Complex geometry and the theory of quantum strings

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A summation over closed orientable surfaces of genus $p \geqslant 2$ (p-loop vacuum amplitudes in the theory of bosonic strings) in the critical dimension $\mathscr{D} = 26$ reduces to an integration over the moduli space M_p of complex structures of Riemann surfaces of genus p. The analytic properties of the integration measure are studied as a function of the complex coordinates on M_p. It is shown that the measure multiplied by $(\det \operatorname{Im} \hat{\tau})^{13}$ (where $\hat{\tau}$ is the period matrix of the Riemann surface) is the absolute square of a function holomorphic and nowhere vanishing on M_p. This function has a second-order pole at the infinity $D = \overline{M}_p / M_p$ of the compactified moduli space M_p. By these properties the measure is determined uniquely, up to an arbitrary constant factor, fact which allows one to construct explicit formulas in terms of theta functions for surfaces of genus $p = 2, 3$. The theory contains power and logarithmic divergences, related respectively to the renormalization of the tachyon wave function and of the slope. The relation of these results to Mumford's theorem is discussed. The quantum geometry of critical strings turns out to be a complex geometry.

1. INTRODUCTION

The theory of quantum strings undoubtedly deserves the immense interest it has generated. To the old ideas we add a new principle: fundamental objects are not pointlike, but one-dimensional; this theory has achieved a number of successes and is at the present time "the most promising candidate, etc."[1–3]

However, string theory cannot be considered to be free of problems. Since most often one discusses its phenomenology problems, we would like here to recall some of the theoretical problems. First, there is no understanding of the fundamental theoretical principle lying at the basis of the theory. The existing description is analogous to the description of particles in the language of sums over trajectories, rather than a field theory. Second, in connection with this absence of a profound fundamental understanding, we do not know how many string theories exist, and perceive the theories which exist for $D = 10$ and $D = 26$ as unique. On the other hand, these theories are special cases of two-dimensional conformal field theories (conformal bootstrap), Ref. 4. At the present time an infinite number of such theories is known. The use of (present and future) achievements of the conformal bootstrap may be beneficial for understanding string theories and increase our possibilities in the construction of new models. Third, there exist two languages for the description of strings: the algebraic language, making use of representations of Virasoro algebras and of other infinite-dimensional Lie algebras, and the geometric language, the language of sums over surfaces, which, as will be shown in the present paper, leads to the complex-analytic geometry of moduli spaces of Riemann surfaces. A relation between these two approaches does not exist, in the large. In the geometric approach the p-loop scattering amplitudes of oriented closed bosonic strings are sums over closed oriented surfaces of genus p ("spheres" with p handles). As will be shown in Sec. 2, in the critical dimension $\mathscr{D} = 26$, the summation reduces to an integration over the moduli space $\overline{M}_p$ of Riemann surfaces of genus p. If one takes string theory seriously it is important to study such sums. Here there arises a fourth problem—that of describing the analyticity properties of multiloop amplitudes as functions of the coordinates on $\overline{M}_p$. These properties determine the structure of divergences of the theory.

In the present paper we investigate just the latter problem and shall see that the analyticity properties turn out to be very simple. Roughly speaking, the amplitudes are constructed in terms of meromorphic, and even rational, functions on $\overline{M}_p$. One can formulate the problem and the result more precisely in the following manner: In the covariant geometric approach of Polyakov[5] the sum over surfaces is a sum over the topologies (the genera p), the internal metrics $g_{ab}(\xi)$, and the embeddings X_μ of the surface with coordinates $\xi_{1,2}$ in a flat $\mathscr{D}$-dimensional spacetime. For $\mathscr{D} = 26$ the conformal anomaly cancels[5] and the complete quantum symmetry group is the product of the Weyl group Conf(S) of conformal transformations: $g_{ab}(\xi) \to \lambda(\xi) g_{ab}(\xi)$ with the diffeomorphism group Diff(S) of general coordinate transformations of the surface S. Thus, for each p we must integrate over the orbits of the group $H = \operatorname{Conf}(S) \otimes \operatorname{Diff}(S)$ in the space $G(S)$ of all metrics, i.e., we must integrate over the quotient space $G/H = M_p$. This space is called the moduli space of Riemann surfaces of genus p, and its dimension, as shown by Riemann, is finite and equal to 0 for $p = 0$, equal to 2 for $p = 1$, and equal to $6p - 6$ for $p \geqslant 2$. In the papers of Teichmüller, Ahlfors, and Bers[6] it was shown that M_p has a natural complex structure. Moreover, M_p is an algebraic manifold.[7]

Let $y_1, \dots, y_{3p-3}$ denote some complex-analytic coordinates on M_p. Then for $\mathscr{D} = 26$ the sum over surfaces of genus p, after separating the volume of the gauge group H, will have the following form (see below, Sec. 2).

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$$Z_p = \int_{M_p} d\Omega \exp \tilde{W}(y_i, \bar{y}_i), \quad d\Omega = (i/2)^{3p-3} dv \wedge d\bar{v},$$

$$dv = dy_1 \wedge \ldots \wedge dy_{3p-3}, \tag{1.1}$$

where $\tilde{W}$ is some function of the coordinates $y_i, \bar{y}_i$. The natural question arises whether the complex structure on M_p manifests itself in the analytic properties of $\tilde{W}(y_i, \bar{y}_i)$? Recall that the one-loop calculation ($p = 1$) (Ref. 8) yields:

$$Z_1 = \int_{M_1} d^2y |y|^{-2} |\Delta(y)|^{-2} (\log |y|)^{-14};$$

$$\Delta(y) = y \prod_{n=1}^{\infty} (1-y^n)^{24}, \tag{1.2}$$

where $y = \exp(2\pi i\tau)$ and τ is the ratio of the two periods of a torus which ranges over fundamental domain of the modular function of the group $SL(2, \mathbb{Z})$. Equation (1.2) suggests that these properties may turn out to be sufficiently simple also in the case when $p > 1$. Up to a power of the logarithm the measure in (1.2) is the absolute square of a nowhere vanishing analytic function y which has a pole of second order at $y = 0$, where the torus degenerates. Our basic claim, proved in detail in Secs. 3 and 4, is that for $p > 1$ the measure exhibits almost the same properties:

$$\text{A) } \exp \tilde{W} = |F(y_1, \ldots, y_{3p-3})|^2 (\det \operatorname{Im} \hat{\tau})^{-13}, \tag{1.3}$$

where $F(y)dv$ is a holomorphic $(3p-3, 0)$-form which vanishes nowhere on M_p, and $\hat{\tau}$ is the period matrix of the Riemann surface with coordinates $y_1, \bar{y}_1, \ldots, y_{3p-3}, \bar{y}_{3p-3}$ in M_p.

B) The form $F(y)dv$ has a second-order pole at the point at infinity D of the space M_p, where the surfaces degenerate.[1)]

This pole leads to divergences in the expression (1.1), and its presence is closely related to the circumstance that the ground state of the bosonic string is a tachyon.

It is easy to show that the conditions A) and B) determine the form $F(y)dv$ uniquely, up to a constant factor. This allows, in particular, to express the $F(y)$ in the case $p = 2, 3$ (Ref. 10), as well as for $p = 4$, in terms of the Riemann theta functions. These results, together with necessary facts from the theory of automorphic forms of C. L. Siegel, form the contents of Section 6. We also indicate that recently, making use of the results of Faltings[11] and of the properties A) and B) of the measure,[12] Manin[13] has succeeded in expressing the measure in terms of theta functions and Abelian differentials by means of a more complicated formula, but in exchange, for arbitrary genus.

We note that in recent papers[2)] (Ref. 15) multiloop amplitudes have been constructed with the aid of the Selberg ζ function.[16] The only shortcoming of these elegant formulas is the fact that they yield an expression in terms of *real* coordinates on M_p (more precisely, on the Teichmüller space which covers it). This hides the simple complex-analytic structure of the theory, which is a serious obstacle to the study of supersymmetric (*SS*) and heterotic (*HS*) strings. We discuss briefly our approach to the theories of *SS* and *HS* in Section 7, where the vanishing of vacuum amplitudes is related the putative absence of parabolic forms of weight 8 on $\bar{M}_p$.

The occurrence of a complex-analytic structure in string theory is closely related to the conformal invariance and the cancellation of the gravitational anomaly separately in the sectors of right-movers and left-movers of the string.[18] Here $F(y)$ [respectively $\overline{F(y)}$] is the contribution to the measure of left-movers (right-movers). The three anomalies—the conformal, gravitational, and analytic ones—cancel simultaneously.

2. FROM A SUM OVER SURFACES TO INTEGRATION OVER MODULI SPACE

According to Polyakov[5] the sum over surfaces is defined as

$$\sum_{\text{surf}} e^{-(\text{area})} \overset{\text{def}}{=} \sum_{p=0}^{\infty} \int \mathscr{D}g_{ab}(\xi) \mathscr{D}X_\mu(\xi) \exp\{-S[X_\mu, g^{ab}]\}, \tag{2.1}$$

where $g_{ab}(\xi)$ is the intrinsic metric of a surface with coordinates ξ_1, ξ_2, and $X_\mu(\xi)$ defines the embedding of the surface in $\mathscr{D}$-dimensional spacetime; S is the Nambu-Goto action

$$S = \int d^2\xi\, g^{1/2} (g^{ab} \partial_a X^\mu \partial_b X_\mu + m_0^2). \tag{2.2}$$

We shall assume below that $\mathscr{D} = 26$. The integration measure in (2.1) is defined by means of integrals in function space:

$$\|\delta g\|^2 = \int g^{aa_1} g^{bb_1} \delta g_{ab} \delta g_{a_1 b_1} g^{1/2} d^2\xi, \quad \|\delta X\|^2 = \int (\delta X)^2 g^{1/2} d^2\xi. \tag{2.3}$$

Each metric determines a volume 2-form $ds = g^{1/2} d\xi^1 \wedge d\xi^2$ and a complex structure compatible with the metric

$$J_a{}^b = g^{1/2} \varepsilon_{ac} g^{cb}, \tag{2.4}$$

where $\varepsilon_{11} = \varepsilon_{22} = 0$, $\varepsilon_{12} = -\varepsilon_{21} = 1$. With this complex structure are associated harmonic coordinates z and $\bar{z}$; z is determined from the solution of the Beltrami equation:

$$J_a{}^b \frac{\partial z}{\partial \xi^b} = i \frac{\partial z}{\partial \xi^a}. \tag{2.5}$$

In these coordinates the metric takes on the conformal form:

$$\hat{g} = g_{ab} d\xi^a d\xi^b = \rho dz d\bar{z}, \quad \rho = e^\varphi. \tag{2.6}$$

For infinitesimal conformal transformations $\delta\rho = \delta\varphi\rho$ and general coordinate transformations $z \to z + \varepsilon(z, \bar{z})$, the variation of the metric is

$$\begin{gathered}\delta\hat{g} = \rho\delta\tilde{\varphi} dz d\bar{z} + \rho\bar{\partial}\varepsilon (\partial\bar{z})^2 + \rho\partial\bar{\varepsilon} (dz)^2, \\ \delta\tilde{\varphi} = \delta\varphi + \partial(\rho\varepsilon) + \bar{\partial}(\rho\bar{\varepsilon}); \\ \partial = \partial/\partial z, \quad \bar{\partial} = \partial/\partial\bar{z},\end{gathered} \tag{2.7}$$

and its norm in the sense of Eq. (2.3) equals

$$\|\delta\hat{g}\|^2 = \int \rho(\delta\tilde{\varphi})^2 d^2\xi + \int \rho(\partial\bar{\varepsilon})(\bar{\partial}\varepsilon) d^2\xi. \tag{2.8}$$

Here and henceforth $d^2\xi$ should be understood as $(1/2) i dz \wedge d\bar{z}$.

To separate in the sum (2.1) the volume of the group of general coordinate and conformal gauge transformations it is necessary, according to Ref. 5, to change over from an integration over the $g_{ab}(\xi)$ to an integration with respect to φ and ε. For these fields the norms in function space are, taking account of the scalar nature of φ and of the vector field character of ε:

$$\|\delta\varphi\|^2=\int\rho(\delta\varphi)^2d^2\xi,\quad \|\varepsilon\|^2=\int\varepsilon\bar{\varepsilon}\rho^2d^2\xi. \tag{2.9}$$

From Eqs. (2.8) and (2.9) we find that[3)]

$$\mathscr{D}g_{ab}(\xi)=\det(-\rho^{-2}\partial\rho\bar{\partial})\mathscr{D}\varphi\mathscr{D}\varepsilon\mathscr{D}g_{ab}^{\perp}. \tag{2.10}$$

In this formula $\mathscr{D}g_{ab}^{\perp}$ denotes integration with respect to those directions in the function space of the metrics which are "orthogonal" to the variations (2.7). To prove the existence of such directions we consider the infinitesimal variation of the metric $\delta\hat{g}^*$ [not to be confused with $\delta\hat{g}$ in Eq. (2.7)]:

$$\delta\hat{g}^*=\delta\varphi^*\rho dzd\bar{z}+f(dz)^2+\bar{f}(d\bar{z})^2. \tag{2.11}$$

From the orthogonality condition

$$\|\delta\hat{g},\delta\hat{g}^*\|=\int\delta\varphi\,\delta\varphi^*\rho\,d^2\xi+\int\bar{\partial}\varepsilon f\,d^2\xi+\int\partial\bar{\varepsilon}\bar{f}\,d^2\xi=0$$

we obtain

$$\delta\varphi^*=\bar{\partial}f=\partial\bar{f}=0. \tag{2.12}$$

The variations of the metric which are orthogonal to the orbits of the gauge group, i.e., satisfy the condition (2.12), are called holomorphic quadratic differentials. It is known that for a surface of genus $p\geqslant 2$ the complex dimension of the linear space V of such differentials is $3p-3$ (i.e., 1 for $p=1$ and 0 for $p=0$). Thus, the integration with respect to $\mathscr{D}g_{ab}^{\perp}$ is in fact an integration over the finite-dimensional space M_p of complex structures of Riemann surfaces with genus p (the moduli space), related to the variations of the metric of the form (2.12).

Complex-analytic coordinates are introduced in the space M_p in the following manner.[6] We choose a basis $f_1,\dots,f_{3p-3}$ in V and a dual basis $\eta^1,\dots,\eta^{3p-3}$ in the space of Beltrami differentials[4)]:

$$\int\eta^kf_j\,d^2\xi=\delta_j^k. \tag{2.13}$$

Then any complex structure J close to the structure J_0 which is compatible with the metric $\rho dzd\bar{z}$ can be parametrized by means of the complex parameters $y_1,\dots,y_{3p-3}$, and is compatible with the metric

$$\hat{g}_y=\rho|dz+y_i\eta^id\bar{z}|^2=\tilde{\rho}dud\bar{u}, \tag{2.14}$$

where the coordinate u is determined by the Beltrami equation

$$\frac{\partial u}{\partial\bar{z}}=y_i\eta^i\frac{\partial u}{\partial z} \tag{2.15}$$

and has a *holomorphic* dependence on the y_i.[6]

The conditions (2.13) determine η^k up to a total derivative

$$\eta^k\to\tilde{\eta}^k=\eta^k+\bar{\partial}\varepsilon^k, \tag{2.16}$$

however, complex structures corresponding to $\delta y_k\eta^k$ and $\delta y_k\tilde{\eta}^k$ and which are infinitesimally close to J_0 coincide. The arbitrariness (2.16) is fixed by the condition of orthogonality (2.14) of the metric to the variations (2.7), leading to the choice

$$\eta^k=\rho^{-1}(N_2^{-1})^{ki}\bar{f}_i,\quad (N_2)_{ik}\overset{\text{def}}{=}\int\bar{f}_if_k\rho^{-1}d^2\xi. \tag{2.17}$$

Consequently

$$\|\delta\hat{g}_y{}^{\perp}\|^2=\delta y_i\delta\bar{y}_k(N_2^{-1})^{ik}$$

and

$$Dg_{ab}{}^{\perp}=(\det N_2)^{-1}d\Omega,\qquad d\Omega=(i/2)^{3p-3}dv\wedge d\bar{v},\quad dv=dy_1\wedge\dots\wedge dy_{3p-3}. \tag{2.18}$$

We now substitute (2.18) into (2.10), carry out in (2.1) the Gaussian integration with respect to $X_\mu(\xi)$ (taking into account the zero mode $X_\mu^{(0)}(\xi)=\text{const}$) and separate the infinite volume of the group of general coordinate and conformal transformations $\int D\varepsilon D\varphi$. After this the problem reduces to the calculation of the following integral over the moduli space M_p:

$$Z_p=\int d\Omega\frac{\det\Delta_{-1}}{\det N_2}\left(\frac{\det N_0\det N_1}{\det'\Delta_0}\right)^{13}(\det N_1)^{-13}$$

$$\overset{\text{def}}{=}\int d\Omega\exp W(y_i,\bar{y}_i)(\det N_1)^{-13}. \tag{2.19}$$

Here $\Delta_j=-\rho^{j-1}\partial\rho^{-j}\bar{\partial}$ is the Laplace operator acting in the space of j-differentials (i.e., tensors $\Phi_{+\underset{j}{\dots}+}$, transforming like $(dz)^{-j}$), and

$$(N_j)_{\alpha\beta}=\int\rho^{1-j}\bar{\Phi}_\alpha\Phi_\beta\,d^2\xi \tag{2.20}$$

is the matrix of the inner products of the zero modes Φ_α of the operator Δ_j. We note that $\det N_1$ does not depend on ρ and $\det N_{-1}$ is missing in Eq. (2.19), since for $p\geqslant2$ the operator Δ_{-1} has no zero modes. On account of the cancellation of the conformal anomaly[5] the product of the remaining terms in (2.19) is also independent of ρ and can thus be only a function of $y_i,\bar{y}_i$. One can convince oneself of this by use of the formula[5,19,20]

$$\delta_\rho\log\frac{\det'\Delta_j}{\det N_j\det N_{1-j}}=\frac{C_j}{6\pi}\int\delta\rho\rho^{-1}\partial\bar{\partial}(\log\rho)d^2\xi,\qquad C_j=6j^2-6j+1, \tag{2.21}$$

where $\det N_k$ is set equal to 1 by definition if Δ_k has no zero modes. [In Eq. (2.21) 6π appears in place of the usual 24π, since $\partial\bar{\partial}=(1/4)\Delta$!].

3. THE HOLOMORPHY OF $F(Y)$

We now prove that $\exp W(y_i,\bar{y}_i)$ in Eq. (2.19) is the square of the absolute value of a holomorphic function of the y_i. For this we have to calculate the variation of W for an infinitesimal variation of the complex structure engendered by a variation of the metric having the form

$$\delta\hat{g}=\rho[\eta(y)\,d\bar{z}^2+\overline{\eta(y)}\,dz^2],\quad \eta=\sum_{i=1}^{3p-3}\delta y_i\eta^i\ll 1. \tag{3.1}$$

The function $\exp W(y_i,\bar{y}_i)$ will be the absolute square of a holomorphic function if (and only if) the second variation of W does not contain terms in $\eta\bar{\eta}$:

$$\delta_\eta\delta_{\bar\eta}W=0. \tag{3.2}$$

We shall show below that

$$\delta_\eta\delta_{\bar\eta}\log\frac{\det'\Delta_j}{\det N_j\det N_{1-j}}=-\frac{C_j}{6\pi}\int\rho^{-2}[\bar\partial f\,\partial\bar f+f\bar f\,\partial\bar\partial\log\rho]\,d^2\xi,$$

$$f=\rho\bar\eta, \tag{3.3}$$

whence, taking account of (2.19), the result (3.2) follows. The analytic anomaly (3.3) cancels and therefore

$$\exp W(y_i,\bar{y}_i)=|F(y_i)|^2. \tag{3.4}$$

Moreover, from (3.3) it follows that any expression of the form

$$\prod_j\left(\frac{\det'\Delta_j}{\det N_j\det N_{1-j}}\right)^{n_j} \tag{3.5}$$

will be the absolute square of a holomorphic function on M_p provided

$$\sum_j C_jn_j=0. \tag{3.6}$$

We now go on to prove Eq. (3.3). Assume that after the variation (3.1) the metric has a conformal form in terms of the coordinates $u,\bar u$. Then

$$\hat g'=\tilde\rho(u,\bar u)\,du\,d\bar u=\tilde\rho(u,\bar u)\,u_z\bar u_{\bar z}|dz+(u_{\bar z}/u_z)\,d\bar z|^2=\rho(z,\bar z)\,dz\,d\bar z+\rho\eta\,(dz)^2+\rho\bar\eta\,(d\bar z)^2, \tag{3.7}$$

whence

$$u_{\bar z}=\eta u_z,\quad \rho(z,\bar z)=\tilde\rho(u,\bar u)\,u_z\bar u_{\bar z}(1+\eta\bar\eta), \tag{3.8}$$

where $u_z\equiv\partial u/\partial z$, etc. In addition,

$$\frac{\partial}{\partial\bar u}=(1-\eta\bar\eta)^{-1}\frac{1}{\bar u_{\bar z}}\left(\frac{\partial}{\partial\bar z}-\eta\frac{\partial}{\partial z}\right). \tag{3.9}$$

We now represent the determinant as a functional integral in the coordinates $u,\bar u$:

$$(\det'\Delta_j(u))^{-1}=\int D'\Phi_j\exp\left[-\int\tilde\rho^{-j}\partial_u\bar\Phi_j\partial_{\bar u}\Phi_j\,i\,du\wedge d\bar u/2\right], \tag{3.10}$$

where the measure is defined by means of the integral

$$\|\delta\Phi_j\|_u^2=\int\overline{\delta\Phi_j}\,\delta\Phi_j\,\tilde\rho^{1-j}i\,du\wedge d\bar u/2. \tag{3.11}$$

In order to find out how the operator Δ_j has changed, we transform in Eqs. (3.10), (3.11) to the coordinates $z,\bar z$. Taking into account the fact that

$$\Phi_j(u,\bar u)=(u_z)^{-j}\Phi_j(z,\bar z),$$

and Eqs. (3.8), (3.9), we find that the action in the integral (3.10), to the required accuracy, has the expression

$$S=\int[1+(j+1)\eta\bar\eta]\rho^{-j}|[\partial-\eta\partial-j(\partial\eta)]\Phi_j|^2i\,dz\wedge d\bar z/2$$

so that the measure (3.11) equals

$$\|\delta\Phi_j\|_z^2=\int|\delta\Phi_j(z,\bar z)|^2(1+(j-2)\eta\bar\eta)\rho^{1-j}i\,dz\wedge d\bar z/2. \tag{3.12}$$

Thus

$$\Delta_j(u)=-\rho^{j-1}[1+(2-j)\eta\bar\eta][\partial-\bar\eta\bar\partial-(1-j)(\bar\partial\bar\eta)]\rho^{-j}[1+(j+1)\eta\bar\eta]\times[\bar\partial-\eta\partial-j(\partial\eta)]. \tag{3.13}$$

We first calculate the second variation of $\log\det N_j$. For this purpose we introduce some notation. Let A^j denote the space of j-differentials, $\mathbf{P}_j$ the projector onto the subspace H^j of holomorphic j-differentials (the zero modes of Δ_j), $\{\Phi_\alpha^{(j)}\}$ is a basis in H^j which we choose as having a holomorphic dependence on the y_j. This is possible, since it follows from Eq. (3.9) that the equation $\bar\partial\Phi_\alpha=0$ satisfied by Φ_α does not depend on the $\bar y_i$. Let, in addition, the index of the symbol Tr denote the space over which the trace is calculated. Then it follows from Eq. (3.12) that

$$\delta\bar\delta\log\det N_j=\mathrm{Tr}_{H^j}(j-2)\eta\bar\eta+\int\rho^{1-j}(\delta\bar\Phi_\alpha^{(j)})(\delta\Phi_\beta^{(j)})(N_j^{-1})^{\alpha\beta}i\,dz\wedge d\bar z/2. \tag{3.14}$$

We make use of the fact that $\delta\bar\Phi_\beta^{(j)}=\bar\delta\Phi_\beta^{(j)}=0$. The variation $\delta\Phi_\alpha$ can be determined from the equation

$$(\Delta_j+\delta\Delta_j)(\Phi_\alpha^{(j)}+\delta\Phi_\alpha^{(j)})=0.$$

It is orthogonal to H^j and has the expression

$$\delta\Phi_\alpha^{(j)}=-\frac{1}{\Delta_j}\delta\Delta_j\Phi_\alpha^{(j)}.$$

As a result we have

$$\delta\bar\delta\log\det N_j=\mathrm{Tr}_{H^j}(j-2)\eta\bar\eta+\mathrm{Tr}_{H^j}\bar\delta\Delta_j\left(-\frac{1}{\Delta_j}\right)\frac{M^2}{M^2+\Delta_j}\delta\mathbf{P}_j,$$

$$M^2\to\infty, \tag{3.15}$$

where we have regularized Δ_j^{-1} and have made use of the hermiticity of Δ_j; $\bar\delta\Delta_j$ denotes the term in (3.13) which is linear in $\bar\eta$.

Now everything is ready for the calculation of the variation of the determinant. We regularize it *à la* Pauli-Villars and keep track only of the terms which do not depend on the regulator mass M:

$$\delta\bar\delta\log[\det'\Delta_j/\det(M^2+\Delta_j)]=\mathrm{Tr}_{A^j}(1-\mathbf{P}_j)(2-j)\eta\bar\eta\frac{M^2}{M^2+\Delta_j}$$
$$+\mathrm{Tr}_{A^{1-j}}(1-\mathbf{P}_{1-j})(j+1)\eta\bar\eta\frac{M^2}{M^2+\Delta_{1-j}}$$
$$+\delta\,\mathrm{Tr}_{A^j}\rho^{j-1}(\bar\delta\partial_{1-j})(1-\mathbf{P}_{1-j})(\rho^{j-1}\partial)^{-1}\cdot$$
$$\times\frac{M^2}{M^2+\Delta_j}(1-\mathbf{P}_j)=\mathrm{Tr}_{A^j}(2-j)\eta\bar\eta\frac{M^2}{M^2+\Delta_j}$$

$$+ \mathrm{Tr}_{A^{1-j}}(j+1)\eta\bar{\eta}\frac{M^2}{M^2+\Delta_{1-j}}$$

$$+(j-2)\mathrm{Tr}_{H^j}\eta\bar{\eta}-(j+1)\mathrm{Tr}_{H^{1-j}}\eta\bar{\eta}$$

$$+\mathrm{Tr}_{H^j}\delta\Delta_j\left(-\frac{1}{\Delta_j}\right)\frac{M^2}{M^2+\Delta_j}\delta\mathrm{P}_j$$

$$+\mathrm{Tr}_{H^{1-j}}\delta\Delta_{1-j}\left(-\frac{1}{\Delta_{1-j}}\right)\frac{M^2}{M^2+\Delta_{1-j}}\delta\mathrm{P}_{1-j}$$

$$+\mathrm{Tr}_{A^j}\rho^{1-j}(\delta\partial_{1-j})\frac{M^2}{M^2+\bar{\Delta}_{1-j}}\rho^{-j}(\delta\bar{\partial}_j)\frac{1}{M^2+\Delta_j}, \qquad (3.16)$$

where $\delta\bar{\partial}_j \equiv -\eta\partial - j(\partial\eta)$. We denote the last term by Y and making use of the equations (3.15) and of

$$\mathrm{Tr}_{A^j}\eta\bar{\eta}\frac{M^2}{M^2+\Delta_j}=\frac{3j-1}{6\pi}\int\eta\bar{\eta}\partial\,\bar{\partial}\varphi\,d^2\xi,\quad \varphi=\log\rho, \qquad (3.17)$$

we obtain[5]

$$\delta\bar{\delta}\log\det{}'_{Reg}\Delta_j=-\frac{6j^2-6j}{6\pi}\int\eta\bar{\eta}\partial\,\bar{\partial}\varphi\,d^2\xi+Y$$

$$+\delta\bar{\delta}\log(\det N_j\det N_{1-j}). \qquad (3.18)$$

Equation (3.17) is derived in the following manner. We calculate the trace choosing in the space A^j the basis formed by the "functions"

$$\delta(\xi-\xi_0)=\int\frac{d^2p}{(2\pi)^2}\exp[i(p\bar{z}+\bar{p}z)],$$

where $z=(\xi-\xi_0)_1+i(\xi-\xi_0)_2$ is a coordinate in the vicinity of ξ_0; and $2p=p_1+ip_2$. For an arbitrary operator $V(\xi)$ we have

$$\mathrm{Tr}\,V(\xi)=\int d^2\xi_0\int\frac{d^2p}{(2\pi)^2}V^{\bullet}(\xi_0,p,\bar{p}), \qquad (3.19)$$

where in the neighborhood of each point ξ_0 we have passed to the momentum representation

$$V(\xi)=V(\xi_0,z,\bar{z};\partial,\bar{\partial})$$
$$=V(\xi_0,i\partial/\partial\bar{p},i\partial/\partial p;i\bar{p},ip)$$
$$\equiv V^{\bullet}(\xi_0,p,\bar{p}). \qquad (3.20)$$

In Eq. (3.17) the unique term which does not depend on M contributes to expansion of Δ_j a term proportional to $\partial\bar{\partial}\varphi=1/4\Delta\varphi$:

$$\Delta_j^{\bullet}(\xi_0,p,\bar{p})=\rho_0^{-1}p\bar{p}+\rho_0^{-1}[(1-j)(\partial\bar{\partial}\varphi)$$
$$\times(1+p\partial_p)+(\partial\bar{\partial}\varphi)(\bar{p}\partial_{\bar{p}}+p\bar{p}\partial_p\partial_{\bar{p}})]+\ldots,$$
$$\rho_0\equiv\rho(\xi_0),\quad \partial_p=\partial/\partial p. \qquad (3.21)$$

Therefore one can omit the remaining terms in Eq. (3.21) and we have

$$\mathrm{Tr}_{A^j}\eta\bar{\eta}\frac{M^2}{M^2+\Delta_j}=-\int d^2\xi_0\,\eta\bar{\eta}\partial\,\bar{\partial}\varphi\int\frac{d^2p}{(2\pi)^2}\frac{M^2}{M^2+\rho_0^{-1}p\bar{p}}$$
$$\times\rho_0^{-1}[(1-j)(1+p\partial_p)+\bar{p}\partial_{\bar{p}}+p\bar{p}\,\partial_p\,\partial_{\bar{p}}]$$
$$\times\frac{1}{M^2+\rho_0^{-1}p\bar{p}}=\frac{3j-1}{6\pi}\int\eta\bar{\eta}\partial\,\bar{\partial}\varphi\,d^2\xi,$$

Q.E.D.

The quantity Y in Eq. (3.18) can be calculated similarly. A complication arises only through the fact that one must expand in z, $\bar{z}$ not only φ, but also η and $\bar{\eta}$, retaining not only the terms $\partial\eta\bar{\partial}\bar{\eta}$ but also the terms linear in φ of the type $\eta\partial\varphi\bar{\partial}\bar{\eta}$, $\eta\bar{\eta}\partial\bar{\partial}\varphi$, etc. [In Eq. (3.18) there appears also a term $\partial\varphi\bar{\partial}\varphi\eta\bar{\eta}$, but if the coefficients of all the other terms are known, its coefficient is fixed by the general covariance requirement.] After tedious calculations we find:

$$Y=-\frac{6j^2-6j+1}{6\pi}\int(\bar{\partial}\bar{\eta}\,\partial\eta+\bar{\partial}\bar{\eta}\,\partial\varphi\,\eta+\bar{\eta}\,\bar{\partial}\varphi\,\partial\eta+\bar{\eta}\eta\,\partial\varphi\,\bar{\partial}\varphi)$$
$$\times d^2\xi-\frac{1}{6\pi}\int\eta\bar{\eta}\partial\,\bar{\partial}\varphi\,d^2\xi. \qquad (3.22)$$

Substituting into Eq. (3.18) we arrive at (3.3). With the help of Eq. (3.17) it is also easy to verify Eq. (2.21).

Summarizing; we have proved Eq. (3.2) and the measure in Eq. (2.19) is indeed, up to the factor $(\det N_1)^{-13}$ the square of the modulus of a holomorphic function, provided that the basis $\Phi_\alpha^{(1)}$ in the space of holomorphic 1-differentials is chosen so as to have a holomorphic dependence on the y_i. This can be realized in the following manner. We choose on the surface S of genus p a symplectic basis consisting of $2p$ closed, noncontractible, oriented paths a_i, b_i, $i=1,\ldots,p$ such that

$$a_i\circ a_j=b_i\circ b_j=0,\quad i\neq j;\quad a_i\circ b_j=\delta_{ij}, \qquad (3.23)$$

where $a\circ b$ denotes the algebraic number of intersections (the intersections are counted with their natural signs). It is known that the space of holomorphic 1-differentials (Abelian differentials of the first kind) has complex dimension p, and that one can select in it a basis $\omega_i(z)=\Phi_i^{(1)}(z)dz$ of normalized differentials, such that

$$\oint_{a_i}\omega_j=\delta_{ij}. \qquad (3.24)$$

Then the matrix

$$\tau_{ij}=\oint_{b_i}\omega_j \qquad (3.25)$$

is called the period matrix of the surface S. In this basis

$$(N_1)_{kj}=\frac{i}{2}\int_S\omega_k\wedge\bar{\omega}_j=\mathrm{Im}\,\tau_{kj}. \qquad (3.26)$$

Substituting into (2.19) and recalling Eq. (3.4), we obtain Eq. (1.3). The holomorphy and absence of zeros of the function $F(y)$ follows from the fact that the regularized determinants in Eq. (2.19) must not vanish on nondegenerate surfaces (since the number of zero modes of each of them is constant: one zero mode for Δ_0 and no zero modes for Δ_{-1}), nor do they become infinite.[6] Consequently, we have proved property A of the measure, stated in the Introduction.

We now briefly discuss the connection between holomorphy of the measure and conformal invariance. The second variation $\delta\bar{\delta}\tilde{W}$ of the effective action of the ghosts and fields X_μ can be expressed in terms of the correlation functions of the energy-momentum tensor operator $T_{\mu\nu}=T_{\mu\nu}^{\text{ghost}}+T_{\mu\nu}^{X}$:

$$\delta\bar{\delta}\bar{W} = \int d^2\xi\, \eta(\xi)\, d^2\xi'\, \overline{\eta(\xi')} \langle T_{++}(\xi) T_{--}(\xi')\rangle. \quad (3.27)$$

From the naive conservation law it follows that $\partial_- T_{++} = \partial_+ T_{--} = 0$, hence, up to zero modes, $\langle T_{++}(\xi) T_{--}(\xi')\rangle = 0$. On account of the conformal anomaly this is not separately true for the ghosts and X_μ fields, so that $\partial_- \langle T_{++}\rangle = \partial_+ \langle T_{-+}\rangle$ and $\langle T_{-+}\rangle \neq 0$. However, for $D = 26$ the anomaly cancels, which has as a consequence $\langle T_{++}(\xi) T_{--}(\xi')\rangle = 0$ up to zero modes; taking the latter into account, we are again led to the result (3.3).

We now go over to an analysis of the behavior of the measure at the point at infinity D of the space M_p, where the surfaces degenerate, and shall prove property B).

4. DIVERGENCES

In this section it will be convenient to deal not with determinants, but with functional integrals. We shall study the divergences of the following integral:

$$Z_p = \int \frac{d\Omega}{\det(f_i, f_j)} \frac{\left[\int \mathcal{D}\varphi \exp\left(-\int \partial\bar{\varphi}\, \bar{\partial}\varphi\, d^2\xi\right)\right]^{13}}{\int \mathcal{D}\varepsilon \exp\left(-\int \rho\partial\bar{\varepsilon}\, \bar{\partial}\varepsilon\, d^2\xi\right)}, \quad (4.1)$$

where φ is a complex scalar field, ε is the complex ghost vector field, f_i is a basis in the space of quadratic holomorphic differentials, related to the deformations of the complex structure

$$dz \to dz + \delta y_i \eta^i d\bar{z} \quad (4.2)$$

by the relation

$$\int \eta^j f_i\, d^2\xi = \delta_i^j, \quad (4.3)$$

the y_i are complex coordinates in the moduli space M_p defined in a neighborhood of the given complex structure dz by Eq. (4.2). The inner product is

$$(f_i, f_j) \overset{\text{def}}{=} \int f_i \bar{f}_j \rho^{-1}\, d^2\xi. \quad (4.4)$$

Eq. (4.1) is also valid for $p = 1$. The measure in the expression of Z_p diverges in two cases.

Case I. The surface of genus p degenerates into two surfaces of genus q and $p - q$ with removed points, with the surfaces glued together at these points (Fig. 1). The set of such surfaces in the moduli space $\bar{M}_p$ will be denoted by D_q, $q = 1, 2, \ldots, [p/2]$.

Case II. The surface of genus p degenerates into a surface of genus $p - 1$ with two points glued together—the remnant of a degenerate handle (Fig. 2). In $\bar{M}_p$ such surfaces are situated on a manifold denoted by D_0. We determine the codimensions (i.e., $\dim M_p - \dim D_\alpha$) of D_q and D_0 in $\bar{M}_p$.

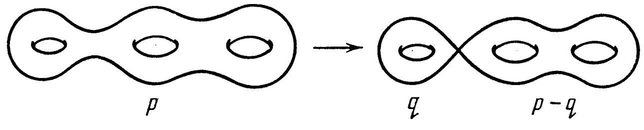

FIG. 1.

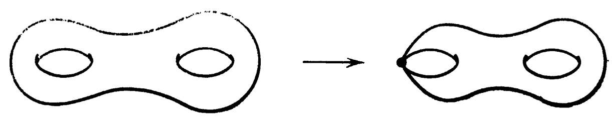

FIG. 2.

For this we make use of the fact that the complex dimension of the moduli space of a surface of genus $p \geqslant 1$ with n distinguished points equals $3p - 3 + n$ [the n coordinates of points of a polygon in the Lobachevsky plane + $(6p - 6)$ parameters of the polygon], therefore the dimensions of D_q and D_0 are

$$\dim D_q = 3q - 3 + 1 + 3(p - q) - 3 + 1 = 3p - 4,$$
$$\dim D_0 = 3(p - 1) - 3 + 2 = 3p - 4. \quad (4.5)$$

Thus, all the D_α have complex codimension 1 in $\bar{M}_p$ and do indeed complete the moduli space of nonsingular surfaces M_p making is a compact space $\bar{M}_p$. For the analysis of the behavior of the measure in Z_p in a neighborhood of $D = D_0 \cup D_1 \cup \ldots \cup D_{[p/2]}$ we choose in this neighborhood the coordinate y_1 transverse to D and the coordinates $y_2, \ldots, y_{3p-3}$ along D, so that locally D is given by the equation

$$y_1(D) = 0. \quad (4.6)$$

We study the measure as a function of y_1 for fixed $y_2, \ldots, y_{3p-3}$. One can show that in a neighborhood of $y_1 = 0$ by means of a conformal transformation of the metric one can transform a degenerating strangulation into a very long cylinder[7] (Fig. 3). This representation is convenient for the investigation of asymptotic behavior. Recall that the measure in Z_p is *independent* of the choice of conformal metric.

In both cases we shall consider the surface S_q as glued together from a cylinder and one (case I) or two (case II) "lids" (Fig. 4). We choose the coordinates on the cylinder as indicated in Fig. 5. To the surfaces in D corresponds the limit $T \to \infty$ in which we are interested. It will be seen in what follows that the "natural" coordinate y_1 is

$$y_1 = \exp[-(T + i\delta)], \quad (4.7)$$

where δ is the rotation angle of the right-hand edge of the cylinder K relative to the left edge, when glued to the lid.

We first estimate the functional integrals. For this we must fix the boundary conditions on the loops Γ_1, Γ_2 and Γ_3, Γ_4, calculate the integrals with prescribed boundary conditions, after that multiply them and integrate over all boundary conditions. In the case I) we obtain for the scalars

$$I_0 = \int \mathcal{D}\varphi \exp\left(-\int_{S_p} \partial\bar{\varphi}\, \bar{\partial}\varphi\, d^2\xi\right)$$
$$= \int \mathcal{D}\varphi(0, \sigma)\, \mathcal{D}\varphi(T, \sigma) \exp\{-v_1[\varphi(0, \sigma)]\}$$

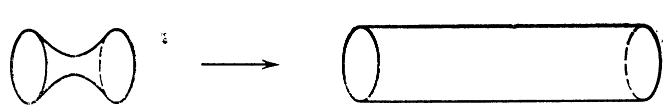

FIG. 3.

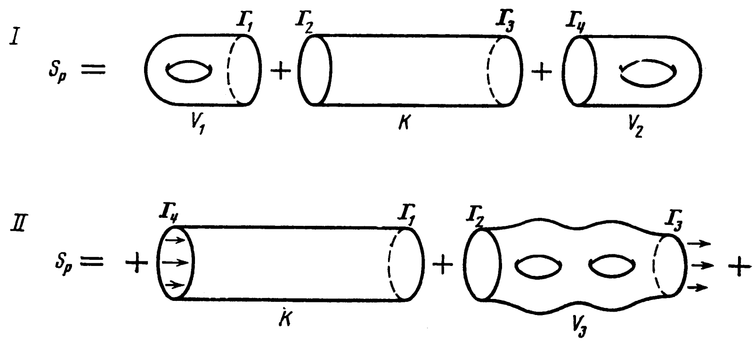

FIG. 4.

$$\times G[\varphi(0,\sigma),\varphi(T,\sigma)]\exp\{-v_2[\varphi(T,\sigma)]\},\tag{4.8}$$

where

$$\exp\{-v_1[\varphi(0,\sigma)]\} = \int_{\text{given }\varphi(0,\sigma)} \mathcal{D}\varphi \exp\Big(-\int_{V_1} \partial\varphi\,\bar\partial\varphi\, d^2\xi\Big),$$

$$G = \int_{\substack{\varphi(0,\sigma)\\ \varphi(T,\sigma)}} \mathcal{D}\varphi \exp\Big\{-\int_K \partial\varphi\,\bar\partial\varphi\, d\sigma\, d\tau\Big\}.\tag{4.9}$$

Similarly one determines $\exp\{-v_2[\varphi(T,\sigma)]\}$ and $\exp\{-v_3[\varphi(0,\sigma),\varphi(T,\sigma)]\}$ for the case II. The same holds for the ghosts, however the action is not $\int \partial\bar\varphi\bar\partial\varphi d^2\xi$, but $\int \rho\partial\bar\varepsilon\bar\partial\varepsilon d^2\xi$.

Since on the cylinder K we have $\rho = 1$, the expression for G is the same for the scalars and for the ghosts:

$$G[\varphi(0,\sigma),\varphi(T,\sigma)] = \exp\{-S_{cl}[\varphi(0,\sigma),\varphi(T,\sigma)]\}\times(\det_{Dir}\Delta_0)^{-1},\tag{4.10}$$

where S_{cl} is the classical action evaluated for the solution of the Laplace equation $\partial\bar\partial\varphi = 0$ with the boundary conditions

$$\varphi(0,\sigma) = \sum_{n=-\infty}^{\infty}\varphi_n(0)e^{in\sigma},\quad \varphi(T,\sigma) = \sum_{n=-\infty}^{\infty}\varphi_n(T)e^{in\sigma},\tag{4.11}$$

and $\det_{Dir}\Delta_0$ is the determinant of the Laplace operator with Dirichlet boundary conditions on the cylinder K. After uncomplicated calculations we find

$$G = T^{-1}e^{T/6}\prod_{n=1}^{\infty}(1-e^{-2nT})^{-2} \times \exp\Big\{-\sum_{n=-\infty}^{\infty}\frac{n}{\operatorname{Sh} nT}e^{nT}|\varphi_n(0)-e^{-nT}\varphi_n(T)|^2\Big\}.\tag{4.12}$$

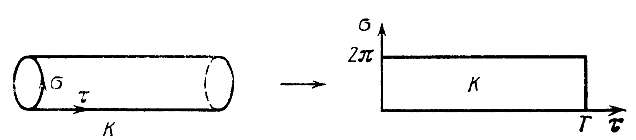

FIG. 5.

Thus, in the case I we must find the asymptotic behavior for $T\to\infty$ of the following expression:

$$I_0 = T^{-1}e^{T/6}\prod_{n=1}^{\infty}(1-e^{-2nT})^{-2}\int\prod_{n=-\infty}^{+\infty}\mathcal{D}^2\varphi_n(0)\mathcal{D}^2\varphi_n(T) \times\exp[-v_1[\varphi_n(0)]-S_{cl}[\varphi_n(0),\varphi_n(T)]-v_2[\varphi_n(T)]].\tag{4.13}$$

For $T\to\infty$ one can neglect in S_{cl} the vanishing terms, if only this does not lead to additional degeneracies in the quadratic form $v_1 + S_{cl} + v_2$, i.e., to independence of the expression on $\varphi_n(0)$ or $\varphi_n(T)$.

From Eq. (4.12) for G it can be seen that

$$S_{cl}|\underset{T\to\infty}{\longrightarrow}\frac{1}{T}|\varphi_0(0)-\varphi_0(T)|^2+\sum_{n>0}2n|\varphi_n(0)|^2 + \sum_{n<0}2|n|\,|\varphi_n(T)|^2+\ldots.\tag{4.14}$$

Where ... denotes exponentially small terms. We have retained $|\varphi_0(0)-\varphi_0(T)|^2/T$ since there are scalar zero modes on V_1 and V_2 and v_1 and v_2 do not depend on $\varphi_0(0)$ and $\varphi_0(T)$. One may neglect the exponentially small terms if v_1 does not degenerate on any vector of the form

$$\sum_{n<0}a_n e^{in\sigma},$$

and v_2 does not degenerate on vectors of the form

$$\sum_{n>0}b_n e^{in\sigma}.$$

We prove that this is indeed so. Let us assume the contrary. In this case there is a solution φ^* of the equation $\bar\partial\varphi = 0$ on V_1 which on Γ_1 becomes equal to

$$\sum_{n<0}a_n e^{in\sigma}.$$

Let us now imagine that the cylinder K extends to the right to infinity ($\tau\geqslant 0$) and the metric ρ on it differs from one and equals $e^{-2\tau} = \rho^*$. In the coordinates

$$u = e^{-(\tau+i\sigma)} \equiv e^{-z}$$

the cylinder $0\leqslant\sigma\leqslant 2\pi$, $0\leqslant\tau$ becomes the disk $|u|\leqslant 1$ with constant unit metric

$$\rho^* dz\wedge d\bar z = |u|^2 d\log u\wedge d\log\bar u = du\wedge d\bar u.$$

With this disk the surface V_1 is glued into a compact surface V_1^* of genus $q\geqslant 1$.

The solution φ^* can be extended to the cylinder maintaining its holomorphy:

$$\varphi^*(\tau+i\sigma)=\sum_{n<0} a_n e^{n(\tau+i\sigma)}.$$

Consequently on the surface V_1^* there exists a *holomorphic* function φ^* which is *different from a constant*. It is known, however, that such functions do not exist. This contradiction proves our assertion.

Summarizing, the form v_1 has no zero vectors of the type

$$\sum_{n<0} a_n e^{in\sigma}.$$

Similarly v_2 has no zero vectors of the type

$$\sum_{n>0} b_n e^{in\sigma}$$

and it is indeed legitimate to neglect in S_{cl} the exponentially small terms. Therefore

$$I_0\Big|_{T\to\infty} \to T^{-1}e^{T/6}\int d^2\varphi_0(0)\,d^2\varphi_0(T)$$

$$\times\exp[-|\varphi_0(0)-\varphi_0(T)|^2/T]\cdot\text{const}$$

$$\sim e^{T/6}\int d^2[\varphi_0(0)+\varphi_0(T)]/2. \qquad (4.15)$$

As should have been expected, there remains an integral over the zero mode—the volume of the "Universe." In the sequel we shall set it equal to one. There remains

$$I_0(T)\Big|_{T\to\infty}\sim e^{T/6}\quad(\text{on } D_q,\ q\neq 0). \qquad (4.16)$$

We now deal with the integral over the ghosts

$$I_1(T)=\int\mathscr{D}\varepsilon\,\exp\Big(-\int\rho\,\partial\bar\varepsilon\,\bar\partial\varepsilon\,d^2\xi\Big). \qquad (4.17)$$

The only difference compared to the scalars consists in the fact that in S_{cl} in Eq. (4.10) one need not regain $|\varepsilon_0(0)-\varepsilon_0(T)|^2/T$, since now v_1 has no zero vectors of the form

$$\sum_{n\leqslant 0} a_n e^{in\sigma}$$

($n\leqslant 0$, and not <0, as was the case for scalars!), and v_2 has none of the form

$$\sum_{n\geqslant 0} b_n e^{in\sigma}.$$

To prove this we again assume that the opposite is true, like in the scalar case. Then on the surface V_1 there exists a solution ε^* of the equation $\bar\partial\varepsilon$ which is continued to the cylinder K by the holomorphic function

$$\varepsilon^*(\tau+i\sigma)=\sum_{n\leqslant 0} a_n e^{n(\tau+i\sigma)}.$$

However, ε now is not a scalar, but a vector. In the coordinates u

$$\varepsilon^*(u)=\varepsilon^*(z)\frac{du}{dz}=-\varepsilon^*(z)\,u=-\sum_{n\leqslant 0} a_n u^{1-n}$$

vanishes for $u=0$. Consequently for the glued surface V_1^* there exists a holomorphic *nonconstant* vector field $\varepsilon^*(z)$. But we know that such fields do not exist if the genus of the surface is equal to or larger than one. Moreover, for $q>1$ there are no holomorphic vector fields at all, and for $q=1$ there exists one holomorphic nowhere vanishing vector field. From this we conclude that v_1 does not admit null vectors of the type

$$\sum_{n\leqslant 0} a_n e^{in\sigma}.$$

Since in this case one need not retain in S_{cl} the term $|\varepsilon_0(0)-\varepsilon_0(T)|^2/T$, the whole leading part of the T-dependence is determined by $\det_{\rm Dir}\Delta_0(T)$ in G:

$$I_{-1}(T)\Big|_{T\to\infty}\sim T^{-1}e^{T/6}\quad(\text{on } D_q,\ q\neq 0). \qquad (4.18)$$

The ratio of functional integrals appearing in the measure Z_p behaves as[8]

$$I_{(\mathrm{I})}=\frac{I_0^{13}}{I_{-1}}\Big|_{T\to\infty}\sim Te^{2T}\quad(\text{on } D_q,\ q\neq 0). \qquad (4.19)$$

The case II is treated in exactly the same manner, only V_3 is glued up with two disks—one on the right and one on the left. The only difference is the fact that now one may neglect the term $|\varphi_0(0)-\varphi_0(T)|^2/T$ in S_{cl} in the case of the scalar field. This is related to the fact that for v_3 the zero mode is the sum $\varphi_0(0)+\varphi_0(T)$ and not the terms $\varphi_0(0)$ and $\varphi_0(T)$ separately, as was the case with v_1+v_2 in case I.

Thus $I_0\sim I_1$ in case II and the ratio of the functional integrals is

$$I_{(\mathrm{II})}=\frac{I_0^{13}}{I_{-1}}\Big|_{T\to\infty}\sim(I_{-1})^{12}\sim T^{-12}e^{2T}\quad(\text{on } D_0). \qquad (4.20)$$

We still need to estimate the volume form

$$\frac{d\Omega}{\det(f_i,f_j)}=\prod_{i=1}^{3p-3}\frac{i}{2}\,dy_i\wedge d\bar y_i/\det(f_i,f_j). \qquad (4.21)$$

For this we use the fact that with the variation of T there is associated a Beltrami differential which is constant on K. Indeed, the variation of the complex structure generated by such a differential: $dz\to dz+a d\bar z$, may be considered as a transformation $z\to\tilde z=z+a\bar z$, which maps the rectangle $0\leqslant\sigma\leqslant 2\pi$, $0\leqslant\tau\leqslant T$ $(z=\tau+i\sigma)$ into a parallelogram. The new value of the coordinate $(T+i\delta)/2\pi i$ in the space $\overline{M}_p$ is the complex ratio of the periods of this parallelogram

$$d\Big(\frac{T+i\delta}{2\pi i}\Big)=\frac{\tilde z(T)}{\tilde z(2\pi i)}-\frac{T}{2\pi i}=\frac{Ta}{i\pi},$$

whence $a=d(T+i\delta)/2T$. Consequently, to the coordinate $\tilde y_1=T+i\delta$ in the moduli space $\overline{M}_p$ corresponds the Beltrami differential $\eta^1=1/2T$, since to a shift by $d\tilde y_1$ in $\overline{M}_p$

corresponds, as we just showed, a variation of the complex structure

$$dz \to dz + d\tilde{y}_1 \eta^1 d\bar{z}.$$

From Eq. (4.3) we find that the quadratic differential[9] f_1 on K is equal to one. The coordinate $\tilde{y}_1$ has a direction transverse to D. The other quadratic differentials can be chosen so that $(f_1, f_1) \sim 1$ and the corresponding coordinates are along D. In this case

$$\det(f_i f_j) \sim (f_1, f_1) \sim T, \tag{4.22}$$

$$\frac{d\Omega}{\det(f_i, f_j)} \sim \frac{d\tilde{y}_1 \wedge d\bar{\tilde{y}}_1}{T} \wedge d\Omega_\parallel = \frac{dT\, d\delta}{T} d\Omega_\parallel$$

$$= \frac{e^{2T}}{T} dy_1 \wedge d\bar{y}_1 \wedge d\Omega_\parallel$$

$$= \frac{dy_1 \wedge d\bar{y}_1}{|y_1|^2 \log(1/|y_1|)} \wedge d\Omega_\parallel,$$

where

$$y_1 = e^{-\tilde{y}_1} = e^{-(T+i\delta)}.$$

The surface D is given by the equation $y_1(D) = 0$. This coordinate is good in the sense that, as can be seen from (4.12), the ratio of determinants can be expanded in a series in y_1 and $\bar{y}_1$ and the divergences which seem to be exponential will be powers in the coordinates y_1 ... .

Collecting Eqs. (4.19)–(4.22) we find the asymptotic behavior of the measure in the neighborhood of D_q, $q \neq 0$:

$$\frac{d\Omega}{\det(f_i, f_j)} \frac{I_0^{13}}{I_{-1}} \bigg|_{|y_1| \to 0} \sim \frac{d^2 y_1}{|y_1|^4} \sim dT e^{2T} \tag{4.23}$$

and in the neighborhood of D_0:

$$\frac{d\Omega}{\det(f_i, f_j)} \frac{I_0^{13}}{I_{-1}} \bigg|_{|y_1| \to 0} \sim \frac{d^2 y_1}{|y_1|^4 [\log(1/|y_1|)]^{13}} \sim \frac{dT}{T^{13}} e^{2T}. \tag{4.24}$$

In order to determine the order of the pole of the form $F(y)dv$ from Eqs. (1.3) and (3.4) one still has to estimate the period matrix (3.25) in the neighborhood of the surface D. In the case I of the decomposition into two surfaces S_q and S_{p-q} of genera q and $p-q$, the p holomorphic 1-differentials ω_i, Eq. (3.24) go over into the holomorphic 1-differentials ω'_α, $\alpha = 1, ..., q$ on S_q and ω''_β, $\beta = 1, ..., p-q$ on S_{p-q}. The period matrix $\hat{\tau}$ then takes on a block form and $\det \operatorname{Im} \hat{\tau}(y)$ has a finite limit for $y_1 \to 0$. Thus, in a neighborhood of D_q, $q \neq 0$,

$$\det \operatorname{Im} \hat{\tau}(y) | \underset{y_1 \to 0}{\longrightarrow} \det \operatorname{Im} \hat{\tau}' \cdot \det \operatorname{Im} \hat{\tau}'', \tag{4.25}$$

where $\hat{\tau}'$ (respectively $\hat{\tau}''$) is the period matrix of S_q (respectively S_{p-q}). In order to estimate the period matrix in the case II—degeneracy of the handle—we choose the basis of cycles (3.23) in such a manner that the cycle a_p should pass transversely to the cylinder K in Fig. 4, II, and the cycle b_p should be along the cylinder. We select on K the coordinate $z = \tau + i\sigma$ (Fig. 5). Then from the relation

$$\oint_{a_i} \omega_j = \delta_{ij},$$

and the holomorphy condition $\bar{\partial}\omega_i = 0$, it follows that for $T \gg 1$ all differentials except ω_p decay exponentially on the cylinder:

$$\omega_i \leqslant e^{-\tau} + e^{-(T-\tau)}, \quad i = 1, \dots, p-1,$$

and $\omega_p = 1/2\pi i$ for $\tau \gg 1$, $T - \tau \gg 1$. Hence the only matrix element which diverges for $T \to \infty$ is

$$\tau_{pp} = \oint_{b_p} \omega_p \approx T/2\pi i$$

and in a neighborhood of D_0 we have[10]

$$\det \operatorname{Im} \hat{\tau} | \underset{y_1 \to 0}{\sim} T = \log(1/|y_1|). \tag{4.26}$$

Substituting (4.25), (4.26) into Eq. (1.3) and comparing, respectively, with Eqs. (4.23), (4.24), we find that the form $F(y)dv$ has a pole of second order on D:

$$F(y)dv | \underset{y_1 \to 0}{\sim} y_1^{-2} dy_1 \wedge dv_\parallel. \tag{4.27}$$

Thus, we have proved for the measure the property B formulated in the Introduction.

We now show that the condition that there be no zeros in M_p and Eq. (4.27) determines the form $F(y)dv$ uniquely, up to a constant factor. Indeed, the ratio of two forms F' and F'' satisfying these conditions is a meromorphic function on $\bar{M}_p$ and does not vanish or become infinite except possibly at the intersections of the components D_i of the surface D (i.e., at those points where the coefficient of y_1^{-2} in Eq. (4.27) can have singularities). This implies that either $F'/F'' = \text{const}$, or that the variety of zeros and poles of the function F'/F'' has complex dimension 2 in $\bar{M}_p$. It is however known that the variety of zeros and poles of a nonconstant meromorphic function on a compact algebraic manifold has complex dimension 1. Consequently $F'/F'' = \text{const}$.

The asymptotic behaviors (4.23) and (4.24) have a very simple meaning. If one considers string theory as the theory of an infinite number of interacting particles, then K in Fig. 5 can be interpreted as the propagator in the proper time representation (with proper time T) and the integral of the measure with respect to dT in the case II can be written in the form

$$Z_p^{\text{div II}} = \int d^{\mathscr{D}} p_\mu \int_0 dT \sum_r \exp[-(p_\mu^2 + m_r^2) T] V_s(r, p_\mu), \tag{4.28}$$

where the summation is over all the particles corresponding to different excited states of the string (p_μ is the momentum flowing along the loop). For large T in the momentum integral in (4.28) small p_μ^2 are important, and the measure in (4.28) has the asymptotic behavior

$$dT \cdot T^{-\mathscr{D}/2} \sum_r \exp(-m_r^2 T) V_s(r, 0), \tag{4.29}$$

from which it follows that the leading contribution to the integral (4.28) for large times T comes from tachyons and massless states:

$$Z_p^{\text{div II}} \sim \int^{\infty} dT \cdot T^{-\mathscr{D}/2} \sum_{m_r^2 \leqslant 0} \exp(-m_r^2 T)\, V_3(r,0). \quad (4.30)$$

We now recall that on a closed bosonic string the ground state is a tachyon[22] with $m_0^2 = -2$ (in our normalization), and all the excited states have $m_r^2 \geqslant 0$ (the multiplet of massless excitations contains the graviton $g^{\mu\nu}$, the tensor $A^{\mu\nu}$ and the dilaton Φ), then, taking into account the fact that $\mathscr{D} = 26$, we obtain from Eq. (4.30) the asymptotic behavior

$$Z_p^{\text{div II}} \sim \int^{\infty} dT \cdot T^{-13} e^{2T},$$

coinciding with Eq. (4.24). Note that for $\mathscr{D} > 2$ the massless states do not contribute to the divergences of the integral (4.30).

In the case I the particles propagate between the vertices V_1 and V_2 (Fig. 4, I) with zero momentum, and in place of Eq. (4.30) we obtain

$$Z_p^{\text{div I}} \sim \int^{\infty} dT \sum_{m_r^2 \leqslant 0} \exp(-m_r^2 T)\, V_1(r)\, V_2(r) \quad (4.31)$$

and the leading contribution to the divergence in Eq. (4.31) is again related to a tachyon and has the form

$$\int^{\infty} dT e^{2T},$$

which coincides with Eq. (4.23). It is, however, clear that there also exists a divergence related to the massless dilaton ($g^{\mu\nu}$ and $A^{\mu\nu}$ are not created from the vacuum) which presumably leads to a renormalization of the Regge slope.

Summarizing, the order γ of the pole of the form $F(y)dv$ equals

$$\gamma = 1 - m_0^2/2. \quad (4.32)$$

The results obtained here allow one to assume that the model is renormalizable, and that there are only two renormalizations: that of the slope, and of the tachyon-vacuum vertex (the tachyon tadpole).

5. THE BOSONIC STRING MEASURE AND MUMFORD'S THEOREM

In this section we give a mathematically rigorous reformulation of the results obtained above and discuss their relation to Mumford's theorem.[7] It follows from Eqs. (2.19) and (3.4) that $F(y)$ is not a function, but a section of a line bundle (i.e., a vector bundle in which the complex dimension of the fiber is one) E over M_p. More precisely, $F(y)$ is the contribution to the measure from the left-moving excitations of the string:

$$F(y) = \det \bar{\partial}_{-1} \cdot (\det \bar{\partial}_0)^{-13}, \quad (5.1)$$

where $\bar{\partial}_j$ acts on the space of j-differentials; $\det \bar{\partial}_j$ is a section of a line bundle[23] with the fiber generated by the form*)

$$\Phi_1^{(j)} \wedge \ldots \wedge \Phi_{\alpha_j}^{(j)} \wedge \Phi_1^{(1-j)} \wedge \ldots \wedge \Phi_{\alpha_{1-j}}^{(1-j)},$$

where the $\Phi_\alpha^{(j)}$ form a basis in the space $\text{Ker}\, \bar{\partial}_j$ of holomorphic j-differentials, and $\Phi_\alpha^{(j-1)}$ is a basis in $(\text{CoKer}\, \bar{\partial}_j)^* \approx \text{Ker}\, \bar{\partial}_{1-j}$. Thus, E is a tensor product of two line bundles over M_p

$$E = \mathscr{K} \otimes \lambda^{-13}, \quad (5.2)$$

where $\mathscr{K}$ is the bundle of $(3p-3,0)$ forms with fiber generated by $dv = dy_1 \wedge \ldots \wedge dy_{3p-3}$, and λ is the bundle of modular forms with fiber generated by $\omega_1 \wedge \ldots \wedge \omega_p$, where $\{\omega_i\}$ is a basis in the space of holomorphic 1-forms. We have chosen it as at the end of Sec. 3. The bundle λ is not trivial, since as one goes around a loop γ in M_p the basis of cycles may change. The section $F(y)$ is well-defined only if the gravitational anomalies cancel[24] in Eq. (5.1). This does indeed occur[18] and the condition of cancellation of the gravitational anomaly is in fact equivalent[19] to the condition of cancellation of the conformal anomaly in the ratio $\det \Delta_{-1}/(\det \Delta_0)$.[13]

A theorem[11)] proved by Mumford[7] by means of a calculation of the Chern class $c_1(E)$ of the bundle E, asserts that this bundle is trivial over M_p (which reflects, in particular, the absence of topological obstructions to anomaly cancellations). Moreover, the calculation of $c_1(E)$ implies that E admits a holomorphic nonvanishing section F over M_p which has a second order pole at infinity (the point D). Furthermore, Wolpert's theorem[14] on the independence of the components $D_0, \ldots, D_{[p/2]}$ of the infinity D in the homology group $H_{6p-8}(\bar{M}_p, Q)$ of the space $\bar{M}_p$ allows one to conclude that any holomorphic nonvanishing section of E over M_p differs from F by a constant factor. As was noted by Beilinson and Drinfel'd, the absolute square of the section F allows one to define a measure on $\bar{M}_p$:

$$d\mu = dv \wedge d\bar{v}\, |F(y)|^2 [\det(\omega_i, \omega_j)]^{-13}, \quad (5.3)$$

where $(\omega_i, \omega_j) \overset{\text{def}}{=} (i/2) \int \omega_i \wedge \bar{\omega}_j$, and $\det(\omega_i, \omega_j)$ forms a natural hermitian metric on λ and coincides in the basis chosen at the end of Sec. 3 with $\det \text{Im}\, \hat{\tau}$. We have thus proved the following.

Theorem. The integration measure in the theory of closed bosonic strings is the absolute square of a global holomorphic section without zeros over M_p of the bundle $\mathscr{K} \otimes \lambda^{-13}$, divided by the 13th power of the natural metric of λ.

Since the holomorphic structure on the moduli space arises out of its algebraic structure, any holomorphic object on this space, and in particular, the section f which appears in string theory, is algebraic (according to the GAGA principle[26]). The following conjecture generalizes our results:

Conjecture. The multiloop amplitudes (not only the vacuum amplitudes) of any conformally invariant string theory (such as the bosonic string in $D = 26$ or the SS and HS in $D = 10$) can be expressed in terms of algebraic objects (functions or sections of holomorphic bundles) over the moduli space of Riemann surfaces.

Thus, quantum geometry is the complex geometry of the space $\bar{M}_p$.

6. MULTILOOP AMPLITUDES AND THETA FUNCTIONS

In this section we list the explicit formulas for the measure in the cases $p = 2$ and $p = 3$ obtained in Ref. 10, and formulate a conjecture on the form of the measure for $p = 4$. It is possible to write simple formulas for the genera $p = 2, 3$ because in these cases there exists an explicit parametrization of the space M_p by means of the period matrices, parametrization which we describe below.

On an arbitrary Riemann surface S_p of genus p one can, in terms of the symplectic basis of cycles (closed loops) a_i, $b_i = 1, \dots, p$

$$a_i \circ a_j = b_i \circ b_j = 0, \quad i \neq j; \quad a_i \circ b_j = \delta_{ij}, \tag{6.1}$$

introduced at the end of Sec. 3, and the dual basis of holomorphic 1-differentials ω_i:

$$\oint_{a_i} \omega_j = \delta_{ij} \tag{6.2}$$

$$\tau_{ik} = \oint_{b_i} \omega_k,$$

construct the period matrix, which satisfies the Riemann relations[27]

$$\tau_{ik} = \tau_{ki}, \quad \operatorname{Im} \hat{\tau} > 0. \tag{6.4}$$

These relations are consequences of the formula

$$\int_{S_p} \omega \wedge \overline{\omega}' = \sum_{i=1}^{p} \left(\int_{a_i} \omega \overline{\int_{b_i} \omega'} + \int_{b_i} \omega \overline{\int_{a_i} \omega'} \right),$$

where ω and ω' are arbitrary holomorphic 1-differentials, and of the fact that the squared norm of the nonvanishing differential ω is positive

$$\|\omega\|^2 = \frac{i}{2} \int_{S_p} \omega \wedge \overline{\omega} > 0.$$

Torelli's theorem asserts that a complex structure is uniquely determined up to a diffeomorphism by the period matrix. Thus, complex structures can be parametrized by means of the matrices $\hat{\tau}$. However, infinitely many matrices $\hat{\tau}$ may correspond to one and the same surface. Indeed, the basis $\{a_i, b_i\}$ is not uniquely determined by the conditions (6.1). We can select a different basis:

$$b_i' = A_{ik} b_k + B_{ik} a_k, \quad a_i' = C_{ik} b_k + D_{ik} a_k, \tag{6.5}$$

which will satisfy the conditions (6.1), if the integer-valued matrices A, B, C, D satisfy the conditions:

$$AB^T - BA^T = CD^T - DC^T = 0, \quad AD^T - BC^T = 1, \tag{6.6}$$

i.e., the matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(p, \mathbb{Z}) \equiv \Gamma_p.$$

The group Γ_p is called the Siegel modular group of degree p (after C. L. Siegel). Under the transformations (6.5) the basis of differentials (6.2) goes over into

$$\omega_i' = \omega_k (C\hat{\tau} + D)_{ki}^{-1}, \tag{6.7}$$

from where it follows that the period matrix $\hat{\tau}'$ has, in the basis (6.5) the form

$$\hat{\tau}' = (A\hat{\tau} + B)(C\hat{\tau} + D)^{-1}. \tag{6.8}$$

Thus, in order to avoid counting the same surface several times, we must restrict our attention to the quotient space

$$\mathfrak{S}_p = \mathscr{H}_p / \Gamma_p,$$

where $\mathscr{H}_p$ denotes the space of all symmetric $p \times p$ matrices with positive-definite imaginary part, and is called the Siegel upper half-plane. The modular group Γ_p acts on it by the Möbius transformations (6.8). The manifold $\mathfrak{S}_p$ has complex dimension $p(p+1)/2$, which for $p = 1, 2, 3$ coincides with the dimension of the space M_p. Indeed, in these cases $\mathfrak{S}_p$ and M_p coincide. To summarize: for genera $p = 1, 2, 3$ the space M_p may be parametrized by means of the period matrices ranging over the fundamental domain $\mathfrak{S}_p$ of the group Γ_p in the Siegel upper half-plane $\mathscr{H}_p$.

It follows from Eqs. (1.1) and (1.3) that in the cases $p = 2, 3$ the measure must have the form[10]

$$Z_p = \int_{\mathfrak{S}_p} \prod_{k \leq j} \frac{i}{2} d\tau_{kj} \wedge d\bar{\tau}_{kj} |\chi_{12-p}(\hat{\tau})|^{-2} (\det \operatorname{Im} \hat{\tau})^{-13}. \tag{6.9}$$

One can prove[28] that the natural modular-invariant measure on $\mathfrak{S}_p$ is

$$d\mu_p = \prod_{k \leq j} \frac{i}{2} d\tau_{jk} \wedge d\bar{\tau}_{jk} (\det \operatorname{Im} \hat{\tau})^{-(p+1)}.$$

Furthermore, it follows from Eq. (6.6) that

$$\det \operatorname{Im} \hat{\tau}' = |\det(C\hat{\tau} + D)|^{-2} \det \operatorname{Im} \hat{\tau}. \tag{6.10}$$

Therefore, the condition that the measure (6.9) should be a measure on $\mathfrak{S}_p$, i.e., should be modular-invariant, has the form

$$\chi_k(\hat{\tau}') = [\det(C\hat{\tau} + D)]^k \chi_k(\tau), \quad k = 12 - p \tag{6.11}$$

[for $p = 3$ Eq. (6.11) requires some explanation, cf. infra]. Furthermore, the form $\prod_{i<j} d\tau_{ij}$ has in the component D_0 of the point at infinity ($\operatorname{Im} \tau_{11} \to \infty$) a first-order pole, and in the component D_1 (for $p = 2, 3$ there are no further components), where $\hat{\tau}$ takes on a block form, it has a zero of order $p - 2$. Therefore, the property B) in the Introduction and Eq. (6.11) imply that $\chi_k(\hat{\tau})$ is a parabolic modular form of weight $k = 12 - p$ on $\mathfrak{S}_p$, and on D_1 it has a zero of order p. A (Siegel) modular form of weight k on $\mathfrak{S}_p$ is defined to be a function $\hat{\tau}$ holomorphic on with the transformation property (6.11). For odd p, k is necessarily even. A modular form which vanishes on D_0 is called parabolic. If both p and k are odd, the form must be defined by an additional multiplication by a character of the group Γ_p, since under a simultaneous change of signs of A, B, C, and D in Eq. (6.8) $\hat{\tau}'$ remains unchanged, whereas the right-hand side of Eq. (6.11) changes sign.

For $p = 1, 2$, the space $\mathfrak{S}_p$ of modular forms is thoroughly studied. Thus, for $p = 1$, all forms are linear combinations of forms of weight 4 and 6, and their number is determined by the formula.

$$p_1(t)=\sum_{k=0}^{\infty} d_{2k}(1)t^{2k}=(1-t^4)^{-1}(1-t^6)^{-1}, \qquad (6.12)$$

where $d_{2k}(p)$ denotes the number of linearly independent modular forms of weight $2k$ on $\mathfrak{S}_p$. For $p=2$ the situation is analogous, albeit more complicated.[29] If one restricts one's attention to forms of even weight, then there are 4 fundamental forms with weights 4, 6, 10, and 12:

$$p_2(t)=(1-t^4)^{-1}(1-t^6)^{-1}(1-t^{10})^{-1}(1-t^{12})^{-1}. \qquad (6.13)$$

In Ref. 29 one can find the expressions for the fundamental forms in terms of Eisenstein series and the theta-constants. In the same paper it is proved that there exists a *unique* parabolic form of weight 10. This form must, consequently, coincide with the form χ_{10} from Eq. (6.9) and has on D_1 a second-order zero, which is easily seen with the help of the equation[29]

$$\chi_{10}=-2^{-14}\prod_{\mathbf{m}}\theta_{\mathbf{m}}^2(\hat{\tau}), \qquad (6.14)$$

where the theta-constants $\theta_{\mathbf{m}}(\hat{\tau})$ are defined as

$$\theta_{\mathbf{m}}(\mathbf{z};\hat{\tau})=\sum_{\mathbf{n}\in\mathbb{Z}^p}\exp\left\{\pi i\left(\mathbf{n}+\frac{\mathbf{m}'}{2}\right)^T\hat{\tau}\left(\mathbf{n}+\frac{\mathbf{m}'}{2}\right)+2\pi i\left(\mathbf{n}+\frac{\mathbf{m}'}{2}\right)^T\left(\mathbf{z}+\frac{\mathbf{m}''}{2}\right)\right\},$$

$$\theta_{\mathbf{m}}(\hat{\tau})=\theta_{\mathbf{m}}(0;\hat{\tau}),\quad \mathbf{m}\equiv(\mathbf{m}',\mathbf{m}''), \qquad (6.15)$$

and the components of the vectors $\mathbf{m}'$, $\mathbf{m}''$ of the characteristic $\mathbf{m}$ take on the values 0, 1. The quantity

$$e(\mathbf{m})=(\mathbf{m}'\mathbf{m}'')\ (\mathrm{mod}\ 2) \qquad (6.16)$$

is called the parity of the characteristic $\mathbf{m}$, and in Eq. (6.14) the product is taken only over even characteristics. For genus p there exist $2^{p-1}(2^p-1)$ odd characteristics. If $e(\mathbf{m})=1$, then $\gamma_{\mathbf{m}}(0;\tau)=0$. It follows from Eqs. (6.14) and (6.15) that for $\tau_{12}\to 0$

$$\chi_{10}\sim\exp(2\pi i\tau_{11})\exp(2\pi i\tau_{22})(\pi\tau_{12})^2+\ldots, \qquad (6.17)$$

as required. It also follows from the results of Ref. 21 that $\chi_{10}(\hat{\tau})$ does not have zeros in the interior of $\mathfrak{S}_2$. The transformation properties of the theta functions (6.15) with respect to the group Γ_p are described in detail in Ref. 30.

The formulas (6.9), (6.14) solve the problem of determining the measure for $p=2$.[10] For $p=3$ the space of modular forms has a more complicated structure. However, it is known that up to weight 10 it is generated by forms of the weights 4, 6, and 10, just like for $p=2$, i.e., there exist only 5 linearly independent forms:

$$\psi_4,\ \psi_6,\ (\psi_4)^2,\ \psi_4\psi_6,\ \psi_{10}. \qquad (6.18)$$

The expressions of the forms ψ_k of weight k in terms of the $\theta_{\mathbf{m}}(\hat{\tau})$ are listed in Ref. 31, where one can also find the assertion that there are no parabolic forms of weight <12. What is specific about the case $p=3$ is the fact that we need a form of odd weight 9, i.e., with values in a character of the group Γ_p. Its square $\chi_9^2=\chi_{18}$ is an ordinary complex-valued form of weight 18. The form χ_{18} must be parabolic and must have in D_0 a second-order zero, and in D_1 a sixth-order zero. In addition, one can show that the measure $d\mu_3$ in Eq. (6.10), for a correct definition of the integration domain, has a zero on the manifold of hyperelliptic surfaces D_*, and therefore χ_{18} must also vanish on D_* and can have no other zeros. Such a form exists and has the form[31]

$$\chi_{18}=\prod_{\mathbf{m}}\theta_{\mathbf{m}}(\hat{\tau}), \qquad (6.19)$$

where the product runs over all 36 even characteristics. Since the modular form is determined uniquely by the position and order of its zeros, up to a constant factor, we conclude that Eq. (6.19) is the form which is the square of the form χ_9 in the measure (6.9). As a result of this we have[10]

$$Z_3=\int_{\mathfrak{S}_3} d\mu_3|\chi_{18}(\hat{\tau})|^{-1}(\det\operatorname{Im}\hat{\tau})^{-9}. \qquad (6.20)$$

In our opinion it makes sense to complement the equations (6.9), (6.14) and (6.19), (6.20) with formulas for the scattering amplitudes of tachyons. For this we need, following Ref. 5, to calculate the Gaussian integral:

$$\int DX^\mu\exp\left[-\frac{1}{\pi}\int dX^\mu\bar{\partial}X_\mu d^2\xi\right]\times\prod_{k=1}^{N}\int\exp[ip_k^\mu X_\mu(\xi_k)]\rho(\xi_k)d^2\xi_k$$

$$=\left(\frac{\det N_0}{\det'\Delta_0}\right)^{13}\exp\left[-\frac{1}{\pi}\int\partial X_{cl}^\mu\bar{\partial}X_{\mu,cl}d^2\xi\right]\times\prod_{k=1}^{N}\int\exp[ip_k^\mu X_{\mu,cl}(\xi_k)]\rho(\xi_k)d^2\xi_k$$

$$\stackrel{\text{def}}{=}\left(\frac{\det N_0}{\det'\Delta_0}\right)^{13}K(p_1,\ldots,p_N;\hat{\tau}), \qquad (6.21)$$

where $\chi_{cl}^\mu(\xi)$ is the solution of the equation

$$-2\partial\bar{\partial}X_{cl}^\mu(\xi)=\sum_{k=1}^{N}i\pi p_k^\mu\delta(\xi-\xi_k). \qquad (6.22)$$

On account of the momentum conservation law

$$\sum_{k=1}^{N}p_k^\mu=0 \qquad (6.23)$$

the solution of (6.22) is easily expressed in terms of the theta functions (6.15):

$$X_{cl}^\mu(\xi)=-\sum_{k=1}^{N}ip_k^\mu[\log|\theta_{\mathbf{m}}[\mathbf{z}(\xi)-\mathbf{z}(\xi_k);\hat{\tau}]|^2+4\pi\operatorname{Im}\mathbf{z}(\xi)^T(\operatorname{Im}\hat{\tau})^{-1}\operatorname{Im}\mathbf{z}(\xi_k)], \qquad (6.24)$$

where **m** is any odd characteristic. The argument of $\mathbf{z}(\xi)$ in Eq. (6.24) is the integral of the vector $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_p)$ formed of holomorphic Abelian differentials along a vector path joining the point ξ to a fixed ξ_0:

$$\mathbf{z}(\xi) = \int_{\xi_0}^{\xi} \boldsymbol{\omega}. \qquad (6.25)$$

Regulating the function (6.24) at $\xi = \xi_k$ as in Ref. 5, we find that on the mass shell $p^2 = 2$ the dependence on ρ in Eq. (6.21) cancels, and after simple calculations the factor $K(p_i, \hat{\tau})$ reduces to the form

$$K(p_i; \hat{\tau}) = \int \prod_{k=1}^{N} \frac{i}{2} \varkappa_{\mathbf{m}}(\xi_k) \wedge \overline{\varkappa_{\mathbf{m}}(\xi_k)} \prod_{i<j} |\chi_{ij}|^{2p_i p_j}, \qquad (6.26)$$

$$\chi_{ij} \equiv \theta_{\mathbf{m}}(\mathbf{z}_{ij}; \hat{\tau}) \exp(-\pi \operatorname{Im} \mathbf{z}_{ij}^T)(\operatorname{Im} \tau)^{-1}(\operatorname{Im} \mathbf{z}_{ij}),$$

where $\mathbf{z}_{ij} \equiv \mathbf{z}(\xi_i) - \mathbf{z}(\xi_j)$ and the 1-differential $\varkappa_{\mathbf{m}}(\xi)$ has the form

$$\varkappa_{\mathbf{m}}(\xi) = \omega_i(\xi) \left.\frac{\partial \theta_{\mathbf{m}}(z; \hat{\tau})}{\partial z_i}\right|_{z=0}. \qquad (6.27)$$

The expression (6.26) does not depend on the choice of the odd characteristic **m** and it can be substituted into the measure for the determination of the amplitudes:

$$A(p_1, \ldots, p_N) = \int_{M_p} d\Omega |F(y)|^2 (\det \operatorname{Im} \hat{\tau})^{-13} K(p_1, \ldots, p_N; \hat{\tau}). \qquad (6.28)$$

For the case $p = 1$ one can reproduce by means of Eqs. (6.26) and (1.2) the known result of Ref. 8.

We now consider the case $p = 4$. The complex dimension of $\mathfrak{S}_4$ is by one larger than the dimension of $\overline{M}_4$, therefore the matrix $\hat{\tau}$ is subject to only one relation. It is called the Schottky relation[32] and consist in the condition that some parabolic form J_8 of weight 8 should vanish:

$$J_8(\hat{\tau}) = 0. \qquad (6.29)$$

Strictly speaking, Schottky has proved that any matrix τ of a Riemann surface of genus 4 satisfies (6.29), whereas the converse was proved only recently[34]—a paper to which we refer the reader for a formula expression J_8 in terms of $\theta_{\mathbf{m}}(\hat{\tau})$. The results of Ref. 33 allow one to formulate the following conjecture.

Conjecture 1. The measure for $p = 4$ has the form

$$Z_4 = \int_{\mathfrak{S}_4} d\mu_4 |\delta(J_8)|^2 (\det \operatorname{Im} \hat{\tau})^{-8} \underline{\underline{\mathrm{def}}}$$

$$\times \int_{M_4} \operatorname{res} dv J_8^{-1}(\hat{\tau})' \wedge \overline{\operatorname{res} dv J_8^{-1}(\tau)} (\det \operatorname{Im} \hat{\tau})^{-13} \qquad (6.30)$$

$$dv \equiv \prod_{i \leq j} d\tau_{ij}.$$

7. DISCUSSION: STRUCTURE OF THE MEASURE FOR A SUPERSTRING

In this section we discuss the structure of the measure for the theory of closed orientable superstrings (SS)[1] and for the heterotic string (HS) model.[2] The results of the preceding sections and of the one-loop calculations of Refs. 1, 2, allow us to assume that in the case $p > 1$ the measure has the following form:

$$Z_p^{SS} = \int_{M_p} d\Omega |F(y)|^2 |\chi_8^{SS}(y)|^2 (\det \operatorname{Im} \hat{\tau})^{-5}, \qquad (7.1)$$

$$Z_p^{HS} = \int_{M_p} d\Omega |F(y)|^2 \chi_8^{SS}(y) \overline{f_8^{HS}(y)} (\det \operatorname{Im} \tau)^{-5}, \qquad (7.2)$$

where $F(y)\chi_8^{SS}(y)$ is the contribution to the *measure* of the Green-Schwarz left-movers, and $\overline{F(y) f_8^{HS}(y)}$ is the contribution of the right-moving excitations of the heterotic string obtained by compactification of a bosonic string into an appropriate torus.[2] In effect Eq. (7.2) is the most general form of the measure for a ten-dimensional string theory with noninteracting left-movers and right-movers. We shall explain below that, apparently, there are only three such theories: the SS, the HS, and the theory with the measure (7.2), where χ_8^{SS} is replaced by f_8^{HS}, with χ_8^{SS} and f_8^{HS} fixed uniquely by some natural conditions.

An important role in the derivation of Eqs. (7.1) and (7.2) is played by the circumstance that, in order to single out the supersymmetric sector in the Neveu-Schwarz string, we were forced to sum over the boundary conditions imposed independently on the right-handed and left-handed fermions (while taking only those conditions for which there are no fermion zero modes). This is why the absolute square appears in Eq. (7.1), rather than the sum of the squares, and the right-moving and left-moving sectors decouple completely, allowing the HS construction.

Since the ground state of the SS is massless[1] and the right-mover sector of the HS contains a tachyon, the analysis at the end of Sec. 3 and the condition of modular invariance of (7.1) and (7.2) imply that:

a) χ_8^{SS} and f_8^{HS} are modular forms of weight 8 on $\overline{M}_p$;

b) χ_8^{SS} is a parabolic form vanishing on D.

As a consequence of b) the tachyon does not contribute to the divergences in the right-hand side of the measure, since $\int d^2y y^{-1}(\bar{y})^{-2} = 0$.

The fact that the conditions a) and b) determine the forms χ_8^{SS} and f_8^{HS} uniquely can be seen[12)] already on the examples of $p = 1, 2, 3$. In these cases $M_p = \mathfrak{S}_p$, and it follows from Eqs. (6.12), (6.13), and (6.18) that there exists a *unique* form of weight 8. It is not parabolic and must coincide with f_8^{HS}. It then follows from b) that

$$\chi_8^{SS} = 0. \qquad (7.3)$$

There exist two representations for the form f_8^{HS}:

$$f_8^{HS} = \left[\frac{1}{N} \sum_{\mathbf{m}} \theta_{\mathbf{m}}^8(\hat{\tau})\right]^2 = \frac{1}{N} \sum_{\mathbf{m}} \theta_{\mathbf{m}}^{16}(\hat{\tau}), \qquad (7.4)$$

where N is some normalizing factor and the summation is only over even characteristics. The representations (7.4) correspond to the groups $E_8\times E_8$ and $SO(32)$, respectively. This can be shown in the following manner. After fermionization[2,34] the contribution of the right-movers to the HS measure differs from their contribution to the measure of the bosonic string by the factor

$$f_8^{\mathrm{HS},SO(32)}=\frac{1}{N}\sum_{\mathbf{m}}[\det{}_{\mathbf{m}}\partial_{1/2}(\det\partial_0)^{1/2}]^{16} \tag{7.5}$$

for the group $SO(32)$, and

$$f_8^{\mathrm{HS},E_8\otimes E_8}=\left\{\frac{1}{N}\sum_{\mathbf{m}}[\det{}_{\mathbf{m}}\partial_{1/2}(\det\partial_0)^{1/2}]^{8}\right\}^2 \tag{7.6}$$

for the group $E_8\times E_8$. The characteristic $\mathbf{m}=(\mathbf{m}',\mathbf{m}'')$ of the determinant $\det_{\mathbf{m}}\partial_{1/2}$ parametrizes the fermion boundary conditions: after surrounding a cycle a_i the fermion field acquires a factor $(-1)^{m_i'+1}$, and after surrounding b_i it acquires the factor $(-1)^{m_i''+1}$. In both cases one needs to sum only over even characteristics (as in the case of the left-movers). The product of the determinants in the square brackets in Eqs. (7.5) and (7.6) is anomaly-free and the following formula holds for it[13)]

$$\det{}_{\mathbf{m}}\bar\partial_{1/2}(\det\bar\partial_0)^{1/2}=\theta_{\mathbf{m}}(\hat\tau), \tag{7.7}$$

is valid for arbitrary genus p. In the case of odd characteristic $\bar\partial_{1/2}$ acquires a zero mode $\varkappa_{\mathbf{m}}^{1/2}$, where $\varkappa_{\mathbf{m}}$ is defined in Eq. (6.27) (one can prove that the square root can be extracted) and (7.7) vanishes. Substituting (7.7) into (7.5) and (7.6) we obtain Eq. (7.4), with the equality

$$f_8^{\mathrm{HS},E_8\otimes E_8}=f_8^{\mathrm{HS},SO(32)}, \tag{7.8}$$

valid for $p=1, 2, 3$ is a nontrivial identity and follows from the fact that for these values of p there is only one form of weight 8 on $\bar M_p$.

The relation of Eq. (7.4) to the groups $E_8\otimes E_8$ and $SO(32)$ appears in the following manner. We consider a lattice Λ in $\mathbb{R}^n$ and relate it to the following theta-series:

$$\theta_\Lambda(\hat\tau)=\sum_{\mathbf{r}_\alpha\in\Lambda}\exp[\pi i(\mathbf{r}_\alpha\mathbf{r}_\beta)\tau_{\alpha\beta}], \tag{7.9}$$

where each of the vectors $\mathbf{r}_\alpha$, $\alpha=1,\dots,p$ runs over the lattice Λ, and

$$\mathbf{r}_\alpha\mathbf{r}_\beta=\sum_{i=1}^{n}\mathbf{r}_\alpha{}^i\mathbf{r}_\beta{}^i.$$

The series (7.9) converges and determines a modular form of weight $n/2$ on $\mathfrak{S}_p$ if and only if the lattice Λ is even and self-dual, i.e., the matrix $g_{ij}=\mathbf{e}_i\mathbf{e}_j$ of scalar products of the generating vectors $\mathbf{e}_i$, $i=1,\dots,n$ is a positive-definite integer-valued matrix with even principal diagonal and $\det g_{ij}=1$. Such lattices exist only for $n=8k$, $k\in\mathbb{Z}$, and for the weight $n/2=8$ there are exactly two such lattices: $\Gamma_8\otimes\Gamma_8$ and Γ_{16} (Ref. 35); Γ_8 is the weight lattice of the group E_8 and Γ_{16} is obtained by taking the union of the weight lattice of the group $SO(32)$ with the same lattice shifted by some spinor weight. It is easy to show that for arbitrary p the following identities hold

$$f_8^{\mathrm{HS},E_8\otimes E_8}=\left[\frac{1}{N}\sum_{\mathbf{m}}\theta_{\mathbf{m}}{}^8(\hat\tau)\right]^2=\theta_{\Gamma_8\otimes\Gamma_8}(\hat\tau)=[\theta_{\Gamma_8}(\hat\tau)]^2, \tag{7.10}$$

$$f_8^{\mathrm{HS},SO(32)}=\frac{1}{N}\sum_{\mathbf{m}}\theta_{\mathbf{m}}{}^{16}(\hat\tau)=\theta_{\Gamma_{16}}(\hat\tau), \tag{7.11}$$

expressing the equivalence of the fermion and boson representations of the right-moving excitation sector of the HS.

We now consider the case $p>3$. For this case the space of modular forms has not been sufficiently studied. Moreover, there is the additional complication related to the fact that for $p>3$ $\mathfrak{S}_p$ no longer coincides with $\bar M_p$, and the spaces of modular forms on $\mathfrak{S}_p$ and $\bar M_p$ will, in general, be different. Nevertheless, the following conjectures seem to us to be correct.

Conjecture 2. For $p\geqslant 4$ there are exactly two forms of weight 8 on $\mathfrak{S}_p$: $\theta_{\Gamma_8\otimes\Gamma_8}$ and $\theta_{\Gamma_{16}}$.

Conjecture 3. For arbitrary p the forms $\theta_{\Gamma_8\otimes\Gamma_8}$ and $\theta_{\Gamma_{16}}$ coincide on $\bar M_p$ and there are no other modular forms $\bar M_p$.

Since neither $\theta_{\Gamma_8\otimes\Gamma_8}$ nor $\theta_{\Gamma_{16}}$ are parabolic forms, (7.3) follows from conjecture 3 for arbitrary p, i.e., the vacuum diagrams on the theory of closed orientable superstrings and heterotic strings vanish to any order.

Another consequence of the conjectures 3 and 2 is the proportionality condition:

$$\theta_{\Gamma_8\otimes\Gamma_8}-\theta_{\Gamma_{16}}=\mathrm{const}\,J_8 \tag{7.12}$$

for $p=4$. Here J_8 is the Schottky parabolic form mentioned in Section 6. The left-hand side of (71.2) does not identically vanish on $\mathfrak{S}_4$ (Ref. 31), as was the case for $p\leqslant 3$, but is a parabolic form. We also note that the validity of conjecture 2 for $p\geqslant 17$ follows from Theorem 1.3 of Freitag's paper (Ref. 36), which asserts that each modular form of weight k on $\mathfrak{S}_p$ for $p>2k$ is a linear combination of theta series (7.9) with even, self-dual[14)] lattices Λ. We hope that the conjectures 1–3 will be proved soon.

In conclusion of this section we would like to make the following remark in relation to the one-loop calculations in SS and HS theories. In the covariant approach[5] to the model of the fermion string two of the ten fermions ψ^μ: ψ^0 and ψ^9 cancel against the fermionic ghosts. One can show that summation over nonperiodic boundary conditions (imposed separately on the right-handed and left-handed ψ^μ) of the contributions from the eight remaining fermions ψ^i, $i=1,\dots,8$, is equivalent to going over to the fermions S^A in the spinor representation of $SO(8)$, but with *periodic* boundary conditions. The operators $\psi_L^i\psi_L^j$ ($\psi_R^i\psi_R^j$) in the vertex operators are replaced by $S_L^A S_L^B\sigma_{AB}^{ij}$ $(L\leftrightarrow R)$, where L, R denote the two-dimensional chirality. For a fermion S_L^A (S_R^A) on a torus with periodic boundary conditions there are 8 zero modes and the functional integral with respect to S_L^A vanishes if the total number of operators S_L^A at the vertices is smaller than 8 (i.e., if the number of gravitons $N<4$). This is exactly the circumstance expressed by the trace identities of Ref. 1. For $N=4$ the one-loop correction is finite.[1]

This, together with the preceding arguments about the vanishing of all vacuum loops forces one to believe in the hypothesis[1] that superstring theory is finite.

We would like to express our profound indebtedness to A. Beĭlinson and V. Drinfel'd for multiple discussions of various mathematical questions related to this work and to making more precise our working hypothesis on the holomorphy of the measure, which has stimulated a study of its singularities. The rigorous mathematical formulation (Sec. 5) of the assertion proved in this paper also belongs to them.

We are also grateful to A. Zamolodchikov, O. Ogievetskiĭ, A. Polyakov, B. Feĭgin, and V. Shekhtman for stimulating discussions.

[1)] When the point D is added to M_p the latter becomes a compact manifold, denoted by $\overline{M}$ (see Ref. 9).

[2)] See also the pioneering papers Ref. 17.

[3)] In the case of general $p \geqslant 2$ considered here the operator $-\rho^{-2}\partial\rho\bar{\partial}$ has no zero modes, and therefore the usual "prime" may be omitted from the determinant.

[4)] A Beltrami differential is a quantity $\eta(z,\bar{z})$ whose relation to the complex structure J is defined by following expression of the metric compatible with J: $g = \rho|dz + \eta d\bar{z}|^2$.

[5)] In Eq. (3.17) we have not written the divergent terms of the type $\eta\bar{\eta}M^2 \log M^2$. One can get rid of them by making use of several regulating masses. This does not affect the magnitude of the M-independent terms.

[6)] In principle, Eq. (3.2) does not exclude the possibility that $F(y)$ in Eq. (3.4) should acquire a nonzero phase when one goes around any closed path γ in $\overline{M}_p$, and thus that it is a function defined not on $\overline{M}$ but on some covering of this space. However, if $F(y)dv$ is a meromorphic form (which will be proved by proving assertion B), then the path γ must be noncontractible. But is is known (Refs. 7, 21) that such paths do not exist: $H_1(\overline{M},\mathbb{Z}) = 0$, and therefore this possibility is excluded. The authors are indebted to A. Yu. Morozov who has brought this circumstance to their attention.

[7)] We choose along the cylinder of length T the coordinate τ, $0 \leqslant \tau \leqslant T$. Then for $T \gg 1$ the multiplication of the flat metric of the cylinder by the conformal multiplier $\lambda = \exp(-2\tau) + \exp(2\tau - 2T)$ converts it into two disks of unit radius connected at the centers by a "strangulated tube" of radius $e^{-T} \equiv |y_1| \ll 1$. More rigorously, the complex structure of the degenerate surface at the strangulation point is constructed in the same manner as the "strangulation" of the hyperbola $uv = y_1$ in $\mathbb{C}^2$, which for $y_1 \to 0$ degenerates into two planes $u = 0$ and $v = 0$ which intersect transversally at the point $u = v = 0$. The metric $\hat{g} = |du/u|^2$ transforms the "strangulation" of the hyperbola into a cylinder of length $T \sim \log(1/|y_1|)$. In effect, $u = 0$, and $v = 0$ have the meaning of the equations of those surfaces into which the initial surface decomposes.

[8)] The case when the genus of V_1^* is zero (compactification by a lid) was discussed by A. Polyakov, to whom we are indebted for explanations. In this case $I_{(1)}$ does not depend on T.

[9)] We recall that Eqs. (4.2) and (4.3) are the definitions of the directions η^i of the corresponding coordinates δy_i in $\overline{M}_p$ in the basis f_i.

[10)] In this case the degenerate surface can be imagined as a surface of genus $p - 1$ with two removed points R and Q. The cycle a_p surrounds one of them and the cycle b_p surrounds the other one. The differential ω_p is a normalized Abelian differential of the third kind with poles at R and Q. For a detailed discussion, see the paper by Alessandrini.[17]

[11)] The puzzling coincidence of the number 13 in Mumford's theorem with the number 26/2 in string theory was pointed out by Yu. I. Manin.[25] We are grateful to him for calling to our attention the work of Ref. 7.

[12)] For $p = 1$ the form $F(y)dv$ is a form of weight 14, which is compensated by the extra power of $(\mathrm{Im}\,\tau)^{-1}$ in Eq. (1.2), which appears on account of the integration over the zero mode of the ghost field $\varepsilon_0 = \text{const}$ in Eq. (4.1). The weights of χ_8^{SS} and f_8^{HS} remain 8, as before.

[13)] The proof together with formulas for the correlation functions of the fermions will be published.

[14)] The authors are grateful to A. N. Andrianov, for communicating this theorem to them.

[*)] *Translator's note*: In the original the authors call this expression a vector. This is true to the extent that the fiber is a one-dimensional vector space; however, it was felt that the term "form" is more appropriate for this object.

[1] J. H. Schwarz, Phys. Reports **89**, 223 (1982). M. B. Green, Surveys in High Energy Phys. **3**, 127 (1983). L. Brink, CERN Preprint, 1984.

[2] D. J. Gross, J. Harvey, E. Martinec, and R. Rohm, Phys. Rev. Lett. **52**, 502 (1985); Nucl. Phys. **B256**, 253 (1985).

[3] M. B. Green and J. H. Schwarz, Phys. Lett. **149B**, 117 (1984); **151B**, 21 (1985). P. Candelas, G. Horowitz, A. Strominger, and E. Witten, Nucl. Phys. **B258**, 46 (1985). E. Witten, Nucl. Phys. **B258**, 75 (1985).

[4] A. Belavin, A. Polyakov, and A. Zamolodchikov, Nucl. Phys. **B241**, 333 (1984). D. Friedan, Z. Qui, and S. Shenker, Phys. Rev. Lett. **52**, 1575 (1984). V. Knizhnik and A. Zamolodchikov, Nucl. Phys. **B247**, 83 (1984). A. Zamolodchikov and V. Fateev, Zh. Eksp. Teor. Fiz. **89**, 380 (1985) [Sov. Phys. JETP **62**, 215 (1985)]; Nucl. Phys. **B**, in press (1986).

[5] A. M. Polyakov, Phys. Lett. **103B**, 207, 211 (1981).

[6] L. Bers, Bull. Amer. Math. Soc. **5**, 131 (1981).

[7] D. Mumford, L'Enseignement Math. **23**, 39 (1977).

[8] J. A. Shapiro, Phys. Rev. **D5**, 1945 (1972).

[9] P. Deligne and D. Mumford, Publ. Math. de l'IHES **36**, 1 (1969).

[10] A. Belavin, V. Knizhnik, A. Morozov, and A. Perelomov, Pis'ma Zh. Eksp. Teor. Fiz. **43**, 319 (1986) [JETP Lett. **43** (1986)].

[11] G. Faltings, Ann. Math. **119**, 387 (1984).

[12] A. Belavin and V. Knizhnik, Preprint, Landau Institute, 1985.

[13] Yu. I. Manin, FAP, in press (1986); Phys. Lett. **B**, in press (1986).

[14] S. A. Wolpert, Bull. Amer. Math. Soc. **11**, 189 (1984).

[15] S. Mandelstam, Preprint UCB-PTH-85/47, Berkeley, Oct. 1985; A. Restuccia and J. G. Taylor, Phys. Lett. **162B**, 109 (1985). G. Gilbert, Univ. of Texas Preprint UTTG-23-85, Austin, 1985. M. A. Namazie and S. Rajeev, Preprint CERN-TH 4327/85. E. D'Hoker and D. H. Phong, Columbia Univ. Preprint CU-TP-323, NY 1985. M. A. Baranov and A. S. Schwartz, Pis'ma Zh. Eksp. Teor. Fiz. **42**, 340 (1985) [JETP Letters **42**, 419 (1985)]. E. Gava, R. Jengo, T. Jayaraman, and R. Ramachandran, Trieste Preprint 90/85/E. P.

[16] D. B. Ray and I. M. Singer, Ann. Math. **98**, 154 (1973).

[17] C. Lovelace, Phys. Lett. **32B**, 703 (1970); **34B**, 500 (1971). V. Alessandrini, Nuovo Cimento **2A**, 321 (1971). V. Alessandrini and D. Amati, Nuovo Cimento **4A**, 793 (1971). M. Kaku and L. P. Yu, Phys. Lett. **38B**, 166 (1971); Phys. Rev. **D3**, 2992, 3007, 3020 (1971). M. Kaku and J. Scherk, Phys. Rev. **D3**, 430, 2000 (1971).

[18] O. Alvarez, Berkeley Preprint UCB-PTH-85/39; LBL-20251.

[19] D. Friedan, in: Recent Advances in Field Theory and Statistical Mechanics (Les Houches, Summer 1982), J.-B. Zuber and R. Stora, eds., North-Holland, 1984.

[20] O. Alvarez, Nucl. Phys. **B216**, 125 (1983).

[21] D. Mumford, Russian transl. in the Coll. Matematika **12** Nr. 6, 67 (1968).

[22] J. A. Shapiro, Phys. Lett. **33B**, 361 (1970); M. A. Virasoro, Phys. Rev. **177**, 2309 (1969).

[23] D. Quillen, transl. in FAP **19**, 37 (1985).

[24] L. Alvarez-Gaumé and E. Witten, Nucl. Phys. **B234**, 269 (1984).

[25] Yu. I. Manin, transl. in FAN **20** (1985).

[26] J.-P. Serre, Géometrie Algèbrique et Géometrie Analytique, Ann. Inst. Fourier, Vol. **6**.

[27] I. Kra, Automorphic Forms and Kleinian Groups, Benjamin, NY, 1972 (Russian transl., Mir, 1975).

[28] C. L. Siegel, Amer. J. Math. **65**, 1 (1943).

[29] J. I. Igusa, Amer. J. Math. **84**, 175 (1962); **86**, 392 (1964).

[30] J. I. Igusa, Amer. J. Math. **86**, 219 (1964).

[31] J. I. Igusa, Amer. J. Math. **89**, 817 (1967).

[32] F. Schottky, J. f. reine u. angew. Math. **102**, 304 (1888).

[33] J. I. Igusa, J. Fac. Sci. Tokyo **28**, 231 (1981).

[34] P. Goddard, D. I. Olive, and A. Schwimmer, Phys. Lett. **157B**, 393 (1985).

[35] J.-P. Serre, Cours d'Arithmétique, Paris (Russ. transl., Mir, 1972).

[36] E. Freitag, Math. Z. **156**, 141 (1977).

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