Stability of a Bennett pinch

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The two-fluid approximation in relativistic electromagnetic hydrodynamics is used to analytically calculate the behavior of small isothermal $z$-independent perturbations in an axisymmetric Bennett pinch near the stability threshold. The results confirm the previous numerical findings that the equilibrium configurations are unstable against shearing of the system as a whole but are stable to filamentation. The spectrum of the acoustic Goldstone branch of the long-wave pinch oscillations is calculated.

1. INTRODUCTION

The problem of hot plasma confinement has stimulated work on equilibrium plasma configurations$^1$ and their stability$^2$ in the framework of magnetohydrodynamics (MHD). An energy principle for analyzing plasma MHD stability was developed in Ref. 3 and applied successfully to several complicated systems in Refs. 4–6.

The MHD equations neglect the relative motion of the electrons and ions in the plasma, which is regarded as a continuous, electrically neutral, conducting fluid. Yet it is clear that the passage of current through the plasma necessarily involves a relative motion of the electron and ion subsystems. Their relative velocity $v_0$ (drift velocity) is related to the current $I$ by $I = eN_e v_0$, where $e$ is the electron charge and $N_e$ is the number of electrons per unit length of the current channel.

The applicability of the MHD approximation to high-current plasma systems is discussed in detail by Kadomtsev$^7$. The relative motion of the electron and ion subsystems can be neglected if the drift velocity is small compared to the characteristic velocities of the problem. For time-dependent flows, $v_0$ must be small compared to the MHD velocity of the plasma. On the other hand, because an equilibrium plasma is at rest, the thermal velocities of the electrons and ions are the only characteristic velocities relevant to studies of the equilibrium plasma configurations.

Trubnikov$^8$ was the first to carry out an MHD analysis of the stability of a fully ionized cylindrical plasma column carrying a longitudinal current uniformly distributed over a cross section. He found that such a system is destabilized by axisymmetric ($m = 0$) longitudinally periodic perturbations, and sausage formation and pinching occur. The need to avoid the sausage instability is one of the reasons why the more complicated toroidal plasma configuration was preferred.

However, pinch research over the last three decades has shown that "plasmas in which a strong sausage formation occurs may be more efficient neutron emitters than systems specifically designed to avoid the sausage instability.\textsuperscript{9} Experiments with dense pinched plasmas revealed that the plasma column is stable against axisymmetric perturbations during the entire lifetime of the discharge.\textsuperscript{10,11} The MHD analysis of the nonlinear stage of the sausage instability carried out in Ref. 12 shows that the current is not cut off even though abrupt constriction of the current channel may occur.

Pinch systems are currently not restricted to devices in which the plasma current is high and the drift velocity low. For example, for relativistic electron beams partially neutralized by ions, $v_0$ is comparable to the speed of light and is certainly much greater than the thermal velocities of the particles. It has become clear that the one-fluid MHD model cannot be used to fully analyze even the macroscopic stability of pinch systems.

If the drift velocity is large compared to the thermal velocities, the pinch system can be described macroscopically by the two-fluid MHD equations.$^1^3$ In this case one can neglect the mutual drag exerted by the electrons and ions, which can be regarded as equilibrium subsystems that interact through the electromagnetic field induced by the charged particles themselves. The quasineutrality approximation is also invalid for relativistic drift velocities. Furthermore, the analysis has been limited to small oscillations of the pinch system\textsuperscript{14,15} about the equilibrium state because no general methods for analyzing stability (in particular, no analog of the Bernstein-Frieman-Kruskal-Kulsrud energy principle$^1$) have yet been derived for the two-fluid electromagnetic hydrodynamic (EMHD) model.

The density, pressure, electromagnetic field, and velocities of the charged fluids are all coupled and oscillate together in a pinched system. These oscillations have so far been considered only in a few limiting cases for which an analytic treatment is possible. It was shown in Ref. 15 that in the two-fluid EMHD model, the pinch system is particularly susceptible to destabilization by short-wave ion-sound oscillations which are uniform in the direction of the current and cause the current channel to break up into parallel structures or jets. A condition for pinch systems to be stable against current channel stratification was derived there by analyzing the behavior near the stability threshold. In general, numerical methods are needed to verify if this condition is satisfied; however, an analytic treatment is possible for a Bennett pinch,$^1^6$ in which the electron and ion densities decay radially in the same way. In the present paper we show that for Bennett pinches the two-fluid EMHD model can be used to find the linear response in addition to permitting an analytic treatment of the stability problem.
2. BENNETT EQUILIBRIUM

The crucial assumption in the two-fluid EMHD description of pinching is that the thermal velocities of the electrons and ions are small compared to the drift velocity $v_d$. Because the Coulomb cross sections fall off rapidly as the relative velocity of the colliding particles increases, this assumption enables one to neglect the mutual drag exerted by the electron and ion subsystems. If we neglect radiation, each of these subsystems can be regarded as a one-dimensional idealized axisymmetric system uniform in the direction of the current will have a three-parameter family of equilibrium solutions in the classical Boltzmann approximation can be expressed as

$$n_e(r) = \frac{\Gamma_r}{4\pi \Delta \epsilon_r} e^{\Phi_e(r)} \left\{ \frac{\epsilon_e}{\epsilon_r} \right\},$$

$$n_i(r) = \frac{\Gamma_r}{4\pi \Delta \epsilon_r} e^{\Phi_i(r)} \left\{ \frac{\epsilon_i}{\epsilon_r} \right\},$$

where the discharge radius $r_s$ is chosen as a convenient scaling factor (the condition that the various forces in equilibrium for an isolated pinch system does not alone suffice to determine $r_s$).

In general, Eqs. (1) with the boundary conditions (2) must be solved numerically for arbitrary $\beta, \Lambda, \rho_0$. The value of the physical parameter $\epsilon$ corresponding to specified $\beta, \Lambda$, and $\rho_0$ is then calculated from the solution. For fixed $\beta$ and $\Lambda$, there exists a value $\rho_0 = \rho_0^* = \ln[1 - \epsilon^2/(1 + \epsilon)]$ (3)

for which the radial density distributions for the electrons and ions are identical. The solution in this case was first derived by Bennett; it depends on two parameters ($\beta$ and $\Lambda$ are arbitrary) and is given by

$$\rho_e(r) = \rho_i(r) = \rho_0^*,$$

where $\rho_0^* = \ln[1 + \epsilon].$

It is remarkable that for a Bennett distribution, the two-fluid EMHD model can also be solved analytically to find the linear response (assumed to be uniform along the current).

3. STABILITY THRESHOLD

Oscillations in a pinch system were studied analytically in Ref. 15 in the two-fluid EMHD model for the extreme case when the wavelength is very small or very large compared to the radius of the pinch. The long-wave oscillations were associated with net displacements of individual regions of the current channel. Owing to the "elasticity" of the electromagnetic field, the pinch system is stable against long-wave perturbations.

For axisymmetric plasma structures that are uniform in the direction of the current (i.e., $z$-independent), the perturbations are of the form

$$f(r, z, t) = f(r) \exp[i(\omega t - k z - \xi)],$$

After linearization, the two-fluid EMHD equations for a nonrelativistic, nondissipative system have the following form. The equation of continuity for the electron $(a = e)$ and ion $(a = i)$ subsystems:

$$\frac{1}{r} \frac{d}{dr} \left( r \frac{db_{en,a}}{d \rho} \right) = \left( \frac{1}{m_a} \frac{d^2}{d r^2} - \frac{\beta B^2}{c^2} \right) b_{en,a} + \frac{e_a}{c} \frac{d}{d r} \left( \frac{d b_{en,a}}{d \rho} \right) - \frac{e_a}{c} \frac{d^2 b_{en,a}}{d r^2} + \epsilon_a \frac{d b_{en,a}}{d \rho} + \epsilon_a \frac{d^2 b_{en,a}}{d \rho^2},$$

the Euler equations:

$$\frac{1}{r} \frac{d}{dr} \left( r \frac{db_{en,a}}{d \rho} \right) + \frac{e_a}{c} \frac{d}{d r} \left( \frac{d b_{en,a}}{d \rho} \right) + \epsilon_a \frac{d b_{en,a}}{d \rho} + \epsilon_a \frac{d^2 b_{en,a}}{d \rho^2} = 0,$$

where the functions $\psi_a$ satisfy the boundary conditions

$$\psi_e(0) = \psi_i(0) = \Phi_e(0) = 0,$$

The parameter $r_s$ appears here in place of $v_s$ and $\beta, \Lambda$, and $\rho_0$ completely specify all the possible configurations. The electron and ion densities $n_{e,i}(r)$ are given in terms of $\psi_e(z)$ by

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the entropy conservation equations
\[ f_\alpha \delta S + \delta P_{\alpha \beta} \delta S_{\beta} = 0; \]
the magneto- and electrostatic equations:
\[ \Delta A_{\alpha \beta} = -\frac{4\pi}{c} \sum_r e_r (n_{\alpha t} \delta A_{\alpha t} + n_{\alpha q} \delta A_{\alpha q}); \]
\[ \Delta P_{\alpha \beta} = -\frac{4\pi}{c} \sum_r e_r n_{\alpha q} \delta A_{\alpha q}. \]
Here \( \alpha' = \alpha - k \) is the Doppler-shifted oscillation frequency for particles of species \( \alpha \), and
\[ \Delta = \frac{1}{r} \frac{d}{dr} \frac{d}{dr} \frac{d}{dr} - \frac{m^2}{r^2} \]
is the Laplace operator.

For short-wave perturbations the radial dependence \( f(r) \) may be taken to have the form
\[ f(r) = \exp \left[ -\int \frac{r}{\rho_0} \left( \frac{dz}{dx} \right)^2 \right] \cdot e^{i \omega t}. \]
In this case the oscillations are localized and the linearized equations reduce to an algebraic system. If \( \Omega_0 \gg \omega \) and \( m \approx q \rho_0 \), the spectrum for longitudinally uniform oscillations is given by\(^{15}\)
\[ \omega_m = \omega_m(Q, \rho_0, \beta^2, \beta^2). \]
Here \( \Omega_0 = \left( e_\alpha / m_\alpha c \right) \Delta A_{\alpha \beta} / \partial r \) is the local Larmor frequency of the electrons, \( \rho_0 \) is the pinch radius, \( \Omega \) is the ion Langmuir frequency, \( Q_R = q^2 + m^2 / r^2 \), and \( r_\rho \) is the Debye radius.

The localization condition \( q R_\rho > 1 \) for the perturbations breaks down near the stability threshold \( Q_R r_\rho ^2 = \beta^2 \), and their wavelength becomes comparable to the radius of the discharge. Thus the local approximation cannot be used near threshold, and in general the equations must be solved numerically. However, the Bennett case \( \rho_0 = \rho_0(\beta) \) is exceptional in that the existence of a stability threshold can be demonstrated analytically. The following analytic treatment can be used to simplify the numerical analysis of stability for all values of \( \rho_0 \), for which equilibrium is possible.

The system will be unstable if and only if the system of homogeneous linear equations admits a nontrivial solution with zero boundary values.

For \( \omega = k = 0 \), the Euler equations (5) imply that \( \nu_\alpha = 0 \). Since the entropy conservation equation holds identically, we need only consider isothermal perturbations: \( \rho_\alpha = T_\alpha \rho_{\alpha t}, T_\alpha = \text{const} \). The continuity equation with \( \nu_\alpha = 0 \) implies that \( v_{\alpha \beta} = 0 \) if \( m \neq 0 \). Equations (4) and (6) imply that the z-projection \( v_{\alpha \beta} \) of the velocity vanishes, and we get the following system of equations for a pinch near the stability threshold:
\[ \Delta v_{\alpha \beta} - \frac{4\pi}{c} \sum_r e_r \nu_{\alpha \beta}(r) (1 - \beta_j k_j) \delta v_{\alpha \beta} = 0, \quad \alpha, \beta = i, e. \] (7)
Here
\[ \Delta = \left( \frac{1}{r} \frac{d}{dr} \frac{d}{dr} \frac{d}{dr} - \frac{m^2}{r^2} \right) \]
is the Laplace operator, the subscripts \( \alpha, \beta \) take the values \( i, e \) for ions and electrons, \( \nu_{\alpha \beta} = \rho_{\alpha t} / \rho_{\alpha q} \) are the relative density perturbations, and \( \beta_\alpha = v_\alpha / c \), where \( v_\alpha \) is the velocity of subsystem \( \alpha \) and \( c \) is the speed of light.

For a Bennett pinch, the change of variable
\[ z = \ln (r / \varpi) \]
reduces (7) to the form
\[ \left( \frac{d^2}{dx^2} - m^2 \right) v_x = \nu_{\alpha \beta} \delta v_{\alpha \beta}, \] (8)
where summation over the repeated subscript \( \beta \) is understood. In a reference frame moving with the ion subsystem \( \beta_\alpha = 0, \beta_\alpha = \beta \), the tensor \( \nu_{\alpha \beta} \) has the components
\[ \begin{align*}
\nu_{i i} &= \frac{1}{2} \left( 1 + \lambda \right) \left( 1 - \beta^2 \right) / \beta^2, \\
\nu_{e i} &= -\frac{1}{2} \left( 1 + \lambda \right) / \beta^2, \\

\nu_{i e} &= -\frac{1}{2} \left( 1 - \beta^2 \left( 1 + \frac{1}{1 - \beta^2} \right) / \beta^2, \\
\nu_{e e} &= \frac{1}{2} \left( 1 + \lambda \right) / \beta^2. 
\end{align*} \] (9)

They depend only on the two parameters \( \beta = \omega / c \) and \( \lambda = T_e / T_i \), as is to be expected since Bennett equilibrium corresponds to a two-parameter family of equilibrium structures. We will need the following properties of the tensor \( \nu_{\alpha \beta} \):
\begin{align*}
\nu_{i i} + \nu_{e e} &= - \nu_{i e} = - \frac{1}{2}, \\
\nu_{i i} &\leq \nu_{i e} = - \frac{1}{2} \left( 1 + \lambda \right) / \beta^2, \\
\nu_{e i} &\leq \nu_{e e} = \frac{1}{2} \left( 1 + \lambda \right) / \beta^2, \\
\nu_{i e} &\leq \nu_{e e} = \frac{1}{2} \left( 1 - \beta^2 \right) \left( 1 + \frac{1}{1 - \beta^2} \right) / \beta^2. 
\end{align*}

The trace \( S \) and determinant \( \Delta \) of the matrix \( \nu_{\alpha \beta} \) are related by
\[ \Delta = S^2 - 2 \lambda / S. \] (10)

If Eqs. (8) have a nontrivial solution satisfying the zero boundary conditions
\[ v_x (z = -\infty) = v_x (z = 0) = 0, \quad \alpha = e, i, \]
then the system must be unstable.

We have derived Eqs. (8) by using the nonrelativistic two-fluid EMHD model; however, their validity is not limited to \( \beta \ll 1 \). Equations (8) can be derived from the electrodynamostatic equations, as was done in Ref. 19. In this case one uses only the relativistic invariance of the distribution function\(^{16}\) and the fact that when \( k = 0 \), the perturbation leaves the system uniform along the direction of the original current.

The general solution of system (8) can be found if we note that the associated Legendre functions satisfy the equation
\[ \left( d^2 / dx^2 - m^2 \right) P_j^m (\cos x) = -v (v + 1) \cos x P_j^m (\cos x). \] (10)

Assuming a solution of (8) of the form
\[ v_x (z) = \nu_x P_j^m (\cos x), \] (11)
we then find a homogeneous system of algebraic equations
\[ (\lambda \nu_{\alpha \beta} + \nu_{\alpha \beta}) b = 0 \] (12)
for the coefficients \( b \), where \( \chi = v (v + 1) \). The solvability
are actually needed to construct the general solution of (8).

If we note that for nonintegral $\nu$ the functions $P^\nu_m(\cos\theta)$ give linearly independent solutions of Eq. (10), we write the general solution as

$$\nu(x) = b_{-\nu}P^\nu_m(\cos\theta) + b_{\nu}Q^\nu_m(\cos\theta)$$

(13)

Here the coefficients $b_{-\nu}$ and $b_{\nu}$ satisfy Eqs. (12) with $\lambda = \lambda_1 = 2$, while $b_{-\nu}$ and $b_{\nu}$ satisfy Eqs. (12) with $\lambda = \lambda_2 = \Delta/2$.

Of the four linearly independent Legendre functions in (13), only the first satisfies the zero boundary conditions, and then only if $m = 1$. The acoustic branch of the pinch becomes a static shear as $k$ oscillations of a vortex filament in a nonideal Bose gas. Let $SPECTRUM P, (\nu)$ also (8) v. the equations for the ions ($a = i$) and the electrons ($a = e$), respectively. Passing from $r$ to the variable $x = \ln(r/r_+)$, we get the following equations for a Bennett pinch:

$$\frac{d^2 v}{dx^2} - \omega^2 v = 0, \quad \frac{d^2 \omega}{dx^2} - \omega^2 = 0$$

(15)

For $x > 1$, the solutions of Eqs. (15) tending to zero as $x \to \infty$ are expressible in terms of modified Bessel functions:

$v = C_1 K_1(\Omega x), \quad \omega = D_1 I_1(\Omega x)$

(16)

For $x < \ln(1/\lambda)$ the terms proportional to $x^2$ and $\Omega^2$ may be treated as perturbations, and we seek a solution of the form

$v = v_{0}(x) + v_{1}(x)$

(17)

We consider the case when the number $N_e$ of electrons per unit length of the current channel satisfies

$eN_e/m_e c e/1$. We can then neglect the terms containing the azimuthal component $A_\phi$ of the perturbation of the vector potential in the equations of motion. Since $m_e < m_i$, we may also neglect the inertia of the electrons. The acoustic branch of the pinch oscillation spectrum is then the one associated with the azimuthal displacements of the ion component. In a reference frame for which the ion subsystem is stationary at equilibrium, we recover Eqs. (7) with the term

$\delta v = \left( \frac{1}{r} \frac{d}{dr} \frac{d}{dr} - \frac{1}{r^2} \right) v_{\nu}$

replaced by

$$\left( \frac{1}{r} \frac{d}{dr} \frac{d}{dr} - \frac{1}{r^2} \right) v_{\nu}$$

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$v = C_1 K_1(\Omega x), \quad \omega = D_1 I_1(\Omega x)$

(16)

For $x < \ln(1/\lambda)$ the terms proportional to $x^2$ and $\Omega^2$ may be treated as perturbations, and we seek a solution of the form

$v = v_{0}(x) + v_{1}(x)$

(17)
We are interested in the solution of (18) that tends to zero as \( x \to -\infty \) and matches the asymptotic solution (16) for \( x > 1 \). We use the method of variation of constants and seek a solution of the form

\[ v_{\alpha} = \sum_{i=1}^{4} A_i u_i + g_\alpha \sum_{i=1}^{4} A_i u_i, \]

where the \( u_i \) are the four linearly independent solutions of the homogeneous system:

\[ u_i = \frac{\text{ch} x}{x}, \quad u_i = -\text{ch} x, \quad u_i = \text{sh} x, \quad u_i = -\text{sh} x, \]

and the functions \( A_i(x) \) are to be found, \( g, = 1 \), and \( g, = -g, \), where

\[ A_i(x) \text{ are to be found, } g, = 1, \quad g, = -g. \]

We impose the two additional constraints

\[ \sum_{i=1}^{4} A_i = 0, \quad \sum_{i=1}^{4} A_i = 0 \]

on \( A_i(x) \), where the primes denote derivatives with respect to \( x \). Substitution of (19) in (18) together with (20) leads to the equations

\[ \sum_{i=1}^{4} A_i u_i = 0, \quad \sum_{i=1}^{4} A_i u_i = f_{\alpha}. \]

Equations (20) and (21) yield two uncoupled systems of algebraic equations for the pairs \( A_i, A_i' \) and \( A_i, A_i'' \):

\[ \sum_{i=1}^{4} A_i u_i = 0, \quad \sum_{i=1}^{4} A_i u_i' = \frac{f_i + g_{\alpha}}{1 + g}, \quad \sum_{i=1}^{4} A_i u_i'' = \frac{2f_i}{1 + g} \text{ch}^2 x. \]

The solution is readily found to be

\[ A_i = \frac{f_i + g_{\alpha}}{1 + g} u_i, \quad A_i' = \frac{2f_i}{1 + g} u_i', \quad A_i'' = \frac{2f_i}{1 + g} u_i''. \]

where the Wronskians \( u_i = u_i' = u_i'' = 0 \) are nonzero and independent of \( x \) for the linearly independent pairs of functions \( u_i, u_i, \), in fact, \( u_{ii} = 4 \).

We can use (22) to find the functions \( v_{\alpha} \) that vanish at \( x = -\infty \):
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