Axial toroidal moments in electrodynamics and solid-state physics

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We have found a family of multipole moments (axial toroidal moments) that differ in their space-time symmetry from the Maxwell-Lorentz moments known from electrodynamics. We consider the realization of this family in a system of magnetic charges, in electric dipole media, and also in media with magnetic fluxes. We investigate within the framework of a microscopic model a phase transition in a crystal with formation of an axial toroidal moment, and discuss certain interesting properties of the axial toroidal state.

§1. INTRODUCTION

In the study of electromagnetic properties of systems with distributed charges and currents it becomes necessary to choose macroscopic characteristics that describe adequately the interaction of these systems with external fields and currents. A convenient mathematical procedure for this purpose is the formalism of multipole expansions of the microscopic charge density \( \rho(r,t) \) and current density \( j(r,t) \).

For classical electrodynamics, this formalism is consistently developed, e.g., in Ref. 1. It is also shown there that besides the known families of charge and magnetic multipole moments there is produced a third family of toroidal multipole moments. The formal cause of the toroidal moments is the symmetry properties (the behavior of quantities \( P, M, T \) are different and are transformed in accordance with Table I.

It can be seen, however, that to construct the complete vector basis of multipole representations of the space-time inversion groups \( K \circ I \) the set of vectors \( P, M, T \) is insufficient, and in principle we need one more axial vector \( G \) whose symmetry properties are indicated in the lower row of Table I. In the Maxwell-Lorentz classical electrodynamics, where

\[
J(r,t) = \sum s \delta^{s}(r-r_{i}(t)),
\]

there is no place for realization of a vector with such properties. It will be shown nonetheless in the present paper that there are many physical applications of the mathematical formalism of multipole expansions, where the presence of the generating dipole moment \( G \) of a family of its multipoles is indispensable.

In the modification of the multiple-expansion scheme for problems of electrodynamics with a magnetic charge (§2) the vector \( G \) is the toroidal moment of the magnetic-charge current. It is similar in a certain sense to the toroidal moment \( T \) of the electric charges. Owing to the pseudoscalar properties of the magnetic-charge density, however, it turns out that \( G \) is an axial vector whereas \( T \) is a polar vector.

In the electrodynamics of continuous media, \( G \) can be introduced to describe systems with charge dipole moments (§3). In this case \( G \) is the analog of the induction toroidal moment \( \mathbf{T}_{m} \) in media with distributed magnetic dipole moments,\(^{1-4} \)

Starting from the symmetry properties and also from the analogy with toroidal multipoles in the Maxwell-Lorentz electrodynamics, we shall hereafter call the vector \( G \) the axial toroidal moment, and the vector \( T \) the polar toroidal moment.

Introduction of the spin makes it necessary to classify the toroidal moments \( T \) and \( G \) with respect to the inversion transformation \( \mathbf{R}_{i} \) in spin space. Where necessary, we shall label the even (singlet) vectors by a subscript \( s \), and the odd (triplet) by \( t \).

To describe phase transitions in crystals, the vectors \( T \) and \( G \) (or their higher multipoles) can be naturally be chosen to be order parameters that transform in accordance with certain irreducible representations of the magnetic group of a high-symmetry phase. We shall examine the distinguishing features of toroidal types of ordering and the systems for which introduction of these terms makes sense.

From the viewpoint of formal symmetry, many of the known order parameters are transformed in analogy with \( T \) and \( G \). Thus, for example, multipole order parameters (spin densities) were introduced to describe spin magnets, and some of these parameters have transformation properties similar to those of \( T \).\(^{5,6} \) The simplest case is that of a two-
of electrodynamics there is complete symmetry between the multipole source forms and the types of fields, especially should be a vector, and the magnetic-charge current a calculable meaning of given in $4)$. These models illustrate quite clearly the physical nature of its formation in crystals.

Radiation in magnetic theory with magnetic point charges is difficult attention to its specific features (the general derivations are considered, and we deemed it unnecessary to pay particular attention to its specific features (the general derivations are given in $4$). These models illustrate quite clearly the physical meaning of $G$ and permit a better understanding of the symmetry of the charged systems.

An electromagnetic theory with magnetic point charges is difficult attention to its specific features (the general derivations are considered, and we deemed it unnecessary to pay particular attention to its specific features (the general derivations are given in $4$). These models illustrate quite clearly the physical meaning of $G$ and permit a better understanding of the symmetry of the charged systems.

A microscopic model of a phase transition with formation of antiferroelectricity ($2$). Another microscopic model with formation of $T$, can be used to describe a number of spin itinerant antiferroelectrics, the axial toroidal moment describes orbital antiferromagnetic ordering.

The introduction of toroidal moments in a special group of order parameters is much more justified in the case of system with itinerant electrons. In particular, the polar multipole expansion of the current density of the spin itinerant antiferromagnets, the axial toroidal moment describes orbital antiferromagnetic ordering (vide infra), and the vector describes the orientational ordering in spin itinerant magnets.

The multipole-expansion scheme permits a very effective description of the general macroscopic properties of many systems with complex distributions of the charges and currents (in particular, the response to an electric field). No less important a task, however, is the investigation of actual quantum-mechanical models that realize various types of multipole structures. A very clear example is the theory developed in Refs. 14 for polar toroidal ordering in crystals. In this paper, along with a general phenomenological analysis of axial toroidal ordering ($4$), we propose a microscopic model of a phase transition with formation of $G$, ($5$). Another microscopic model with formation of $G$, was considered earlier, and we deemed it unnecessary to pay particular attention to its specific features (the general derivations are given in $4$). These models illustrate quite clearly the physical meaning of $G$ and permit a better understanding of the nature of its formation in crystals.

### Multipole Expansion of a System of Magnetic Charges

It was noted in Ref. 1 that in the dual-invariant scheme of electrodynamics there is complete symmetry between the multipole source forms and the types of fields, especially radiation fields. We note that in an electromagnetic theory invariant to $R$ and $I$ reflections the electric-charge current should be a vector, and the magnetic-charge current a pseudovector, if the customary convention concerning the space-charge properties of the field $F$ and $H$ is adhered to (see, e.g., Ref. 17). Obviously (see Ref. 16), formulation of an electromagnetic theory with magnetic point charges is difficult (see, however, Ref. 18) since, e.g., the relation $\text{div } H = -\sigma(r)$ either the charge is not simply a number, or else the charge is a number and there is no parity conservation in the theory. This difficulty does not arise in the macroscopic formulation, since the function $\rho_m(r,t)$ can be always assumed to be odd, and $\rho_m(r,t)$ in the equation $\text{div } H = -\sigma(r)$ can be regarded as an axial vector. Taking into account this difference between the world of magnetic charges and the world of electric charges, the problem of multipole expansion of the densities $\rho_m(r,t)$ and $\rho_e(r,t)$ can be easily solved by simply making the replacements $\rho_e \rightarrow \rho_m$ and $\rho_m \rightarrow \rho_e$ in the corresponding equations of Ref. 1. We write out these equations and note the identical dual symbolism of $\rho/e$ in the “electric” and “magnetic” worlds. Thus, the expansion of the charge is written in the form

$$\rho_{e/m}(r,t) = \sum_{n=1}^{\infty} \left( \frac{(-1)^{n+1}}{2n+1} \right) \frac{4\pi}{(2n+1)!} \int \frac{d^3r}{r} \rho_{e/m}(r,t) \frac{r}{r} \cdot \mathbf{e}_n \cdot \mathbf{e}_n \, d^3r,$$

where the charge multipole moments

$$Q_{nm}^e/h^l = \int \frac{d^3r}{r} \rho_{e/m}(r,t) \frac{r}{r} \cdot \mathbf{e}_n \cdot \mathbf{e}_n \, d^3r$$

and their radii raised to the power $2n$

$$r_{nm}^e/h^l = \left( \frac{4\pi}{2n+1} \right)^{1/2} \int \frac{d^3r}{r} \rho_{e/m}(r,t) \frac{r}{r} \cdot \mathbf{e}_n \cdot \mathbf{e}_n \, d^3r$$

which complete the multipole parametrization of the initial function $\rho_{e/m}(r,t)$.

The multipole expansion of the current density of the magnetic (electric) charges with the toroidal part singled out takes the form

$$J_{e/m}^l(r,t) = \frac{1}{(2n+1)!} \sum_{l=1}^{\infty} \left( \frac{(-1)^{n+1}}{2n+1} \right) \frac{4\pi}{(2n+1)!} \int \frac{d^3r}{r} \rho_{e/m}(r,t) \frac{r}{r} \cdot \mathbf{e}_n \cdot \mathbf{e}_n \, d^3r.$$

The basis of the expansion ($6$) is introduced as follows:

$$F_{e/m}^{l\alpha} = \frac{\mathbf{e}_n \cdot \mathbf{e}_n}{(2n+1)!} \text{rot}(\mathbf{e}_n \cdot \mathbf{e}_n).$$

The spherical vector are introduced as in Ref. 19:

$$M_{e/m}^{l\alpha} = \frac{1}{(2n+1)!} \int \frac{d^3r}{r} \rho_{e/m}(r,t) \frac{r}{r} \cdot \mathbf{e}_n \cdot \mathbf{e}_n \, d^3r.$$

The magnetic multipole distributions are

$$M_{e/m}^{l\alpha}(r,t) = \frac{1}{(2n+1)!} \int \frac{d^3r}{r} \rho_{e/m}(r,t) \frac{r}{r} \cdot \mathbf{e}_n \cdot \mathbf{e}_n \, d^3r.$$

### Table I

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345 Sov. Phys. JETP 63 (2), February 1986

Dubovik et al. 345
the magnetic multipole moments are
\[ M^m_r (r, t) = \frac{4\pi}{(2l+1)(l+1)!} \int r Y_{lm}(r, t) \, dr; \]
the toroidal multipole distributions are given by
\[ T_r^{\ell \ell'} (r, t) = \frac{-i \ell}{(2\ell+1)i^\ell} \frac{4\pi}{(2\ell+1)(\ell+1)!} r Y_{\ell \ell'}(r, t) \]
and the longitudinal charge multipole moments are
\[ Q_r \ell \ell' (r, t) = \frac{-i \ell}{(2\ell+1)i^\ell} \frac{4\pi}{(2\ell+1)(\ell+1)!} r \tilde{Y}_{\ell \ell'}(r, t) \]
When the conditions for spectral expansion of \( \hat{Q}_r \ell \ell' (r, t) \) and \( \hat{T}_r \ell \ell' (r, t) \) are satisfied (the simplest case is that of harmonic sources) the last definitions reduce to Eqs. (3) and (4), and the expressions (12), (13) are functionally dependent on \( Q_r \ell \ell' \) in all cases, in view of the conservation of the 4-current.

The question of how the "longitudinal" moments \( Q_r \ell \ell' (0, t) \) "turn up" in the expansion of the transverse part of the current (\( \text{div} \, j = 0 \)) [see Eq. (6)] is more complicated and is answered in Ref. 1 (see also the literature cited there).

It is easy to ascertain, by starting from the properties of the vector spherical functions \( F_{\ell \ell'}(r) \) relative to \( \ell \) reflections, that the distributions \( M_r \ell \ell' \) now produce \( E_r \) type fields (in particular, they emit \( E_r \) multipoles), while the charge \( Q_r \ell \ell' \) and toroidal \( T_r \ell \ell' \) distributions produce the \( M_r \ell \ell' \) type fields (in particular, \( Q_r \ell \ell' \) and \( T_r \ell \ell' \) are responsible for the emission of \( M_r \ell \ell' \) multipoles).
scales over which the multipole expansion is carried out are large compared with the characteristic dimensions of the dipoles themselves, so that the latter can be regarded as pointlike. The electric polarization of the medium is introduced in the usual manner:

\[ P(r,t) = \sum \delta(r-r_0(t)) \]  

(18)

Obviously, the electric polarization \( P \) can imitate, by virtue of the dual symmetry, multipole moments of magnetic charges. We introduce an axial "current" that is a pseudovector with respect to time reversal in a medium of distributed electric dipoles:

\[ \mu_\alpha(r,t) = \sum \delta(r-r_0(t)) - \text{rot} P_\alpha(r,t) \]  

(19)

where \( P_\alpha \) is the transverse part of the electric-dipole-moment density (polarization). The longitudinal part \( P \) of the polarization is described by the scalar distribution density of the electric charges

\[ \rho_\alpha(r,t) = \text{div} P_\alpha(r,t) \]  

(20)

We emphasize that the axial current \( \mu_\alpha \) differs in its nature from the polar current \( J_\alpha \) in the Maxwell-Lorentz equation:

\[ j_\alpha(r,t) = \sum \delta(r-r_0(t)) - P_\alpha(r,t) \]  

(21)

In contrast to the magnetization components \( M_\alpha \) and \( M_\beta \), both polarization components \( P_\alpha \) and \( P_\beta \) enter in the Maxwell-Lorentz equations, and \( P_\alpha \) drops out of them only in the static limit.

Returning to Eq. (19), we see that substitution of the effective current \( j^{\text{eff}}_\alpha \) in the definition of \( M_\alpha \), transforms, after appropriate integration by parts, the definition of \( M_\alpha \) into the usual definition of the electric part, in which \( j_\alpha \) is replaced by \( P_\alpha \). In this case, of course, \( E^{\text{em}} = 0 \). On the other hand, the situation with \( T^{\text{eff}}_\alpha \) is more curious. Substitution of \( j^{\text{eff}}_\alpha \) (in the definition (11) and transfer of the derivative lead to an equation similar to (16):

\[ T^{\text{eff}}_\alpha = \text{rot} J^{\text{eff}}_\alpha = \frac{4\pi}{12} \sum \delta(r-r_0(t)) \int r \text{div} P_\alpha(r,t) d^2r \]  

(22)

It follows directly that the elementary "induced" axial toroidal dipole moment is

\[ T^{\text{eff}}_\alpha = \frac{1}{2} \sum \delta(r-r_0(t)) \]  

(23)

and its geometric representation is a closed chain of electric charge dipoles (Fig. 1b). The last equations demonstrate the simplest possibility of imitating, in the dipole representation, symmetry elements that are absent from a system of electric point charges. (In principle, completeness of the properties under \( R \) and \( I \) reflections can be obtained also in media made up of elementary higher multiples of the usual type.)

We indicate one more possibility of realizing the \( T^s \) symmetry in media. Recall that in the problem of particle motion in a centrosymmetric potential one encounters a correlation between the angular-momentum and momentum vectors \( \mathbf{L} \) and \( \mathbf{P} \). This correlation is described by the Runge-Lenz operator

\[ \Pi = [\mathbf{L}, \mathbf{P}] \]  

(24)

The Runge-Lenz operator appears formally even in the analysis of the dynamic symmetry of the nonrelativistic Kepler problem, but it plays a more substantial role in the analysis of the dynamic symmetry of the relativistic Coulomb problem\(^7\) or of the motion of a free relativistic particle that satisfies the Dirac equation.\(^8\) The presence in a medium of a distribution of a vector of type \( H \) (of orbital or spin origin) also requires introduction of axial toroidal moments.

Consider now a medium with distributed moment fluxes described in the general case by the second-rank tensor (dyad)

\[ \Pi_{ij} = \langle L_i P_j \rangle \quad i, j = x, y, z \]  

(25)

We represent (25) as a sum of a symmetric \( \Pi_{ij}^{\text{sym}} \) and antisymmetric \( \Pi_{ij}^{\text{asym}} \) parts. The latter is the dual of the polar vector \( \Pi \) and can be described in analogy with the preceding analysis of the vector \( \mathbf{P} \) in electrodipole media. We introduce a transverse induction (axial) current

\[ \mu_{\text{ind}}^{\text{eff}}(r,t) = \text{rot} J_\alpha(r,t) = \text{rot} (\mathbf{L} \cdot \mathbf{P}) \]  

(26)

Obviously, substitution of this current in Eqs. (10) and (11), just as substitution of \( J_\alpha \), will give rise to a multipole family \( T^{\text{ind}} \). The elementary is in this case

\[ T_\alpha = \frac{1}{2} \sum \delta(r-r_0(t)) \]  

(27)

The ideal geometric picture of this dipole consists of local moments precessing on a circle (Fig. 1c).

\section{Phenomenological Theory of Axial Toroidal Ordering in Crystals}

Consider a system of itinerant electrons, in which a second-order phase transition produces a unique type of long-range order describable by an axial vector that is even with respect to time reversal. Before turning to actual microscopic models that explain the mechanisms of this ordering, let us dwell on some of its phenomenological consequences that involve only formal symmetry considerations.

Among all magnetic symmetry classes, the following 43 allow existence of an axial vector that is even with respect to time reversal \( R \): 1) thirteen ordinary crystal classes that do not contain \( R \) at all:

\[ C_1, C_3, C_4, C_6, C_{2v}, C_{3v}, C_{4v}, C_{6v}, C_{2h}, C_{3h}, C_{4h}, C_{6h} \]  

(28)

2) the same classes supplemented by the operation \( R \); 3) seventeen proper magnetic classes:

\[ C_1(C_1), C_3(C_3), C_4(C_4), C_6(C_6), C_{2v}(C_{2v}), C_{3v}(C_{3v}), C_{4v}(C_{4v}), C_{6v}(C_{6v}), C_{2h}(C_{2h}), C_{3h}(C_{3h}), C_{4h}(C_{4h}), C_{6h}(C_{6h}) \]  

(29)

We consider hereafter only systems that do not contain nontrivial translations, in which specification of the magnetic class is the necessary and sufficient condition that determines the existence of a vector \( G \). As in the case of the polar vector \( T \), all the following classes can be easily obtained from the tables of irreducible representations of point groups.

The establishment of an axial toroidal order in a crystal can be associated with relaxation of some collective electron...
oscillation mode. We name this an axial toroidal mode, in analogy with the previously considered polar toroidal oscillations. Assume that for some reason the frequency of a toroidal mode became anomalously small and a tendency to establish a toroidal long-range order set in.

It is convenient to analyze the general properties of such systems by the effective-Lagrangian method. Assuming that the symmetry group of the high-symmetry phase admits as a subgroup one of the axial magnetic groups listed above, we consider small low-frequency axial toroidal oscillations above the phase-transition point. In the absence of external field, the Lagrangian of the system takes the form

$$\mathcal{Z} = K - U,$$  \hfill (30)

where

$$K = \frac{1}{2M_0} \left( \tilde{G}(\tilde{G})^* + D_\alpha (\tilde{G})^\alpha \right)^2,$$  \hfill (31)

and

$$U = iG G + \theta (\text{rot} G)^\alpha,$$  \hfill (32)

where the coefficients $M_\alpha$, $D_\alpha$, $\alpha, \beta > 0$, and the system symmetry above the transition point is assumed for the sake of argument to be cubic. We have retained in the kinetic energy (31) a term of type $(G)^2$, allowance for which will be justified presently. From symmetry consideration, we express the interaction with the external field in the form

$$\Delta \mathcal{Z} = -i\varepsilon G \text{rot} \text{A},$$  \hfill (33)

where A is the vector potential, $\varepsilon$ the speed of light, and $\lambda$ a coefficient. Equation (33) can be represented also in the form of two equivalent expressions

$$\Delta \mathcal{Z} = -i\varepsilon \text{GB},$$  \hfill (34)

and

$$\Delta \mathcal{Z} = i\lambda \text{rot} \text{E},$$  \hfill (35)

where $B = \text{curl} \text{A}$, $E = -\text{A}/c$, and we used the Maxwell equation for the solenoidal component of the electric field $E$: $\text{rot} \text{E} = -B/c$. \hfill (36)

It can be seen from (33) and (34) that the addition to the dynamic magnetic susceptibility $\Delta \chi(\omega)$ takes at low frequencies the form

$$\Delta \chi(\omega) = \frac{\lambda^2}{\varepsilon^2} \left( \frac{\omega}{\Omega_c^2} \right)^2,$$  \hfill (37)

where $\Omega_c^2 = 2M_0 \omega^2$ is the natural frequency of the axial toroidal oscillations. The vanishing of $\Omega_c$ corresponds to a second-order phase transition. Note that we are dealing throughout only with transverse axial toroidal oscillations, while the longitudinal ones do not interact with the electric and magnetic fields. In principle, axial toroidal modes could react to a current of magnetic charges (were they to exist), just as polar toroidal modes react to an ordinary electric current.$^{14}$

What is noteworthy is the nontrivial frequency dependence of $\Delta \chi'(\omega)$ (the numerator is proportional to $\omega^3$) just as in the case of polar toroidal oscillations, where the dynamic dielectric constant has an anomaly similar to (37), Ref. 24. Expression (37) is valid only at low frequencies, when the second term of (31) can be neglected. To obtain the correct asymptotic form at $\omega \Omega_c$, it is necessary to take this term into account, and then the contribution $\Delta \chi'(\omega)$ vanishes at high frequencies, as it should. In the microscopic model ($\S$) the second term becomes appreciable at frequencies $\omega - \Omega_c$, where $\Omega_c$ is a characteristic one-electron energy of the order of the semiconductor band gap.

Curious effects can take place in systems in which the axial toroidal ordering is accompanied by some other type of magnetic long-range order. For example, in the case of antiferromagnets with spin density waves (SDW) the appearance of axial toroidal order leads to “weak” ferromagnetism.$^{15}$

In the case of antiferromagnets that contain localized moments besides itinerant electrons, the axial toroidal order also introduces in the Lagrangian of the system a term responsible for the weak ferromagnetism of the local moments:

$$\Delta \mathcal{Z}_s = iG [L, M],$$  \hfill (38)

where $L$ is the antiferromagnetism vector, $M$ the average magnetic moment of the cell, and $g$ a coefficient. We note that in the model of Ref. 15 the entire effect is of purely exchange origin and does not contain a relativistic smallness. On the other hand, the usual Dzyaloshinskii-Moriya weak-ferromagnetism mechanism is connected with spin-orbit or magnetodipole interaction.$^{25}$

If the axial toroidal moment $G$ has an incommensurate structure below the phase-transition point, inhomogeneous spontaneous polarization sets in:

$$P = \lambda \text{rot} G,$$  \hfill (39)

as a direct result of writing the term representing the interaction with the electric field in the form (35). The transverse static dielectric constant diverges at the phase-transition point.$^{16}$

On a certain wave vector $q = q_0$, obtained when the frequency of the natural transverse oscillations vanishes, we have $\Omega_c (q_0) = 0$, $\Delta \chi (q_0) \rightarrow \infty$ (at $q_0 \neq 0$).

Axial toroidal oscillations can interact with other collective excitations in crystals. Consider, for example, a ferromagnet with local moments, in which axial toroidal ordering is not realized in the ground state. At the same time, the collective toroidal oscillations are intermixed with the ordinary magnons, since the effective Hamiltonian of the system contains terms of the type

$$\Delta \mathcal{Z}_0 = -iM \text{rot} G,$$  \hfill (41)

and

$$H_{\text{tor}}(G) = \lambda G,$$  \hfill (42)

where $\lambda$ is a proportionality coefficient and $M$ is the magnetic moment. We write down the Bloch equation, with (41) and (42) taken into account, for small deviations $m(r, t)$ of the magnetic moment $M$ from the equilibrium value $M_0$:

$$m(r, t) = -i M_0 (M + m, H_{\text{tor}}),$$  \hfill (43)

and

$$M - M_0 + \Delta M - H_{\text{tor}} = H_{\text{tor}},$$  \hfill (44)

where

$$\Delta M \equiv M - M_0 = \left[ M_0, H_{\text{tor}}(G) \right] = -i \frac{\hbar}{\varepsilon} \text{rot} \text{G},$$  \hfill (45)

and $\lambda$ is a proportionality coefficient and $M$ is the magnetic moment. We write down the Bloch equation, with (41) and (42) taken into account, for small deviations $m(r, t)$ of the magnetic moment $M$ from the equilibrium value $M_0$: $m = -i \frac{\hbar}{\varepsilon} [M_0, M]$.\hfill (43)
Here $g = g/2mc$, where $g$ is the gyromagnetic ratio and $H_6$ is the magnetic anisotropy contribution, which we shall not write down here explicitly (see Ref. 12). Putting $	ilde{a} = a_n \hat{n}$, where $\hat{n}$ is a unit vector in the direction of the wave vector $q$, $mLM_n$, and $g = 2$, we obtain for the Fourier components of $m$,

$$\omega_n = \gamma_0 q \pm \omega_0 q \hat{n} + i \omega_n \hat{n},$$

and use for $G_n$ an equations that follows from the variation of the effective Lagrangian with allowance for (33) and (34):

$$\frac{\omega_n^2}{2M_\xi} = \omega_n^2 - q \hat{n} \cdot \omega_n \
+ \frac{\alpha_n}{2} m_n = 0.$$  

The equation for the dispersion law of the magnon-toroidal oscillations is of the form

$$\omega_0^2 \frac{d^2 m}{d^2_\xi} = \frac{1}{\beta_m} (q^2 \pm q \omega_0 \hat{n} + \omega_0^2 q \hat{n} \cdot \omega_n).$$

As $q \to 0$ Eq. (53) takes the simpler form

$$\omega_0^2 \frac{d^2 m}{d^2_\xi} = \omega_n^2 (q \hat{n} \cdot \omega_n),$$

and at $q \to 0$ we have the asymptotic relations

$$\omega_0^2 - \omega_n^2 (q \hat{n} \cdot \omega_n).$$

Note that Eqs. (51)–(55) are valid only at low frequencies and small momenta ($\omega, c, q < E$) in the microscopic model of §5. If $\omega, c, q \gg E$, we must retain in the Lagrangian (31) the terms with higher derivatives of the order parameter $G$, in analogy with the case of polar toroidal oscillations. As a result we have at high energies and momenta the correct asymptotic forms

$$\omega_0^2 - \omega_n^2 (q \hat{n} \cdot \omega_n).$$

Interesting nonlinear optical effects can take place in systems with axial toroidal ordering. Below the transition point, in particular, an anomalous contribution proportional to the electric field $E$ is made to the components of the gyration tensor $g_{\xi}:

$$g_{\xi} \sim \omega_0 E \hat{n}.$$

The anomalous behavior of the electrooptic and magneto-optic characteristics of crystals can be of help in the identification of toroidal transitions.

§5. MACROSCOPIC MODEL OF AXIAL TOROIDAL ORDERING

Consider now a two-band model of a semiconductor or a semimetal with straight extrema at the point $k_0$ of the Brillouin zone. Assume that the matrix element of the interband dipole transition is zero at the point $k_0$ (the wave functions of bands 1 and 2 have like parity but belong to different irreducible representations of the group of the wave vector $k_0$) and the matrix element of the interband transition with respect to the orbital momentum differs from zero. A model with this symmetry was considered in Ref. 26, where it was shown that realization of electron-hole pairing with imaginary singlet order parameter gives rise to orbital ferromagnetism of the occupied-band electrons. We write the Hamiltonian of the system in the $k-p$ approximation in an external electromagnetic field $H$.
where $m_1$ and $m_2$ are the effective masses of the electrons and holes in bands 1 and 2; $E_g$ is the band gap of the semiconductor; $\Lambda(r,t)$, $\Phi(r,t)$ are the vector and scalar potentials of the electromagnetic field, and $\Delta_1(r,t)$ is the order parameter that describes the ordered state below the phase-transition point in a two-band model of the excitonic-dielectric type and has in the general case a tensor structure:

$$\Delta_1 = \Lambda_1 + i \Lambda_2$$

where $\Lambda$ is a unit matrix and $\phi$ is a vector made up of Pauli matrices. It is assumed that the effective interaction constant $g_{\phi e}$ is a maximum in the case of a transition into a state with $\Delta_1$, so that the corresponding transition temperature (or the critical value of the band gap $E_g^*$ in the semiconductor model at $T = 0$) is also a maximum, and the state with $\Delta_1$ is energywise most favored. Explicit forms of the effective interaction constants for all possible structures of the order parameter $\Delta_1$ can be found, e.g., in Ref. 28.

The tensors $g_{\phi e}$ in the one-electron part of the Hamiltonian $H$ are given by

$$g_{\phi e} = -\frac{1}{2} \sum_{\ell,m} \left[ \frac{1}{E_{\ell} - E_m} + \frac{1}{E_m - E_{\ell}} \right] P_{\ell}^e P_{m}^e \delta^{\ell m},$$

where $E_{\ell}$ is the $\ell$-band energy at the point $k_\ell$, while $P_{\ell}^e$ is the momentum matrix element between the and $i = 1,2$ and the remote band $\tau = 1,2$ and $m$ is the electron mass. We consider next the case when the tensor $g_{\phi e}$ is pure real (this occurs, for example, when the Bloch wave functions $\psi_{\ell}(k)$ at the point $k_\ell$ can be chosen real).

The system with Hamiltonian (59) is analyzed by the standard Green's function method, and we shall not dwell on the calculation technique (a detailed exposition of the general calculation procedure in models of the excitonic-dielectric type can be found in Ref. 28). We note only the singularities connected with the reaction to an external magnetic field, since it just these singularities which explain the type of electronic ordering that is produced in the system below the phase-transition point. We write down the effective Lagrangian that describes the transition into the state below the phase-transition point. We denote the change of $\Delta_1$, which describes the transition into the state with $K \neq 0$ for a semiconductor model with a small band gap $E_g \approx E_\tau$, where $E_\tau$ is of the order of the exciton band energy. According to lower terms relative to the parameter $\Delta_1$, $E_\tau < 1$, when $\Delta_1 = (\Delta_1 + \Delta_2)/2$, we obtain in a weak and slowly varying transverse field, after laborious calculations,

$$K = \frac{N_m N_\ell}{2 m_0},$$

$$U = \frac{N}{\alpha (\Delta_1)^2} - \frac{\lambda}{c} \partial t \Delta_1 \partial r,$$

$$M = \varepsilon E_\tau \partial r, \quad \Delta_0 \approx \Delta_1 \cos q_0 r,$$

where $\varepsilon$ is the wave vector of the superstructure ($q_0 = 0$), $\Delta_1$ is the amplitude excitations are in fact longitudinal toroidal oscillations, while the phase excitations are magnons, inasmuch as at small deviations from equilibrium we have in such a system

$$\Delta_1(t) = |\Delta(t)| \exp i(q_0 t) = |\Delta| (1 + i q_0 t),$$

$$\Delta_0 (t) = \Delta_0 + \delta \Delta(t),$$

$$\delta \Delta(t) = \delta \Delta_0 (t),$$

$$G(t) = i \delta \Delta_0 (t), \quad \delta M = i \delta \Delta_0 (t),$$

where $G(t)$ is the density of the axial toroidal moment and $M(t)$ is the density of the orbital magnetic moment. Both oscillations make resonant contributions to the dielectric constant and the magnetic permeability of the system at the corresponding frequencies.

An interesting situation can arise in the case of a non-commensurate structure $\Delta_1$ (soliton lattice). In accordance with the general conclusions of §4, a spontaneous inhomogeneous transverse polarization $P_\tau (r) = \text{curl} \text{ Gr} (r)$ is produced in the system. In the semimetal with Hamiltonian (59) (where the Fermi energy is $E_F = -E_{\tau}/2$), in the region of the incommensurate structure (59) at $T < T_0$, where $T_0$ is the transition temperature, we have

$$P_\tau (r) = \text{rot} (\Lambda_1 \Lambda_1^e r),$$

$$\varepsilon = \frac{4 \pi e_0}{\alpha (m_0)}, \quad N_0 = \frac{m_p c}{4 \pi e_0},$$

$$\Delta_0 (r) = \Delta_1 \cos q_0 r,$$

and $q_0$ is the wave vector of the superstructure ($q_0 = 0$). With decreasing temperature, one more transition can occur and produce an order parameter $\Delta_0 (r)$ against the background of $\Delta_1 (r)$, with the spatial distribution of $\Delta_0 (r)$ shifted by $\pi/2$ relative to $\Delta_1 (r)$ (for details see, e.g., Ref. 29):

$$\Delta_0 (r) = \Delta_1 \cos \pi q_0 r.$$
Thus various types of electronic ordering (ferroelectric, ferromagnetic, or "ferroelectromagnetic") are produced in the region of domain walls of incommensurate structures. Various types of electronic ordering (ferroelectric, ferromagnetic, or "ferroelectromagnetic") are produced in the region of domain walls of incommensurate structures. Various types of electronic ordering (ferroelectric, ferromagnetic, or "ferroelectromagnetic") are produced in the region of domain walls of incommensurate structures. Various types of electronic ordering (ferroelectric, ferromagnetic, or "ferroelectromagnetic") are produced in the region of domain walls of incommensurate structures. Various types of electronic ordering (ferroelectric, ferromagnetic, or "ferroelectromagnetic") are produced in the region of domain walls of incommensurate structures. Various types of electronic ordering (ferroelectric, ferromagnetic, or "ferroelectromagnetic") are produced in the region of domain walls of incommensurate structures. Various types of electronic ordering (ferroelectric, ferromagnetic, or "ferroelectromagnetic") are produced in the region of domain walls of incommensurate structures. Various types of electronic ordering (ferroelectric, ferromagnetic, or "ferroelectromagnetic") are produced in the region of domain walls of incommensurate structures. Various types of electronic ordering (ferroelectric, ferromagnetic, or "ferroelectromagnetic") are produced in the region of domain walls of incommensurate structures.

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