0$.

We note that, for small $|\psi_i|$, $|\chi_i|$, $|\delta I_i|$, the splitting of the states of the transmitted radiation that occurs for $\delta I_i = \delta I'_B(\psi_i, \chi_i)$ is seen mostly in the intensity: $\delta I_t^{(1,2)} = \pm [2A_{II}^{-1}(\delta I_i - \delta I'_B)]^{1/2}$; to within corrections $\sim \delta I_t / I'_0$, the values of ψ_t, χ_t are the same for the two states and are linear functions of ψ_i, χ_i. Well away from symmetric directions, the splitting seen in the polarization of the transmitted radiation is just as significant as the intensity splitting (see Ref. 9).

The criterion for polarizational OB (4) can be reduced with the aid of (6) to

$$\det(\hat{B}) \equiv B_{11}B_{22} - B_{12}B_{21} = 0. \tag{9}$$

This determines the corresponding limiting threshold intensity[2] $I''_0 \equiv I_B(0, 0)$.

When (9) is satisfied, the relation between the incident and transmitted polarizations becomes essentially nonlinear even for small $|\psi_i|$ or $|\chi_i|$: terms of first order in the linear combination θ_t of the parameters ψ_t, χ_t are then no longer present in (6), where

$$\theta_t = \psi_t \cos\alpha_t + \chi_t \sin\alpha_t, \qquad \cot\alpha_t = -B_{12}/B_{11} \tag{10a}$$

(to be specific, we assume that $B_{11} \neq 0$). In view of this, it is convenient to transform in (6) from $\psi_{i,t}, \chi_{i,t}$ to $\theta_{i,t}, \eta_{i,t}$:

$$\theta_i = \psi_i \cos\alpha_i + \chi_i \sin\alpha_i, \qquad \eta_{i,t} = -\psi_{i,t} \sin\alpha_{i,t} + \chi_{i,t} \cos\alpha_{i,t},$$
$$\operatorname{ctg}\alpha_i = -B_{21}/B_{11}, \qquad 0 \leqslant \alpha_{i,t} \leqslant \pi. \tag{10b}$$

The equations for θ_t, η_t are

$$\theta_i = K_\theta[\theta_t; \eta_t], \qquad \eta_i = G\eta_t + K_\eta[\theta_t; \eta_t], \tag{11}$$

$$G = B_{11}^{-1}(B_{11}^2 + B_{12}^2)^{1/2}(B_{11}^2 + B_{21}^2)^{1/2},$$
$$K_\theta[\theta_t; \eta_t] = \delta I_i(\tilde{C}_{11}\theta_t + \tilde{C}_{12}\eta_t) + L_\theta^{(3)}[\theta_t; \eta_t], \tag{12}$$

$$\tilde{C} = \hat{S}_i \hat{C} \hat{S}_t^{-1}, \quad L_\theta^{(3)} = L_\psi^{(3)} \cos\alpha_i + L_\chi^{(3)} \sin\alpha_i - (\tilde{C}_{11}\theta_t + \tilde{C}_{12}\eta_t) L_I^{(2)}$$

[the expression for K_η is analogous to that for K_θ; $\hat{S}_t, \hat{S}_i$ are matrices that implement the canonical transformations (10a) and (10b)]. In the derivation of (11), terms $\sim (\delta I_t)^2$ were discarded in (6), and δI_t was expressed in terms of δI_i, θ_i, and η_t.

The quantity θ_t in (11) corresponds to a "slow" variable ("soft mode" that becomes prominent near the points of bifurcation of solutions of ordinary differential equations.[10] The quantity η_t corresponds to the "fast" variable. The value of η_t is uniquely determined by n_i, θ_t, and δI_i, and can be found from perturbation theory:

$$\eta_t = \eta_t^{(0)} + \eta_t^{(1)} + \ldots, \qquad \eta_t^{(0)} = G^{-1}\eta_i,$$
$$\eta_t^{(1)} = -G^{-1}K_\eta[\theta_t; G^{-1}\eta_i]. \tag{13}$$

Bearing (13) in mind, we obtain

$$\theta_i = K_\theta[\theta_t; G^{-1}\eta_i] \equiv {}^1/_6 a_3\theta_t^3 + {}^1/_2 a_2\theta_t^2\eta_i + \theta_t(a_1\eta_i^2 + c_1\delta I_i) + a_0\eta_i^3 + c_0\delta I_i\eta_i. \tag{14}$$

The cubic equation given by (14) can have either one or three real roots. The existence of three roots $\theta_t^{(1,2,3)}$, i.e., three states of polarization of transmitted radiation, actually corresponds to the onset of polarizational OB. The intensity $\delta I_i = \delta I''_B(\theta_i, \eta_i)$, for which polarizational OB occurs for given polarization of incident radiation θ_i, η_i, can be found from (1), (6), and (11), or directly from (14). It can be written in the reduced form

$$\delta I_B''(\theta_i, \eta_i) = \eta_i^2 F\left(\frac{\theta_i^{1/3}}{\eta_i}\right)$$
$$F(q) = -\frac{1}{c_1}\left(\frac{1}{2}a_3 v^2 + a_2 v + a_1\right), \qquad v \equiv v(q), \tag{15}$$

where $v(q)$ is the solution of the cubic

$$ {}^1/_3 a_3 v^3 + {}^1/_2(a_2 + a_3 b)v^2 + a_2 b v - b_0 + q^3 = 0,$$
$$b = c_0/c_1, \quad b_0 = a_0 - a_1 b. \tag{16}$$

According to (15), analysis of the shape of the surface $I''_B(\theta_i, \eta_i)$ reduces to the analysis of the function $F(\theta_i^{1/3}/\eta_i)$. The form of $F(q)$ is shown in Fig. 1. Depending on the sign of a_3c_1, this function has a minimum (for $a_3c_1 < 0$) or maximum (for $a_3c_1 > 0$). When $|q| \gg 1$, we have the asymptotic result $F(q) \sim q^2$. This asymptotic behavior, which corresponds to $|\theta_i^{1/3}/\eta_i| \gg 1$, describes $\delta I''_B$ throughout virtually the entire range of small θ_i, η_i; here, $\delta I''_B$ is determined almost entirely by the value of θ_i, and depends on θ_i nonanalytically:

$$\delta I_B'' \approx -(3/2c_1)(a_3/3)^{1/3}\theta_i^{2/3}, \qquad |\theta_i|^{1/3} \gg |\eta_i|. \tag{17}$$

In the narrow region $|\theta_i| \ll |\eta_i|^3 \ll 1$, we have

$$\delta I_B'' \approx F(0)\eta_i^2 \tag{18}$$

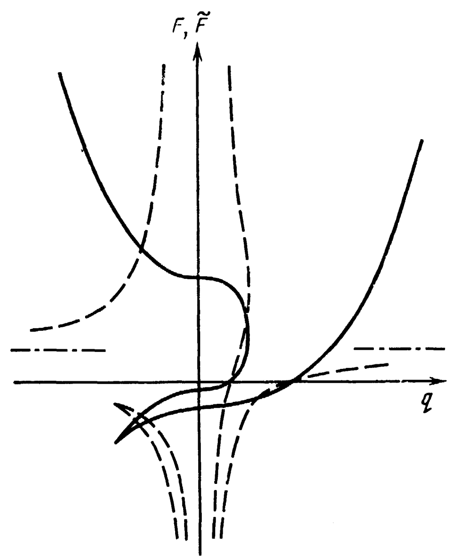

FIG. 1. Schematic form of the functions $F(q)$ (solid lines) and $\tilde{F}(q)$ (broken lines) that determine, near the *s*-threshold for polarizational OB, the sections through the bifurcation surface $I_B(\psi_i, \chi_i)$ by the planes η_i = const and θ_i = const, respectively (the increase in the threshold radiation intensity during a change in its polarization, $\delta I''_B = \eta_i^2 F(q) = \theta_i^{2/3}\tilde{F}(q)$, $q = \theta_i^{1/3}/\eta_i$, where θ_i and η_i are linear combinations of ψ_i and χ_i). The dot-dash lines are the asymptotes of $\tilde{F}(q)$ for $|q| \to \infty$.

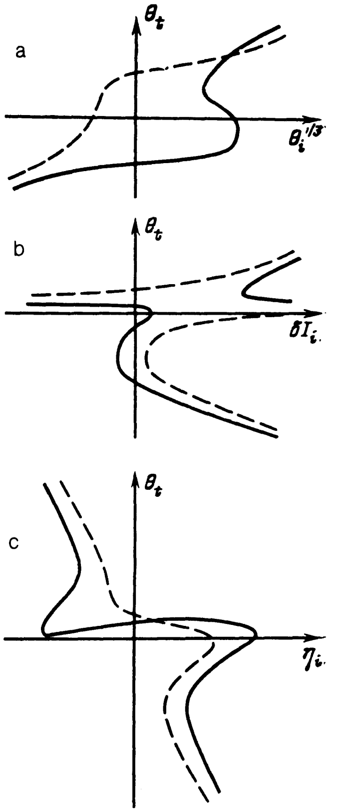

FIG. 2. Examples of the dependence of polarization parameter θ_t of transmitted radiation on the polarization parameters θ_i, η_i and the intensity δI_i of incident radiation near the s-threshold for polarizational OB. The solid and broken curves refer to different values of $\delta I_i/\eta_i^2$ (a), $\theta_i^{1/3}/\eta_i$ (b), and $\delta I_i/\theta_i^{2/3}$ (c).

where $F(0)$ can assume either one of three values; see below.

The most important property of $F(q)$ is that, according to (15) and (16), its extremum is a spinodal point. Near this point, we have

$$F_\pm(q)\approx F_c-\frac{a_3}{2c_1}\Theta(M\delta q)\left[M\delta q\pm\left|\frac{2a_3}{3(a_2-a_3b)}\right|(M\delta q)^{3/2}\right], \tag{19}$$

where $F_c\equiv F(q_c)$ is the extremal value of $F(q)$, given by

$$F_c=(a_2^2-2a_3a_1)/2a_3c_1,\qquad q_c=a_3^{-2/3}(a_3^2b_0+{}^1/_2a_2^2a_3b-{}^1/_6a_2^3)^{1/3},$$
$$M=6q_c^2(a_2-a_3b)^{-1},\qquad \delta q=q-q_c \tag{20}$$

and $\phi(x)$ is a step function. It is clear from (19) that the branches $F_\pm(q)$ diverge as $(q-q_c)^{3/2}$.

The point $q=0$ is a point of inflection of $F(q)$, and the derivative dF_+/dq becomes infinite for $q=q_s$, where

$$q_s=(b_0+{}^1/_2a_2b^2-{}^1/_6a_3b^3)^{1/3},$$
$$F_+(q_s)\equiv F_s=-({}^1/_2a_3b^2-a_2b+a_1)/c_1\;. \tag{21}$$

In the interval between q_s and q_c, the function $F(q)$ assumes three values and this substantially enriches the OB picture (see next Section). We note that the quantity $F(0)$ in (18) assumes three values for $q_s/q_c<0$.

It follows from the foregoing analysis that the surface $\delta I_B''(\theta_i,\eta_i)$ is highly anisotropic. It is corrugated and has a "cusp" (spinodal edge). On this edge,

$$\delta I_B''=\eta_i^2F_c,\qquad \theta_i=\eta_i^3q_c^3. \tag{22}$$

Figure 1 shows (on an enlarged scale, $q\sim\theta_i^{1/3}$) a section of $\delta I_B''(\theta_i,\eta_i)$ by the $\eta_i=$ const plane, which is practically perpendicular to the spinodal edge. It is important to emphasize that the properties of $\delta I_B''(\theta_i,\eta_i)$ are connected with the structure of (15) and (16), and do not depend on the explicit values of the parameters, i.e., they constitute a symmetry property of polarizational OB.

The shape of the bifurcation surface becomes appreciably simpler when the system is isotropic in the plane perpendicular to the direction of propagation of the radiation. The choice of the axis from which the angles ψ_i, ψ_t are measured is then arbitrary. Hence, $L_I^{(2)}$, $L_\chi^{(3)}$ in (6) depend only on χ_t, and the matrix elements become $B_{21}=C_{21}=0$. The bifurcation value of the incident radiation, I_B, depends only on χ_i.

Near the s-threshold for polarizational OB in an isotropic system ($B_{22}=0$), equation (6) yields the following equation for χ_t:

$$^1/_6a_3\chi_t^3+c_1\chi_t\delta I_i-\chi_i=0. \tag{14a}$$

It is readily seen that we then have

$$\delta I_B''(\chi_i)=-(9a_3/8c_1^3)^{1/3}\chi_i^{2/3}. \tag{15a}$$

The function $I_B''(\chi_i)$ is symmetric and single-valued. It has a cusp at $\chi_i=0$.

3. POB IN THE NEAR-THRESHOLD REGION

When (14)–(16) are taken into account, the range of the parameters δI_i, θ_i, η_i near the s-threshold for polarizational OB in which the radiation in the system has three stationary states is determined by the condition

$$a_3c_1[\delta I_i-\eta_i^2F(\theta_i^{1/3}/\eta_i)]<0. \tag{23}$$

For finite η_i, the limiting threshold intensity for polarizational OB is $\delta I_c=\eta_i^2F_c$. On the cusp ($\delta I_i=\delta I_c$, $\theta_i^{1/3}=\eta_iq_c$), the three stationary states merge together.

Let us first consider what happens to the transmitted radiation when the incident polarization parameter θ_i is varied. It is clear from (15) that, for given δI_i, η_i, polarizational OB will occur in a θ_i interval whose limits are determined by the points of intersection of the graph of $F(\theta_i^{1/3}/\eta_i)$ (see Fig. 1) and the straight line $F=\delta I_i/\eta_i^2$. According to (14), the dependence of the transmitted polarization parameter θ_t on θ_i in the polarizational OB region is S-shaped (see Fig. 2a), and the distance between the upper and lower branches is[3] $\Delta\theta_t\sim[(c_1/a_3)(\delta I_c-\delta I_i)]^{1/2}$. The dependence of η_t and δI_t on θ_i is either S-shaped or loop-shaped. It is clear from symmetry considerations that, for $\eta_i=0$, the dependence of δI_t on θ_i is described by a symmetric loop-shaped curve (cf. Ref. 7). The separation between the branches is

$$\Delta\eta_t\sim|\delta I_i|[(c_1/a_3)(\delta I_c-\delta I_i)]^{1/2},$$
$$\Delta I_t\sim|\delta I_i(\delta I_c-\delta I_i)|^{1/2}.$$

It is evident that OB is most clearly defined in terms of the parameter θ_t.

When the incident polarization is such that $\theta_i^{1/3}/\eta_i$ lies between q_c and q_s, then, as is clear from Fig. 1, there are two ranges of δI_i in which (23) are satisfied, and OB sets in. There is only one such region in the opposite case. The dependence of θ_t on δI_i in these cases is shown in Fig. 2b by the solid and broken lines, respectively.

To analyze the onset and the properties of POB as η_i is varied, it is convenient to replace $F(q)$ with the function

$$\tilde{F}(q)=q^{-2}F(q),\quad \delta I_B''(\theta_i,\ \eta_i)=\theta_i^{2/3}\tilde{F}(\theta_i^{1/3}/\eta_i). \qquad (15b)$$

This new function is shown by the broken lines in Fig. 1. $\tilde{F}(q^{-1})$ represents the section of the surface $\delta I_B''(\theta_i, \eta_i)$ by the plane $\theta_i = \text{const}$.

When $q_c q_s < 0$ (shown in Fig. 1), $\tilde{F}(q^{-1})$ describes three continuous curves (one has a spinodal point). When $q_c q_s > 0$, the function $\tilde{F}(q^{-1})$ describes one curve, on which analysis of (15), (15b), and (16) shows that, in addition to the spinodal point, there is also an extremum. Altogether, the equation $\delta I_i = \theta_i^{2/3}\tilde{F}(\theta_i^{1/3}/\eta_i)$ can have 0, 2, 4 (in some cases, even 6) roots η_i. These roots separate intervals of η_i in which radiation in the system has alternately one or three stationary states (see Fig. 2c).

In a two-dimensionally isotropic system, for which $\theta_i = \chi_i, \theta_t = \chi_t$, and $\chi_t, \delta I_t$ do not depend on ψ_i, the dependence of χ_t on χ_i is centrally symmetric, while the dependence of δI_t on χ_i is symmetric (and loop-shaped in the OB region).

4. POLARIZATIONAL OB IN THE NONLINEAR RING INTERFEROMETER

The above general theory can be used to find the conditions for and properties of intensity OB and polarizational OB in specific systems. The specificity of the system manifests itself only in the explicit expressions for the coefficients in (6). They are determined by the mechanism responsible for the optical nonlinearity and the construction of the system. It is very significant that, when the coefficients are calculated, only the field component E_x must be regarded as large, whereas E_y may be taken to be small and examined by perturbation theory. In particular, to establish whether polarizational OB can occur, i.e., whether (4) is satisfied, it is sufficient to consider the linear response of the system to E_{yi}. This produces a considerable simplification of the analysis of systems with the large self-induced anisotropy that is necessary for polarizational OB.

Let us now consider polarizational OB in a medium with locally unique response, placed in a ring cavity. The field in the radiation escaping from the cavity is then determined by the field $\mathbf{E}(l)$ at exit from the medium, and the latter is determined (uniquely) by the field $\mathbf{E}(0)$ at entry to the medium. The fields $\mathbf{E}(0)$, $\mathbf{E}(l)$ are related to the incident field E_i by

$$\mathbf{E}(0)=\tilde{\mathbf{E}}_i+R\exp(i\varphi_R)\mathbf{E}(l),\qquad \tilde{\mathbf{E}}_i=T\mathbf{E}_i, \qquad (24)$$

where R^2, φ_R are, respectively, the resulting reflection coefficient and phase gain in the cavity, and T^2 is the transmission coefficient of the front mirror.

When (24) is taken into account, the conditions for the onset of intensity OB and polarizational OB, given by (3) and (4), can be rewritten in the form

$$\partial|\mathbf{E}_i|^2/\partial|\mathbf{E}(0)|^2=0,\qquad \psi(0)=\chi(0)=0, \qquad (25)$$

$$\partial(\psi_i,\ \chi_i)/\partial(\psi(0),\ \chi(0))=0,\qquad \psi(0)\to 0,\qquad \chi(0)\to 0. \quad (26)$$

Before we can calculate the derivative in (25), we must find the relation between $\mathbf{E}(l)$ and $\mathbf{E}(0)$ for $E_y(0) = 0$, and evaluate the derivatives in (26) for $|E_y(0)| \ll |E_x(0)|$ and $\delta(0) \equiv \operatorname{Arg}[E_y(0)/E_x(0)] = 0$ [it is clear from (5) and (24) that these $\delta(0)$ determine $\partial/\partial\psi(0)$, $\partial/\partial\chi(0)$, respectively].

We can now use (24)–(26) to find the relationships between the parameters for which OB takes place in the general case of nonlinear transparent media. The envelope $\mathbf{P}^{(nl)}$ of the nonlinear polarization of the medium at the field frequency ω can then be written in the form

$$\mathbf{P}^{(nl)}=-{}^1/_2\partial\mathscr{H}(\mathbf{E},\ \mathbf{E}^*)/\partial\mathbf{E}^*,\qquad \mathscr{H}^*=\mathscr{H}, \qquad (27)$$

where $\mathscr{H}(\mathbf{E}, \mathbf{E}^*)$ is proportional to the part of the field energy per unit volume that is due to the nonlinearity of the response of the medium (see Ref. 11). When (27) is taken into account, the truncated Maxwell equations can be reduced to the Hamilton-type equation

$$du/dz=\partial H/\partial\delta,\qquad d\delta/dz=-\partial H/\partial u, \qquad (28)$$

where

$$u=(|E_y|^2-|E_x|^2)/E^2,\qquad \delta=\operatorname{Arg}(E_y/E_x),$$
$$H\equiv H(E, u, \delta) = \mathscr{H}(\mathbf{E}, \mathbf{E}^*)/E^2,\quad E^2 = |\mathbf{E}|^2$$

and z is a coordinate measured in units of $cn(\omega)/2\pi\omega$. The nonlinear increase in the phase Φ_x for $E_y = 0$ is also determined by the function H:

$$\frac{d\Phi_x}{dz}=-\frac{1}{2}\left[\frac{\partial(E^2H)}{\partial E^2}\right]_{u=-1}. \qquad (29)$$

The symmetry of the medium and symmetry under time reversal ensure that the effective Hamiltonian H of the radiation has the following properties:

$$H(E,\ u,\ \delta)=H(E,\ u,\ \delta+n\pi),\qquad n=\pm1,\ \pm2,\ldots, \qquad (30)$$
$$H(E,\ u,\ \delta)=H(E,\ u,\ -\delta),\qquad H(E,\ u,\ \delta)=H(E,\ -u,\ \delta).$$

The following simple condition for the onset of intensity OB (refractive OB, in this case) follows directly from (24), (25), and (29):

$$1+R^2-2R\cos\varphi-\Lambda_I R\sin\varphi=0, \qquad (31)$$
$$\Lambda_I=E^2 l[\partial^2(E^2H)/\partial(E^2)^2]_{u=-1},\quad |\tilde{\mathbf{E}}_i|^2=E^2(1+R^2-2R\cos\varphi).$$

where l is the dimensionless thickness of the medium and φ is the total phase gain given by

$$\varphi = \varphi_R + l\{1 - 1/2[\partial(E^2H)/\partial E^2]_{u=-1}\}.$$

The condition for the appearance of the polarizational OB, given by (26), can be reduced to the following relation by solving (28) and using (24) and (30):

$$\operatorname{Re}[(1-Rt_1e^{i\varphi})(1-Rt_2^*e^{-i\varphi})]=0,\qquad t_{1,2}=t(\delta_{1,2}(0)), \qquad (32)$$

where $|t^2|$ is the transmission coefficient for the y-component of the field by the medium, and

$$t=[\zeta(0)/\zeta(l)]^{1/2}\exp\{i[\delta(l)-\delta(0)]\},$$
$$\zeta(z)=[\partial H(E,u,\delta(z))/\partial u]_{u=-1},$$
$$d\delta/dz=-\zeta(z),\quad \delta_1(0)=0,\quad \delta_2(0)=\pi/2. \tag{33}$$

Formulas (31)–(33) show that intensity OB is related to the dependence of the phase gain (refractive index of the nonlinear medium) on the radiation intensity, whereas polarizational OB is related to the self-induced anisotropy of refraction [for $\mathbf{P}^{(nl)}\,||\mathbf{E}$, we have $t_{1,2}=1$ and (32) is not satisfied]. These formulas determine explicitly the conditions for and type of OB that occurs for a specific mechanism of nondissipative optical nonlinearity.

The expressions given by (32) and (33) become simpler in the case of an isotropic medium. The final expression is similar to (31) except that Λ_I is replaced by the material parameter

$$\Lambda_P=l[\partial H(E,u,\pi/2)/\partial u]_{u=-1} \tag{34}$$

[in deriving (34), we took into account the fact that, in an isotropic medium, H depends only on E and $\cos 2\chi=(\cos^2\delta+u^2\sin^2\delta)^{1/2}$, in which case, $t_1=1$ and $|t_2|=(1+\Lambda_P^2)^{1/2}>1$.

According to (31) and (34), both intensity OB and polarizational OB will occur in a transparent isotopic medium for a certain detuning of the cavity ($\varphi\neq 2\pi n$); the necessary thickness l of the medium decreases as H becomes more dependent on the radiation intensity (in the case of intensity OB) or the degree of ellipticity (in the case of polarizational OB). Equations (31) and (34) describe, in particular, the onset of intensity OB and polarizational OB in the case of the cubic nonlinearity discussed in Ref. 2 (intensity OB and polarizational OB were obtained in this case in Ref. 5 for particular parameters values by numerical calculations).

Nonlinear absorption of light in the medium has an essentially different effect on intensity OB and on polarizational OB. A rapid rise in absorption with intensity will suppress intensity OB (see Ref. 1), but polarizational OB is still possible. In particular, dichroic OB, which is a variety of polarizational OB, can occur[7] in the case of two-photon resonance absorption and an accurately tuned cavity.

The approach developed in this paper can be used to consider polarizational OB when both absorption and refraction are essentially nonlinear. We can illustrate this in the special case of a cubic crystal, which is important from the experimental point of view, in which two-photon absorption takes place as a result of the $A_{lg}\to T_{lu}$ transition. We then have

$$P_\varkappa^{(nl)}=\gamma E_\varkappa(|\mathbf{E}|^2-|E_\varkappa|^2),\quad \varkappa=x,y,z,\quad \gamma=\gamma'+i\gamma'' \tag{35}$$

where the coordinate axes lie along the $\langle 100\rangle$ axes. When radiation propagates along the $\langle 100\rangle$ direction, the minimum threshold field $\tilde{E}_i^{(0)}$ for polarizational OB can be obtained from (24), (26), and (35):

$$\tilde{E}_i^{(0)}=E[1+\tilde{R}^2(1+\nu E^2)^{-1}-2\tilde{R}(1+\nu E^2)^{-1/2}\cos\varphi]^{1/2},$$
$$E^2={}^1/_2(\nu\tilde{R}^2h^2)^{-1}\{(1+\tilde{R}^2)^2-4\tilde{R}^2h\cos\varphi+(1+\tilde{R}^2)[(1+\tilde{R}^2)^2-4\tilde{R}^2(\cos^2\varphi-(\gamma'/\gamma'')^2\sin^2\varphi)]^{1/2}\},$$
$$\tilde{R}=R\exp(-\alpha_0 l/2),\quad \nu=(\gamma''/\alpha_0)[1-\exp(-\alpha_0 l)],$$
$$h=\cos\varphi-(\gamma'/\gamma'')\sin\varphi,\quad h>0. \tag{36}$$

where α_0 is the dimensionless one-photon absorption coefficient. The condition $h>0$ defines the range of φ in which polarizational OB is possible.

When the refraction nonlinearity is small ($|\gamma'|\ll\gamma''$), the smallest value of $\tilde{E}_i^{(0)}$ in (36) arises from exact tuning of the cavity ($\varphi=2\pi n$), and is exactly the same as the result reported in Ref. 7 (where it was obtained by a different method). For small absorption nonlinearity ($\gamma''\ll|\gamma'\sin\varphi|$), we have $\tilde{E}_i^{(0)}\approx[-\gamma'(\nu/\gamma'')\tilde{R}\sin\varphi]^{-1/2}(1+\tilde{R}^2-2R\cos\varphi)^{3/2}$. In general, the minimum of $\tilde{E}_i^{(0)}$ is reached for certain optimum values of φ and l. It is significant that absorption and refraction nonlinearities as mechanisms for polarizational OB do not suppress one another.

5. CONCLUSION

It follows from the foregoing results that, in nongyrotropic systems with a high enough degree of symmetry, we can identify two main types of optical bistability, namely, intensity OB and polarizational OB. They differ in both their physical origin and in their properties. The differences are particularly well-defined when the incident radiation parameters I_i,ψ_i,χ_i lie near the s-threshold for OB (the latter corresponds to radiation polarized along some symmetric direction in the nonlinear medium, $\psi_i=\chi_i=0$, and threshold intensity $I_i=I_B(0,0)$ for this polarization.

In general, we can cross the OB threshold by varying both the intensity and the polarization of the incident radiation. The two additional stationary states of radiation in the system (one is usually unstable) that appear in the course of this process in the neighborhood of the s-threshold for intensity OB differ most strongly from one another in their intensities. The intensity difference is proportional to the square root of the difference between the incident intensity I_i and its threshold (bifurcation) value $I_B(\psi_i,\chi_i)$ for the given polarization. Accordingly, the amplification factor for fluctuations in the incident intensity rises rapidly as the threshold is approached in the system. In the case of intensity OB, the surface $I_B(\psi_i,\chi_i)$ is an elliptic or hyperbolic paraboloid near its symmetry axis $\psi_i=\chi_i=0$.

Near the s-threshold for polarizational OB, there are three closely-spaced stationary states of the radiation in the system where response of the latter is multivalued. These states differ most strongly in their polarization. On the whole, it is precisely the polarization of the radiation in the system (including the multivalued region) near the s-threshold for polarizational OB that undergoes the largest change as the parameters of the incident radiation are varied. In fact, the multivalued response occurs as a result of the development of polarizational instability in the system. In the case of polarizational OB, the bifurcation surface $I_B(\psi_i, \chi_i)$ is highly anisotropic for low values of ψ_i, χ_i, and has a singularity (spinodal edge). The three stationary states of the radiation in the system merge on this edge.

Since the transmission remains qualitatively the same throughout the region of existence of the above stationary states, the results we have obtained provide us with information on the nature of OB in a wide range of incident-parameter values. The criteria for the onset of intensity OB and polarizational OB, obtained by taking into account symmetry considerations, are convenient in specific calculations, and can be used for systems containing isotropic media or crystals whose nonlinear optical response is significantly anisotropic.

We note in conclusion that soft splitting of radiation states near the s-threshold for polarizational OB (in contrast to the break in the case of intensity OB) leads to a relatively high probability of fluctuational transitions between states. The result of this is that a stationary distribution over the states can evolve in a relatively short time in a nonequilibrium system. This occurs independently of past history, and is determined exclusively by the incident radiation parameters and the character of the fluctuations (see Ref. 12).

[1)]When the point $\psi_i = \chi_i = 0$ lies on the line of self-crossing of $I_B(\psi_i, \chi_i)$, then for $I_i = I_B(0,0)$, we find that four states of transmitted radiation with the same I_t merge in pairs at this point, as do states with opposite ψ_t, χ_t $[\psi_t^{(1,2)} = -\psi_t^{(3,4)}, \chi_t^{(1,2)} = -\chi_t^{(3,4)}, \psi_t^2 + \chi_t^2 \sim 1]$.

[2)]The set of values of incident-radiation parameters $I_i = I_0''$, $\psi\theta_i = \chi_i = 0$ that corresponds to a symmetric point of the bifurcation surface $I_B(\psi_i, \chi_i)$ will be called the s-threshold for polarizational OB.

[3)]In the region of single-valued transmission, $|\theta_t| \sim |\theta_i|^{1/3} \gg |\theta_i|$ for $|\theta_i| \gg |c_0 \delta I_i \eta_i|$, $|\eta_i|^3$, i.e., small deviations of the polarization of the incident radiation are strongly amplified by a nonlinear system near the s-threshold for polarizational OB.

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