Decay of correlations in dynamical systems with chaotic behavior

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Dynamical systems (billiards) corresponding to the inertial motion of particles in a two-dimensional region with elastic reflection from its boundary are considered. Depending on the structure of the boundary, the billiards display substantially different types of behavior, ranging from integrable to chaotic. Estimates of the decay of the time correlations of the phase functions are obtained for different classes of billiards with chaotic behavior. In this case the asymptotic form of the correlations is always found to be nonexponential. The asymptotic form of the diffusion coefficient for a periodic Lorentz gas is calculated in the limits of high and low density. The question of the relationship between the rates of decay of correlations in dynamical systems with discrete and continuous time is discussed.

1. INTRODUCTION

There has recently been a considerable growth of interest in the study of dynamical chaos, by which is meant that type of behavior of completely deterministic systems (in the absence of random noise and fluctuations) in which their motion is indistinguishable from random motion, i.e., is extremely irregular and unpredictable. Models displaying such behavior have been discovered in practically all areas of physics, and their number is continuously increasing.[1–4]

In the study of such systems two fundamental questions arise: What is the mechanism of the stochastic motion of the given system, and what statistical properties does the motion possess? In all model systems, by the mechanism of the stochastic motion we mean a local-instability mechanism that causes initially close trajectories to diverge sufficiently rapidly in the phase space with the passage of time. As a result, a prediction of the evolution of such a system can be only statistical, since the state of the system after a long time depends on unimaginably subtle features of the initial state. It is clear that the presence of such a (local) instability is a necessary condition for stochasticity of the system.

In respect of the statistics of the chaotic motion of a dynamical system it is not possible to formulate any single property that would characterize the system completely or sufficiently exhaustively. Dynamical systems have a large number of different statistical properties that cannot be arranged into a chain of mutually amplifying properties.[5,6] In particular, there does not even exist a generally accepted definition of a system with chaotic behavior. The absence of a unified terminology is partly connected with this. Therefore, it is appropriate to stipulate that by a dynamical system with chaotic behavior we shall mean either a conservative (the phase volume is invariant with respect to the dynamics) system for which the correlators of the (nonpathological[1)]) phase functions tend to zero at large times, or a dissipative system for which almost all (with respect to the phase volume) trajectories tend to (one or more) stochastic strange attractors. According to the customary definition of a stochastic attractor,[5] in such systems the correlations also decay with time. Thus, we shall regard as chaotic a dynamical system in which the time correlations are decoupled. For conservative systems this property (which in ergodic theory is called mixing) is equivalent to the fact that a nonequilibrium distribution (absolutely continuous with respect to the phase volume) tends in the course of time to an equilibrium distribution, i.e., in systems with mixing the fluctuations relax. In dissipative systems fluctuations can encompass domains of attraction belonging to different attractors. Therefore (if there are at least two attractors), in such systems there is a whole family of natural equilibrium distributions.

One of the basic problems arising in the study of systems with mixing is that of estimating the rate of decoupling of the correlations. A slow (power-law) decay of correlations was discovered a comparatively long time ago in numerical experiments for a gas of hard spheres, and later also for other models of statistical mechanics.[8] Nevertheless, until now it has been customary to assume[4] that in dynamical systems with chaotic behavior the time correlations should (in every case, as a rule) decay at an exponential rate. One of the aims of this paper is to show that this is not so, or, is not entirely so in every case.

The paper has a further aim: The study of specific physical systems with chaotic behavior is always based on a system of model ideas concerning the mechanisms of the stochasticity of the dynamical systems. With some specific mechanism (model) in mind, one makes approximate qualitative estimates which are then compared with the results of a numerical calculation. But the model ideas are determined by the set of examples of dynamical systems with chaotic behavior that admit a rigorous investigation. In the present paper we consider a broad class of intuitive examples of systems with chaotic behavior that display different rates of decay of the time correlations. The method applied, which consists in estimating the probabilities (phase volume) of sets of trajectories that possess particular properties is at the present time the only method of investigating such systems (it is possible that there can be no other methods).

The results to be described are obtained rigorously. Despite the fact that the corresponding proofs are rather long

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and complicated, the basic ideas are quite intuitive. It is evident that these ideas can also be used to investigate physical models that cannot (at least yet) be subjected to rigorous analysis.

2. BILLIARD SYSTEMS

The models that will be considered belong to the class of billiard systems, or, simply, billiards. A billiard is a dynamical system generated by the inertial motion of a material point inside a certain region Q on a plane.[2), 3)] Upon reaching the boundary ∂Q of the region Q the point is reflected from it in accordance with the law "angle of incidence equals angle of reflection." The billiard is a Hamiltonian system. The corresponding potential is equal to zero inside the region and to infinity on its boundary.

The trajectory of the billiard in Q is a broken line whose segments correspond to the free motion of the particle between two successive reflections from the boundary of the region. In addition to their intuitive aspect, billiards also have a clear physical meaning, being model systems in statistical mechanics, acoustics, optics, and certain quasiclassical problems.[8–11]

Let $q = (q^1, q^2)$ be the coordinates in Q, and let $v = (v^1, v^2)$ be the velocity of the particle. With no loss of generality, we shall assume that the modulus of the velocity is equal to unity. The region Q is the configuration space of the billiard. The phase space M of the billiard consists of the set of all pairs $x = (q,\mathbf{v})$ such that q is a point in Q. The phase flux $\{S^t\}$ $(-\infty < t < \infty)$ generated by the billiard conserves the volume (measure) μ in M, given by the formula $d\mu = (2\pi\overline{Q})^{-1} dq d\omega_q$, where dq is an area element in Q, $d\omega_q$ is a line element on the unit circle of velocities, and $\overline{Q}$ is the area of Q.

In the study of the statistical properties of dynamical systems it is usually found to be convenient to go over to a system with discrete time. Apart from the fact that this decreases the dimensionality of the system to unity, it simplifies, as a rule, the numerical modeling of the system and the interpretation of the results. For billiards the change to discrete time is implemented in a very natural way.

We shall consider phase-space points $x = (q,\mathbf{v})$ for which q belongs to the boundary ∂Q and the velocity $\mathbf{v}$ points into the region Q. We denote the set of all such points by M_1. It is clear that M_1 is the edge (boundary) of the phase space M. We launch a billiard trajectory from a point $x \in M_1$. We now take on this trajectory the point y corresponding to the moment immediately after its first (after x) reflection from the boundary of the region. If this reflection occurs at a nonsingular point of the boundary the point y is determined uniquely. It is easy to see that the set of trajectories that are at some time incident at singular points of the boundary have a volume (μ) equal to zero, and therefore such trajectories can be disregarded.

Let $\tau(x)$ (where $x = (q,\mathbf{v})$) be the free time of a particle starting with velocity $\mathbf{v}$ from the point q on the boundary ∂Q. Then the point $y = (q_1, \mathbf{v}_1)$ corresponding to the first reflection of the trajectory x from the boundary can be written as $y = S^{\tau(x)+0} x$. It is clear that the point q_1 lies on the boundary ∂Q, and the velocity $\mathbf{v}_1$ points into the region Q. In this way, by denoting $y = Tx$, we have defined a transformation T that takes points of the set M_1 into points of the same set. The transformation T generates a dynamical system with discrete time, the phase space of which is the set M_1. The time in this system, as is easily seen, is equal to the number of reflections of the trajectory from the boundary of the region.[4)]

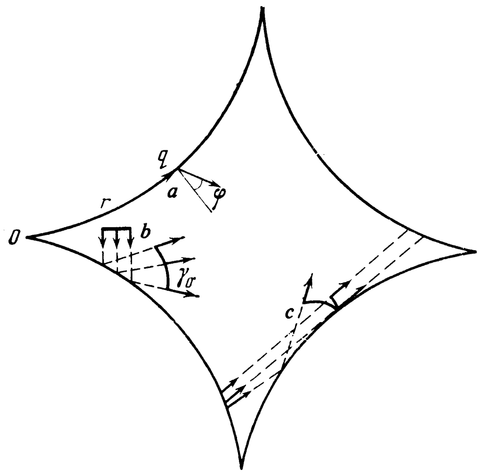

FIG. 1. A scattering billiard: *a*) coordinates in the phase space of a system with discrete time; *b*) reflection of a plane beam from a scattering wall; *c*) breaking of a smooth beam of trajectories in the case of tangency to a wall.

We introduce on M_1 a convenient system of coordinates. Let $x = (q,\mathbf{v})$ be a point of M_1 such that q is a nonsingular point of the boundary ∂Q. We place the point x in correspondence with the three numbers i, r, and φ, where i labels a connected component of the boundary ∂Q, r is the normalized distance along the boundary from any fixed (initial) point on this component along Q, and φ is the angle between the velocity vector $\mathbf{v}$ and the (inward with respect to the region Q) normal to the boundary ∂Q at the point q (Fig. 1). The coordinate i takes a finite number of values (equal to the number of connected components of the boundary), $-\pi/2 \leqslant \varphi \leqslant \pi/2$, and $0 \leqslant r \leqslant 1$.

In these coordinates the phase space M_1 of the system with discrete time is a set of cylinders. Each such cylinder corresponds to one connected component of the boundary ∂Q. For simplicity we assume that ∂Q consists of one connected component—in particular, because this will be the case in almost all the examples considered below.

By projecting the invariant (with respect to the phase flux $\{S^t\}$) measure μ in M on to the boundary M_1 of M, we obtain a certain measure ν in M_1. It is easy to calculate[12] that on each component of M_1 this measure in the coordinates (r, φ) is given by the expression $d\nu = C \cos \varphi dr d\varphi$, where C is a normalization factor chosen in such a way that $\nu(M_1) = 1$. It is not difficult to convince oneself[6] that the measure ν is invariant under a transformation T; i.e., for any (measurable) subset A of the phase space M_1 the equality $\nu(A) = \nu(TA)$ is fulfilled.

3. INTEGRABLE BILLIARDS

Let the region Q be a circle. Then all the segments of any configurational trajectory of the billiard in Q are tangent to the same (for the given trajectory) circle, concentric with ∂Q. Curves of this kind are called caustics. The presence of a continuous family of caustics implies that a billiard in a circle is an integrable system. Its phase space is layered onto Kolmogorov tori[13] (which, in Hamiltonian systems with one degree of freedom, are closed curves). To each torus corresponds the set of all trajectories that are tangent to the same caustic. The additional integral is the angular momentum about the axis perpendicular to the plane of the circle and passing through its center.

Thus, a billiard in a circle is nonergodic. Correspondingly, the correlations of the phase functions for it do not decay. For example, if $f(x) = f(r,\varphi) = \varphi$, then

$$\int_{M_1} f(x) f(T^n x)\, d\nu = \int_{M_1} \varphi^2\, d\nu = \text{const} > 0.$$

Nevertheless, close trajectories lying on different tori diverge in the phase space M_1 at a rate that is linear in the number of reflections.

If the boundary ∂Q is an ellipse, the corresponding billiard is also integrable. It is not difficult to check[6] that in this case there are two continuous families of caustics: The trajectories that do not intersect the segment linking the foci of the ellipse ∂Q are tangent to an ellipse that is confocal with ∂Q, while the trajectories that do intersect this segment are tangent to a hyperbola that is confocal with ∂Q.[5)]

In Ref. 11 it is shown that for a billiard inside a sufficiently smooth convex curve ∂Q there is always a continuous family of caustics close to ∂Q. Here the measure $\mu(\nu)$ of the set of trajectories of the billiard that are tangent to these caustics in the phase space M (M_1) corresponding to a system with continuous (discrete) time is positive. It follows from this that such billiards are nonergodic and the time correlations in them do not decay. However, this does not rule out the possibility that a stochastic component of positive measure can exist in the phase space of a billiard inside a smooth convex curve of general form. At present this question has not been studied at all.

4. SCATTERING (DIVERGING) BILLIARDS (SINAĬ BILLIARDS)

If all the smooth components of the boundary ∂Q are convex inward to the region a billiard in Q is said to be scattering. This class of billiards was introduced by Sinaĭ[12] in connection with an investigation of certain models of nonequilibrium statistical mechanics. At present, scattering billiards are often called Sinaĭ billiards.

In their properties, Sinaĭ billiards are opposite to integrable billiards, in the sense that their trajectories are (locally)[6)] dispersed with an exponential rate. The presence of an instability of this kind in such systems was first noted by Krylov,[14] and an exact formulation and proof have been given by Sinaĭ.[12] The mechanism of local instability of scattering billiards arises from the scattering (diverging) character of the boundary ∂Q.

In fact, if a beam of parallel trajectories is incident on the boundary (Fig. 1), then after reflection from the boundary this beam will become divergent (convex) and will remain so in all subsequent reflections from the boundary (for $t > 0$).[7)]

In the phase space M of the billiard such a beam of trajectories corresponds to a certain convex curve γ_0 (Fig. 1). Let $\varkappa(x_0)$ be the curvature of this curve at the point x_0 and suppose that in the time from 0 to $t > 0$ not one point of this curve has reached the edge ∂M. Then, as is easy to calculate,[12] the curve $\gamma_t = S^t \gamma_0$ at the point $x_t = S^t x_0$ will have the curvature

$$\varkappa(x_t) = \varkappa(x_0) / (1 + t\varkappa(x_0)). \tag{1}$$

Upon reflection from the boundary the curvature of such a curve changes discontinuously, and the formula

$$\varkappa_+(x) = \varkappa_-(x) + 2k(q)/\cos\varphi \tag{2}$$

is valid, where $\varkappa_+(x)$ ($\varkappa_-(x)$) is the curvature of the beam under consideration immediately after (before) reflection of the trajectory of the point x from the edge, q is that point on the boundary ∂Q at which this reflection occurred, $k(q)$ is the curvature of ∂Q at the point q, and φ is the corresponding angle of incidence.

Let γ_0 be a smooth convex curve on M and let $l(\gamma_0)$ be its length. We shall assume that in the time from 0 to t all trajectories corresponding to this curve experience the same number m of reflections, and that these reflections are from the same smooth components of the boundary ∂Q.

We shift γ_0 under the action of the dynamics by the time t. As a result we obtain a convex smooth curve γ_t. From (1) and (2) we obtain (if $l(\gamma_0)$ is small)

$$l(\gamma_t) \geqslant l(\gamma_0)(1 + \tau_1(y)K_1(y)) \dots (1 + \tau_m(y)K_m(y)), \tag{3}$$

where y is an arbitrary point on the curve γ_0, $\tau_y(y)$ ($1 \leqslant i \leqslant m$) is the free time before the ith reflection of its trajectory from the boundary, and the numbers $K_i(y)$ are connected by the recursion relation

$$K_{i+1}(y) = K_i(1 + \tau_i K_i)^{-1} + 2k(q_i)/\cos\varphi_i;$$

$k(q_i)$ is the curvature of ∂Q at the point of the ith reflection, and φ_i is the corresponding angle of incidence.

From (3) it can be seen that if the free time $\tau(x)$ is bounded above and below (by a positive constant) and the boundary ∂Q is strictly convex inward to Q (i.e., its curvature nowhere vanishes), then the length of any convex curve in the phase space of the corresponding billiard grows exponentially in time. It is clear that for a bounded region Q it is impossible to satisfy all these conditions. In fact, if the smooth components of the boundary ∂Q are convex inward to Q, then near the singular points of ∂Q the free time $\tau(x)$ can be arbitrarily small. However, this difficulty can be circumvented by assuming that no two smooth components of the boundary of Q are tangential, i.e., nonzero angles are formed at the intersections. On this case, as is easily verified, in any neighborhood $U(q)$ of a singular point $q \in \partial Q$ a trajectory can have no more than a finite number $n = n(q,U)$ of successive reflections. We shall take nonintersecting neighborhoods of all the singular points of the boundary ∂Q. Let

n_0 be the maximum length of a series of successive reflections inside any of these neighborhoods, and let τ_0 be the minimum free path of a particle outside these neighborhoods. The following bound follows easily from (3):

$$l(\gamma_t) > l(\gamma_0)\left((1+k_{min}\tau_0)^{1/n_0}\right)^{[t/\tau_{max}]}, \quad (4)$$

where $k_{\min}$ is the minimum curvature of the boundary and $\tau_{\max}$ is the maximum free path; $[a]$ denotes the integer part of the number a.

Thus, scattering billiards in regions of this kind possess the property of local exponential instability, from which one can derive ergodicity[12]; i.e., for any ("good") phase function on M (M_1), with probability unity with respect to the measure μ (ν),

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f(S^t x)\,dt = \int_M f(x)\,d\mu,$$
$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} f(T^k x) = \int_{M_1} f(x)\,d\nu. \quad (5)$$

We shall consider for this phase function the time correlation function

$$b_f(\tau, x) = \frac{1}{T}\int_0^T f(x(t))f(x(t+\tau))\,dt - \left(\int_M f(x)\,d\mu\right)^2,$$
$$b_f(m, x) = \frac{1}{n}\sum_{k=0}^{n-1} f(T^k x)f(T^{k+m}x) - \left(\int_{M_1} f(x)\,d\nu\right)^2. \quad (6)$$

By virtue of (5), i.e., because of the ergodicity of the considered class of billiards, at large times $\tau(m)$ the expressions (6) can be replaced, respectively, by

$$b_f(\tau) = \int_M f(S^t x)f(x)\,d\mu - \left(\int_M f(x)\,d\mu\right)^2,$$
$$b_f(m) = \int_{M_1} f(T^m x)f(x)\,d\nu - \left(\int_{M_1} f(x)\,d\nu\right)^2. \quad (7)$$

First we shall consider a simpler question—that of the decay of correlations in dynamical systems with discrete time. The case of continuous time is discussed in the next Section.

In dynamical systems of abstract origin (Anosov systems[8)] and Smale systems[1]) that possess the property of local exponential instability the correlations are damped with an exponential rate,[15] i.e.,

$$|b_f(n)| < C_f e^{-\alpha n}, \quad (8)$$

where the constant α depends on the properties (and parameters) of the system under consideration, and C_f depends on the function f.

This decay law arises from the following mechanism. In conservative systems with local exponential instability one can draw through any typical (i.e., with probability 1 with respect to the measure ν) point x of the phase space a manifold $\gamma^{(c)}(x)$ such that all its points approach each other with an exponential rate under the action of the dynamical system. For a system with n degrees of freedom the dimensionality of $\gamma^{(c)}(x)$ is equal to n, i.e., for $n = 2$ we have a curve in phase space.

Correspondingly, for motion in the time-reversed direction the points of $\gamma^{(c)}(x)$ diverge with an exponential rate. However, since the volume of the phase space M_1 is finite, the curve $T^{-n}\gamma^{(c)}(x)$, with length of the order of const e^n, is arranged in a complicated manner in M_1, and fills M_1 approximately uniformly. The latter implies that any small cube with edge of the order of a constant in the phase space is intersected by the curve $T^{-n}\gamma^{(c)}(x)$ approximately e^n times. In fact, it is this circumstance which makes it possible to obtain the bound (8) for the decay of the correlations in Anosov systems, although the corresponding rigorous proof is rather complicated.[15]

For scattering billiards, despite the presence of local exponential instability, the decay of the correlations turns out to be slower. To be precise, for such billiards the authors of Ref. 16 proved the bound

$$|b_f(n)| < C_f \exp(-\alpha n^\gamma), \quad (9)$$

where $1/3 \leqslant \gamma < 1$. This bound has subsequently been confirmed on several occasions by means of numerical modeling. In particular, for a billiard in the region depicted in Fig. 1, the estimate $\gamma \approx 0.42$ has been obtained in several papers.[17]

The reason for this slower (in comparison with exponential) damping of the correlations is as follows. Dynamical systems with discrete time that are generated by billiards in regions Q whose boundary has at least one segment convex inward to Q are, in contrast to Anosov systems, discontinuous (Fig. 1). It is easy to see that a point x of the phase space M_1 of the billiard belongs to the discontinuity set of the transformation T if the velocity vector corresponding to the point Tx is tangent to the boundary ∂Q. The presence of discontinuities leads to the result that with increase of the time n the curve $T^{-n}\gamma^{(c)}(x)$ not only lengthens but also breaks. At a discontinuity point the direction of this curve is almost exactly reversed and the curve is, as it were, (locally) doubled (Fig. 1). As a result, the rate of filling of the phase space by the curve $T^{-n}\gamma^{(c)}(x)$ is somewhat reduced, and it is this which causes the damping of the correlations in such systems to be not purely exponential. We note that this type of decay of correlations has now also been discovered in certain other models.[18–21]

5. THE PERIODIC LORENTZ GAS

We shall consider a system generated by the inertial motion of an infinite number of noninteracting point particles in the field of an infinite number of stationary spherical particles (scatterers). Upon reaching the boundary of a scatterer a particle is elastically reflected from it. This model, called the Lorentz gas, was introduced by Lorentz in 1905 in order to describe the dynamics of an electron gas in metals.[22] It is also used in problems in the theory of channeling of particles in crystals. In view of the absence of interaction it is sufficient to study the motion of only one point particle.

The Lorentz gas is in the class of Sinaĭ billiards, since

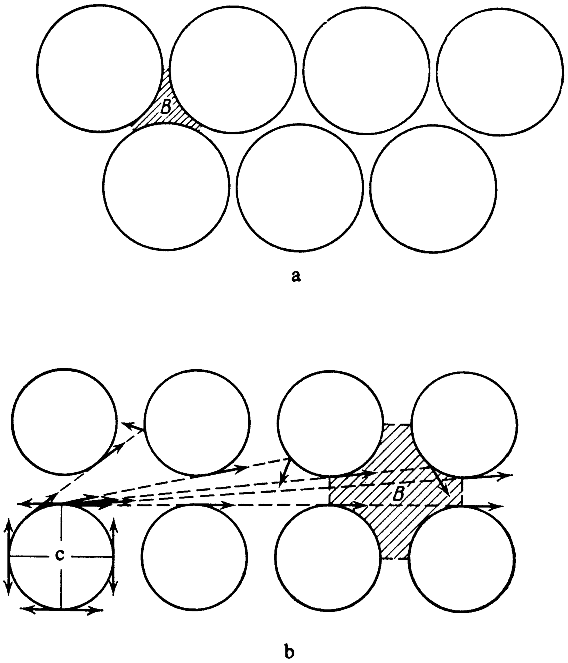

FIG. 2. A periodic Lorentz gas: a) with a bounded free path; b) with an unbounded free path; c) singular trajectories; B is a unit cell of the configuration.

the scatterers are convex. We shall assume that they are arranged periodically and all have the same radius R. By regarding as identical all scatterers that differ by a displacement by a vector of the corresponding lattice, we obtain a scattering billiard with a phase space of finite volume.

The Lorentz gas with a periodic configuration of scatterers displays two essentially different types of behavior, depending on whether the free path (FP) of the particle is bounded or is not bounded (from above). For simplicity and clarity we shall confine ourselves to the two-dimensional case and shall consider two basic configurations of scatterers, when their centers form a triangular and a square lattice on a plane (Fig. 2).

If the scatterers do not intersect, then for the first of these configurations the FP can be either bounded or unbounded. But if the scatterers intersect, a particle is found to be trapped in one of the infinite number of similar regions into which the plane is divided by the scatterers. Thus, this case reduces to that studied in the preceding Section. Therefore, we shall assume that the FP of a particle is bounded from below, i.e., the scatterers do not intersect.

The phase space of the Lorentz gas (with discrete time) consists of an infinite number of cylinders (in the coordinates (r, φ) introduced earlier), each of which corresponds to a fixed scatterer. For the two configurations investigated the transformation T is constructed in the same way on all the cylinders, and therefore as the phase space M_1 we can take one such cylinder.

If the FP is bounded, then on the cylinder M_1 there is only a finite number of discontinuity curves of the transformation T. Thus, for this model all the results of the preceding Section, and, in particular, the bound (9) for the rate of decay of the time correlations, remain valid.

The situation for a periodic Lorentz gas with an unbounded FP turns out to be more complicated. It is easy to see that the FP $\tau(x)$ can be unbounded only in the vicinity of those points if the phase space M_1 which correspond to trajectories in which all reflections are tangent to the boundary ∂Q and, at all the points of tangency, the corresponding scatterers lie on the same side of the trajectory under consideration (Fig. 2b). It is clear that the number of these points is finite. We shall denote them by $\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_p$. (It is easy to see that $p \geqslant 8$.) The entire set of discontinuity curves can be divided into p infinite series of curves, converging to the corresponding points (Fig. 2b). Therefore, a curve that lies in the phase space of a periodic Lorentz gas with an unbounded FP and corresponds to a divergent beam of trajectories can be broken under the action of the transformation T into an arbitrarily large number of parts. Thus, for this model an additional (in comparison with the case of a bounded FP) difficulty is that of investigating how the transformation T is constructed in the neighborhood of the singular trajectories $\hat{x}_i$, $1 < i < p$.

In connection with this we shall make one remark of a general nature. In the investigation of a new class of (more-complicated) systems with chaotic behavior one usually proceeds as follows. One separates out in the phase space of an (arbitrary) system of this class a subset G (called the "good" subset) on which the behavior of this system is analogous to the behavior of simpler (and already investigated) systems, and a "poor" subset $P = M_1 - G$, on which the dynamics of the system is characterized by certain new properties not previously encountered in examples. It is clear that the study of the properties of the system of interest reduces to the investigation of its dynamics on the poor subset and of

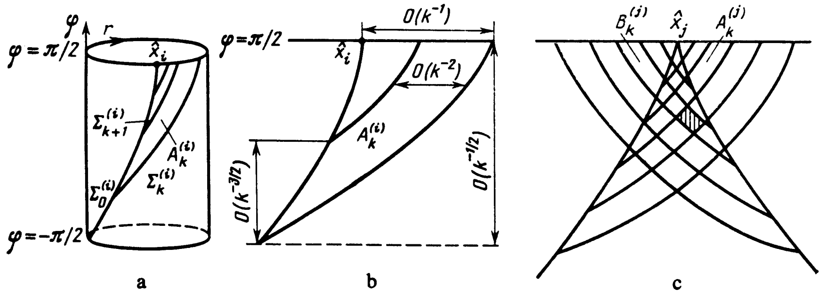

FIG. 3. Discontinuity curves in the neighborhood of the singular trajectories: a) discontinuity curves of the transformation T; b) dimensions of the cell $A_k^{(i)}(B_k^{(i)})$; c) discontinuity curves of T and T^{-1}; the shaded region is $TA_k^{(i)} \cap A_m^{(i)}$.

the character of the transitions from P to G and vice versa.

In particular, the rate of decay of the correlations is determined by the following factors:

1) the measure $\nu(P)$ of the poor set;

2) the probabilities $P_n(P,G)$ that in the time n the system passes for the first time from the poor set to the good set.

We now obtain a bound on the rate of decoupling of correlations for a periodic Lorentz gas with an unbounded FP. We label all curves of the ith series (converging to $\hat{x}_i$) in an infinite sequence $\Sigma_k^{(i)}$, $k = 0, 1, 2, \ldots$. One end of the curve $\Sigma_0^{(i)}$ is $\hat{x}_i$, and its second end belongs to the other edge component of the corresponding cylinder. For each curve $\Sigma_k^{(i)}$, $k = 1, 2, \ldots$, one end lies on the same edge component of this cylinder as the point $\hat{x}_i$, and the other end lies on the curve $\Sigma_0^{(i)}$ (Fig. 3a). The labeling of the set of curves $\Sigma_k^{(i)}$ is such that the curves that are closer to $\hat{x}_i$ have the higher numbers. In the phase space (on the cylinder) we consider the region $A_k^{(i)}$ bounded by the curves $\Sigma_0^{(i)}$, $\Sigma_k^{(i)}$, $\Sigma_{k+1}^{(i)}$, and the edge of the cylinder (Fig. 3b). For all points x belonging to $A_k^{(i)}$ the free time $\tau(x)$ satisfies the inequality (see Fig. 2b)

$$\text{const}\, k < \tau(x) < \text{const}\, k, \tag{10}$$

while for the measure (phase volume) of $A_k^{(i)}$ the following relation holds (see Fig. 3b):

$$\text{const}\, k^{-3} < \nu(A_k^{(i)}) < \text{const}\, k^{-3}. \tag{11}$$

It is easy to see that the constants appearing in (10) and (11) depend only on the configuration of the centers of the scatterers and on their radii.

For the system under consideration it is natural to regard the poor set as the set of cells that have labels larger than some fixed number n_0 and belong to any of the p infinite series. According to (11), the phase volume of such a set is of the order of n_0^{-2}. Thus, we have estimated the first factor in the correlation functions that govern the asymptotic behavior.

We turn now to the estimation of the second factor. We shall consider the transformation T^{-1} inverse to T. For the same reason as for T, the transformation T^{-1} is discontinuous. The corresponding discontinuity curves form, as for T, p infinite series $\Xi_k^{(i)}$, $1 \leqslant i \leqslant p$, $k = 0, 1, 2, \ldots$, that converge to the very same points $\hat{x}_i$, $1 \leqslant i \leqslant p$ (Fig. 2b). In analogy with the preceding case, we introduce into the analysis regions $B_k^{(i)}$ of the phase space M_1 that are bounded by the curves $\Xi_0^{(i)}$, $\Xi_k^{(i)}$, $\Xi_{k+1}^{(i)}$, and the edge of the corresponding cylinder (Fig. 3c). It is not difficult to verify that the mapping T carries each cell $A_k^{(i)}$ into a cell $B_k^{(j)}$ that has the same label k but corresponds to a different singular trajectory $\hat{x}_j$, $i \neq j$. It follows from this (see Fig. 3) that the set $TA_k^{(i)}$ intersects only those cells $A_n^{(j)}$ whose labels satisfy the inequality

$$\text{const}\, k^{1/2} < n < \text{const}\, k^2. \tag{12}$$

Here the probability of a transition (under the action of the transformation T) from $A_k^{(i)}$ to the set of cells A with labels greater than k is of the order of $k^{-1/2}$. In other words, we have the bound

$$\sum_{j=1}^{p} \sum_{n=k}^{\infty} \nu(TA_k^{(i)} \cap A_n^{(j)}) < \text{const}\, k^{-1/2}. \tag{13}$$

We shall now consider how a transition from the poor set to the good set occurs. According to (12), under the action of the transformation T the cell $A_k^{(i)}$, narrowing in one direction and lengthening in another, intersects $\sim k^2$ such cells of another (the jth, $j \neq i$) series. We shall take any one such intersection $TA_k^{(i)} \cap A_m^{(j)}$ and examine what will then happen to it. In analogy with the previous discussion we find (Fig. 3c) that the image $T(TA_k^{(i)} \cap A_m^{(j)})$ is a very narrow band intersecting the cell $\Xi_m^{(j')}$ (with the same label m!) along its entire length from $\Xi_0^{(j')}$ to the edge of the cylinder. Therefore, for each subset $TA_k^{(i)} \cap A_m^{(j)}$ the bounds (12) and (13) also hold (in them we must replace $A_k^{(i)}$ by $TA_k^{(i)} \cap A_m^{(j)}$ and k by m). Analogously, we find that the relations (12) and (13) are valid for all sets of the form $T[(TA_k^{(i)} \cap A_m^{(j)}) \cap A_{m'}^{(j')}]$, $m' > 1$, $1 \leqslant j' \leqslant p$, etc.

From this it is not difficult to derive that for large $n (n > n_0)$ the probability $P(n)$ that trajectories starting from the set

$$\bigcup_{i=1}^{p} \bigcup_{k=n}^{\infty} A_k^{(i)}$$

do not once, in the time $\ln n$, enter the "most good" set

$$M_1 - \bigcup_{j=1}^{p} \bigcup_{k=2}^{\infty} A_k^{(j)}$$

satisfies the inequality

$$P(n) < \delta^n, \tag{14}$$

where $0 < \delta < 1$.

Thus, despite the fact that the poor set has a (power-law) probability n_0^{-2}, a transition from it into the good set occurs (with probability $1 - \delta^{n_0}$) in a very short (logarithmic) time. Therefore, the presence of the poor set does not worsen in this sense the properties of a periodic Lorentz gas with an unbounded FP (with discrete time!), and for the decay of the correlations of its phase functions, as in the case of a bounded FP, the bound (9) holds.

The numerical calculations of Ref. 23 show that when the radius R of the scatterers passes through a value equal to half the period L of the square lattice on the plane the parameter γ changes its value discontinuously from 0.42 (for $R > L/2$) to 0.86 (for $R < L/2$). In both these ranges the value of γ is found to be constant. The increase of γ is explained by the fact that, according to (3), for large free times the rate of dispersal of close trajectories (in the phase space of a system with discrete time) increases. Below we shall show that for the critical value $R = L/2$ the correlations have a power-law decay.

The concepts of a good and a poor set in the phase space of the given system can vary in accordance with which properties we are investigating. We shall consider, e.g., the problem of finding the asymptotic form of the diffusion coefficient (DC) $D = D(R)$ of a periodic Lorentz gas at high $(R \to L/2)$ and low $(R \to 0)$ densities of the scatterers. By virtue of the nature of the interaction, the DC is the only transport coefficient for the Lorentz gas. According to the Einstein formula,[8] we have

$$D=\sum_{n=0}^{\infty}\langle \mathrm{v}(0)\mathrm{v}(n)\rangle_{\mathrm{v}}.$$

It follows from (9) and (14) that the DC exists both for a bounded and for an unbounded free path.

We consider first the high-density limit, i.e., we assume that neighboring scatterers are almost touching (and their centers, as before, form a square lattice). The size of the gap between them will be denoted by l. It is clear that as $l\to 0$ the DC tends to zero, since a particle becomes trapped in a bounded region.

For this problem it is no longer possible to use the theory of scattering billiards,[12] since the free path $\tau(x)$ of a particle in the limit $l\to 0$ is not bounded from below. Therefore, in this case the poor set is the set of trajectories that have long series of reflections in the narrow "mouth" between neighboring scatterers, since the lengthening coefficients (see (3)) in such a series of reflections are of the order of $1+l$. This poor set P in the phase space M_1 lies far from the singular trajectories $\hat{x}_1,\dots\hat{x}_p$ (for small l, as is easily seen, $p=8$), i.e., is far from that set which we regarded as poor when estimating the rate of damping of the correlations. We note that in the phase space of the Lorentz gas (on a cylinder) P has small dimensions in both coordinates.

Let a moving particle start (with a uniform distribution) from the boundary ∂B of any one of the elementary cells into which the plane is divided by the scatterers (Fig. 2). We shall consider the mean square displacement of this particle after n reflections:

$$D_n=\sum_{i=1}^{2}\int_{\partial B}(q^i(T^n x)-q^i(x))^2\,d\mathrm{v}$$

$$=\sum_{i=1}^{2}\Big(n\int_{\partial B}(q^i(Tx)-q^i(x))^2\Big)\,d\mathrm{v}$$

$$+2\sum_{2\leqslant j\leqslant n}\int_{\partial B}(q^i(T^j x)-q^i(T^{j-1}x))(q^i(Tx)-q^i(x))\,d\mathrm{v}.$$

We shall show that

$$\lim_{n\to\infty}D_n/n\sim l. \tag{15}$$

We shall consider a series of successive reflections from the boundaries of two neighboring scatterers. From Fig. 4 we have ($|\alpha_n|<\pi/6$, $|\alpha_{n+1}|<\pi/6$)

$$\varphi_{n+1}=\varphi_n+\alpha_n+\alpha_{n+1},$$
$$\sin\alpha_{n+1}-\sin\alpha_n=[2+l-\cos\alpha_n-\cos\alpha_{n+1}]\,\mathrm{tg}\,(\varphi_n+\alpha_n). \tag{16}$$

From this, by means of elementary transformations, we find that the FP of the particle between the nth and the $(n+1)$th reflection in such a series is equal to

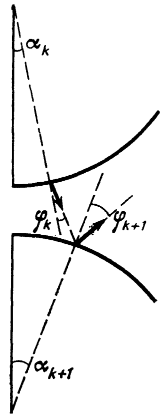

FIG. 4. Segment of a trajectory in the "mouth" between neighboring scatterers.

$$\tau_n=\Big(2\sin^2\frac{\alpha_{n+1}}{2}+2\sin^2\frac{\alpha_n}{2}+l\Big)[\cos(\varphi_{n+1}+\alpha_{n+1})]^{-1}. \tag{17}$$

We note that the probability of trajectories that jump, after one reflection, from the initial elementary cell $B_{0,0}$ to some other cell is of the order of l. At the same time, the set of trajectories that pass, after one reflection, from $B_{0,0}$ to an elementary cell B_{i_1,i_2}, where $|i_1|+|i_2|>1$, is of the order of l^2. We now consider all the trajectories that emerge from close points $x'=(\varphi_1',\alpha_1')$ and $x''=(\varphi_1'',\alpha_1'')$ and make, at the initial time, a small angle. We put

$$T^k x'=(\varphi_k',\ \alpha_k'),\quad T^k x''=(\varphi_k'',\ \alpha_k''),$$
$$\Delta\varphi_k=\varphi_k''-\varphi_k',\quad \Delta\alpha_k=\alpha_k''-\alpha_k'.$$

If n successive reflections of these trajectories have occurred only from one and the same pair of neighboring scatterers, we have from (16)

$$\varphi_{n+1}''-\varphi_{n+1}'=(\varphi_1''-\varphi_1')+(\alpha_1''-\alpha_1')$$
$$+2\sum_{k=2}^{n}(\alpha_k''-\alpha_k')+(\alpha_{n+1}''-\alpha_{n+1}'). \tag{18}$$

We find from the relation (18) that the set of trajectories that remain inside a mouth for a long time, after emerging from the mouth, has in the coordinate φ in the phase space of the Lorentz gas (on a cylinder) a size of the order of a constant. Thus, the trajectories, on emerging from the mouth, forget about their "painful" past spent in the poor set. The required asymptotic relation (15) now follows easily from (9) and (14), since the expression for D_n, which made use of the Einstein formula, can be rewritten in terms of the velocity correlations.

We now consider the low-density limit. For simplicity we assume that the lattice period $L=1$. It is easy to see (Fig. 5) that the mean free path of a particle is of the order of R^{-1}. Therefore, in the expression for the mean square displacement D_n of the position of a particle it is convenient to take the factor R^{-2} outside the integral. With allowance for the given normalization, the average lengthening coefficient in

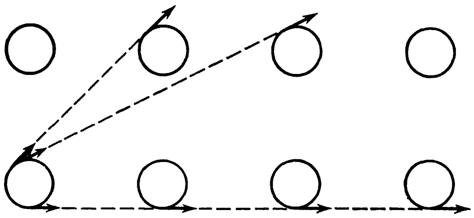

FIG. 5. Appearance of new singular trajectories upon decrease of the radii of the scatterers.

the phase space M_1 becomes of the order of R^{-1} (see (3)).

Furthermore, as $R \to 0$, the number p of singular trajectories grows like R^{-2}. In fact, each such trajectory is determined by a common exterior tangent to a pair of scatterers in the case when the corresponding coordinates of their centers are relatively prime (Fig. 5). At the same time, it is known[24] that the number of numbers that do not exceed N and are relatively prime with N is proportional to N.

It remains to calculate the relation between the measures of regions $A_k^{(i)}$ that have the same label k but belong to different infinite series. All such series can be divided into eight identical groups (enclosed between any coordinate semiaxis and the bisecting line of the corresponding quadrant). In each of these groups $\nu(A_k^{(i)}) \sim i^{-1}\nu(A_k^{(1)})$. It is also not difficult to see that the image $TA_k^{(i)}$ of a cell with label $k \leqslant R^{-1}$ intersects cells of all the series of at least one of the eight identical groups of such series (Fig. 5). In addition, the intersections $TA_k^{(i)} \cap A_m^{(j)}$ of the image of each set with all the series in the group have the same (to within terms of order R^{-2}) measures. Therefore,

$$\lim_{n\to\infty} D_n/nR^{-1} \sim (-R\ln R)^{-1},$$

where the left-hand side has been further divided by R^{-1}, i.e., by a quantity of the order of the mean free path. Thus, the diffusion coefficient $D(R)$ is not an analytic function at the point $R=0$.

6. NOWHERE-SCATTERING BILLIARDS WITH CHAOTIC BEHAVIOR

Let the boundary ∂Q contain at least one component that is convex outward from the region Q. Then upon reflection from this component a parallel beam of rays will become convergent. In this case close trajectories of a billiard not only do not diverge but even have a tendency to come closer together. The examples considered in Sec. 3 also demonstrate the stability of billiards in convex regions. It turns out, however, that there exists a local-instability mechanism that differs from that in the case of scattering (diverging) billiards and leads to stochastization of the dynamics of billiards in a region with a boundary containing focusing components.[25]

This mechanism consists in the following: If we wait long enough, in this time a convergent beam of rays is defocused and becomes divergent. Therefore, if the time τ_c between two successive reflections of this beam from the boundary of the region, during which the beam was convergent, is shorter than the time τ_d during which it was divergent ($\tau_c + \tau_d = \tau$), dispersal of close trajectories will occur in the phase space of the billiard. However, in order that such a situation occur between any two successive reflections from the boundary it is also necessary that after each reflection the beam become strongly focused, since otherwise the condition $\tau_d > \tau_c$ will not be fulfilled.

In analogy with (1) and (2), for a series of successive reflections from a focusing component Γ of the boundary ∂Q we obtain

$$\varkappa_+^{(n)} = \varkappa_-^{(n)} - 2k_n/\cos\varphi_n, \qquad \varkappa_-^{(n+1)} = 1/(\tau^{(n)} + 1/\varkappa_+^{(n)}), \quad (19)$$

where $\varkappa_-^{(n)}$ ($\varkappa_+^{(n)}$) is the curvature of the considered beam of rays at the moment before (after) reflection from Γ, $\tau^{(n)}$ is the FP between the nth and the $(n+1)$st reflections, k_n is the curvature of Γ at the nth reflection, and φ_n is the corresponding angle of incidence;

$$\varkappa_+^{(n)} = (\tau_c^{(n)})^{-1}, \; \varkappa_-^{(n+1)} = (\tau_p^{(n)})^{-1}, \; \tau_c^{(n)} + \tau_p^{(n)} = \tau^{(n)}.$$

It was shown in Refs. 25 and 26 that billiards in regions with boundaries all of whose components are either focusing or neutral possess the property of mixing if [9)]: 1) each focusing component Γ of the boundary ∂Q has a constant curvature; 2) the arc that extends Γ to a complete circle $O_\Gamma \supset \Gamma$ lies strictly inside the region Q. Some examples of such regions are given in Fig. 6.

Thus, such billiards possess chaotic behavior. However, the character of this behavior differs substantially from that in the case of scattering billiards. In Sinaĭ billiards a typical trajectory executes irregular oscillations all the time. In stochastic billiards when the boundary has focusing components the chaos is intermittent. In fact, the trajectory of such a billiard, when undergoing a set of successive reflections from a focusing component of the boundary ∂Q, moves along the surface of a certain Kolmogorov torus (is tangent to the corresponding caustic; see Sec. 3). Then the trajectory breaks away from this torus and for a certain time (up to the next series of reflections from some focusing component of the boundary) executes irregular oscillations, again falls onto a Kolmogorov torus, breaks away from it, and so on. Thus, in such billiards we encounter the possibility of the

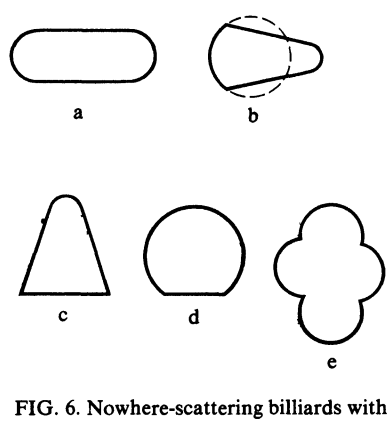

FIG. 6. Nowhere-scattering billiards with chaotic behavior.

existence of dynamical systems that have chaotic behavior and in which the stochasticity is intermittent.

We shall estimate the rate of decay of correlations in such billiards. From (3) and (19) it follows that the poor set in the given case is the set of trajectories A_n (B_n) that have a long series of not fewer than n successive reflections from one and the same focusing component (from a set of neutral components) of the boundary ∂Q.

A trajectory can be reflected for a long time only from neutral components of the boundary if it is close to a periodic trajectory of the billiard in the corresponding polygon, e.g., a trajectory of period equal to two, perpendicular to the parallel sides of a "stadium" (Fig. 6a). Using the expression for the invariant measure ν, we obtain

$$\text{const}\, n^{-2} < \nu(A_n) < \text{const}\, n^{-2}, \quad \text{const}\, n^{-1} < \nu(B_n) < \text{const}\, n^{-1}. \tag{20}$$

We now consider the second factor determining the asymptotic form of the decay of correlations. We note that after a long series of reflections from a focusing component of ∂Q there can begin (after several reflections) a still longer series of reflections from another or from the same focusing component. An analogous situation is also possible for long series of reflections from neutral components of ∂Q.

It follows from the results of Ref. 26 that for the considered class of regions the asymptotic form of the correlations is governed equally by both factors, i.e., by the meausre both of the poor sets A_n and B_n and of the set of trajectories that hop rapidly from one poor set to the other. In particular, for the stadium we have the bound[10)]

$$|b_f(n)| < \text{const}\, n^{-1}, \tag{21}$$

while for a stadium with nonparallel sides (Fig. 6b) and in the absence of neutral components of the boundary we have

$$|b_f(n)| < \text{const}\, n^{-2}. \tag{22}$$

We note that a power-law bound for the damping of correlations in a stadium has also been obtained in numerical experiments with this model.[27]

We turn now to the situation in which the boundary of the region contains both focusing and diverging components. For simplicity we shall assume that there are no neutral components. The mechanisms of local instability of diverging and nowhere-diverging billiards can impede each other in this case. Indeed, divergent beams of rays with large curvature should not be incident on a focusing component Γ of the boundary, since upon reflection from Γ such a beam either remains divergent or becomes convergent, but has no time to be defocused. In Ref. 28 it was shown that when the conditions 1) and 2) are satisfied the time correlations in the corresponding billiards are damped. In this case the bound (22) holds.

7. DECAY OF CORRELATIONS IN SYSTEMS WITH CONTINUOUS TIME

It is well known[6] that when one goes from a dynamical system with discrete time to the corresponding system with continuous time the statistical properties of the system, generally speaking, are changed. The question of what changes can occur in this transition is extremely important from the practical point of view, since in the numerical investigation of dynamical systems with chaotic behavior one usually studies a discrete-time system generated by a Poincaré transformation of some $(d-1)$-dimensional surface M_1 in the d-dimensional phase space M of the system with continuous time. Therefore, it is important to be able to carry the results of the calculation over to the complete system.

Certain rough statistical properties, such as, e.g., ergodicity, are preserved in the transition to a system with continuous time. For the mixing property, however, this is not so. For example, if the time $\tau(x)$ after which a trajectory (of the complete system) emerging from a point $x \in M_1$ returns to M_1 takes not more than a countable number of commensurable values, a discrete component appears in the spectrum of the complete system, and thus the time correlations in the system are not damped.[11)] Therefore, it has always been assumed that in the transition from a discrete system to a continuous system the statistical properties of the system can only be worsened.

It turns out, however, that this is not so. We shall consider, e.g., a stadium with nonparallel sides (Fig. 6b). The correlations of the phase functions of a billiard in this region, as already noted, satisfy the bound (22), i.e., decay with a power-law rate. On the other hand, for the corresponding system with continuous time we have the bound

$$|b_f(t)| < \text{const}\exp(-\alpha_1 t^{\gamma_1}), \tag{23}$$

where $0 < \gamma_1 < 1$. The point is that in the system with continuous time each trajectory having a series of successive reflections from a focusing component of the boundary passes through this series in a finite time (in a system with discrete time, this time can be arbitrarily large).

We obtain another example of such a situation by considering a periodic Lorentz gas at the bifurcation value $R = L/2$. In this case the trajectory of a billiard in an elementary triangular (or quadrangular) cell can have arbitrarily many reflections in a small neighborhood of the corner points of this region. On the other hand, it follows from the relations (16) that in a continuous system the time corresponding to such a series of reflections is finite. Therefore, for the correlation functions of this billiard in the case of continuous time the bound (21) holds, and in the case of discrete time the bound (23) holds. (We note that the decay of the correlations for billiard trajectories in the vicinity of such corners has been investigated qualitatively and numerically in Ref. 30.) The opposite situation obtains for a periodic Lorentz gas with an unbounded FP. In the given model, as follows from (11) and from the expression for the invariant measure μ of a billiard with continuous time, the measure of the set of trajectories that are not once reflected from the boundary ∂Q in the time t is of the order of t^{-1}. Therefore, for this system in the case of continuous time we have the bound.

$$|b_f(t)| < \text{const}\, t^{-1}, \tag{24}$$

and in the case of discrete time, the bound (9). Thus, although the given dynamical system possesses chaotic behavior, i.e., its motion is random, a diffusion coefficient for it does not exist.

In all the remaining examples, considered above, of billiards with chaotic behavior the character of the decay of the correlations is the same for discrete and continuous time. Such will be the case, in particular, for the stadium, for the periodic Lorentz gas with a bounded FP, for scattering billiards (if the smooth components of the boundary ∂Q are not tangent to each other), if at least one focusing component of the boundary ∂Q is greater than a semicircle, etc. It is clear that the asymptotic forms of the correlations functions coincide if the length of a segment of any trajectory in discrete and in continuous time is of the same order i.e., $C_1 n < t < C_2 n$, where n is the number of reflections that the given trajectory has in the time t, and the constants C_1 and C_2 depend only on the geometry of the region Q.

We now consider a random perturbation of the configuration of the scatterers in a periodic Lorentz gas with an unbounded MFP. Then, if this perturbation is so small that certain trajectories that are not once reflected from the scatterers are preserved, their neighborhoods make a contribution of the order of n^{-2} to the asymptotic form of the correlations. We note that precisely this type of decay of correlations for a Lorentz gas with a random configuration of scatterers has been discovered in numerical experiments with this model.[8]

8. CONCLUSION

The examples considered show that the asymptotic form of correlation functions in dynamical systems with chaotic behavior depends on extremely fine properties of these systems and can be not only exponential but also quasiexponential and power-law. Moreover, purely exponential damping of correlations in model physical systems is evidently exotic.

In this connection, we note that a typical Hamiltonian system, as is well known,[31] is neither integrable nor stochastic, and its phase space contains both regions in which the motion is chaotic and "islands of stability" that consist (mainly) of Kolmogorov tori. Both types of region are invariant under the dynamics, i.e., any trajectory is contained as a whole in only one of them. Such systems are customarily called systems with a separated phase space. An important advance in the study of such systems has been made by Chirikov and coworkers.[32] In systems with a separated phase space there is always an additional factor that slows the decay of the correlations, viz., the sticking of a trajectory near the boundaries of islands of stability.[32] Therefore, it is customary to assume that in such systems the correlations are always damped with a power-law rate.

We shall give an example that shows that this is not so. We shall consider a billiard in a region such that all the components of its boundary are scattering and one component is part of an ellipse and contains the ends of the semiminor axis of the ellipse (Fig. 7). The phase space of a billiard in this region contains an island of stability corresponding to a neighborhood of this periodic trajectory. It can be shown

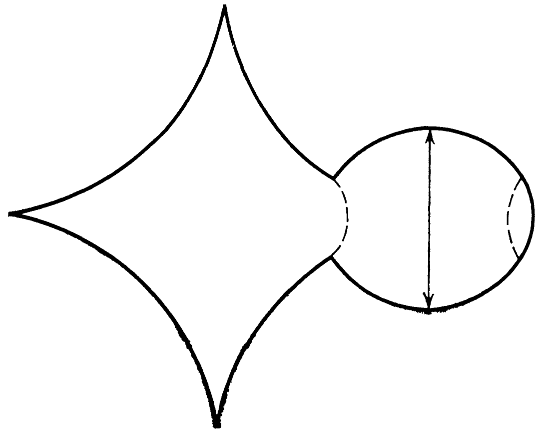

FIG. 7. Billiard with a separated phase space: ↔ is a stable periodic trajectory; ---- is an "island of stability."

that if the eccentricity of this ellipse is small and the ellipse as a whole lies inside the region, then the bound (23) holds, i.e., the time correlations of the billiard in Q decay quasiexponentially. The same is true for the case in which there are several nonintersecting islands of stability. But if inside the islands there lie other islands of stability corresponding to higher resonances, the correlations evidently always decay with a power-law rate.

We note that a slow (power-law) decay of correlations might be regarded from a practical point of view as a complete absence of decay of correlations in the system. However, as shown by the examples given, this is not so. In the phase space there is always a good subset on which mixing occurs rapidly. Here the probability of this good subset is large. But the asymptotic form of the correlations (on average) is determined by the (relatively small) measure of the poor subset, in which the initial conditions are forgotten slowly. Therefore, for most initial conditions the decay occurs rapidly at first, and as soon as the corresponding trajectories begin to sense that there is a poor set the rate of this decay decreases. Precisely this situation is observed in practically all work devoted to the numerical investigation of dynamical systems with chaotic behavior.

As shown by the results of the present paper, a numerical investigation of the asymptotic form of the correlation functions in such systems must necessarily be preceded by a theoretical (qualitative) investigation, the purpose of which is to find the correct analytical expression for this asymptotic form. The numerical-modeling problem is to determine the values of the parameters appearing in this asymptotic form. Indeed, in the study of such a subtle property as the rate of decay of correlations one can practically never be certain that enough calculations have been performed to obtain the correct answer (at least to the question: Is it a quasiexponential or power-law decay?)

Finally, we note that the examples given can also serve as models for dissipative dynamical systems with chaotic behavior. In fact, the asymptotic form of the correlations in such systems is determined by the dynamics on their (strange) attractors. Each stochastic attractor necessarily has its own invariant measure. But if we consider a dynamical system only on an attractor (with this measure), this system, from the point of view of ergodic theory, is in no way different from a conservative system. In particular, for the correlations of the phase functions on a Lorentz attractor, we have the bound (9) (Ref. 33).

The author is grateful to P. V. Sasorov for useful comments that made it possible to improve the account.

[1)]Even of specially prepared "very" stochastic systems one can find phase functions with slowly decaying correlations.[7]

[2)]Of course, one can consider billiards in a space of any number of dimensions, but for clarity we confine ourselves to the two-dimensional case.

[3)]To avoid pathologies that are of no fundamental importance for us we shall assume that each connected component of the boundary ∂Q consists of a finite number of triply continuously differentiable curves. The points of intersection of these curves are the singular points of the boundary.

[4)]There are situations in which certain trajectories are not once reflected from the boundary. However, in all physically meaningful examples the phase volume of the set of such trajectories is equal to zero.

[5)]Here it is possible that it is not the broken-line segment itself that is tangent to the hyperbola, but the straight line containing this segment.

[6)]Since the volume of the phase space $M(M_1)$ is finite, initially close trajectories can move apart only until the distance between them becomes of the order of the spatial scale of $M(M_1)$.

[7)]Throughout, we consider only local (narrow) beams of rays.

[8)]This class also contains certain models of classical mechanics—the so-called geodesic flows on manifolds of negative curvature.[13]

[9)]The condition 1) can be weakened by permitting the curvature of the focusing components to vary slowly. If Γ intersects only components of ∂Q with zero curvature, it is sufficient that the center of O_Γ lie inside Q (Fig. 6b).

[10)]The bound (21) also holds if upon continuation of the rectilinear sides of the boundary ∂Q one obtains a polygon with commensurable angles.

[11)]The conditions under which the mixing property is preserved in the transition to continuous time have been studied in Ref. 29.

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