Classification of degeneracies and analysis of their stability in the theory of elastic waves in crystals

V. I. Al'shits, A. V. Sarychev, and A. L. Shuvalov

Institute of Crystallography, Academy of Sciences of the USSR

(Submitted 26 December 1984)
Zh. Eksp. Teor. Fiz. **89**, 922–938 (September 1985)

The phase velocity degeneracies of elastic waves in crystals are classified according to the type of geometry of the velocity sheet contacts and the nature of the singularities of the polarization vector fields near the degenerate points. An invariant method is proposed for determining the local geometry of the velocity sheets near a degeneracy and the index of the corresponding singular point of the polarization field which does not require solving the wave equation. The behavior of several types of degeneracies is investigated for small perturbations of the elastic tensor of the crystal.

INTRODUCTION

Three bulk elastic modes can propagate along any given wave vector **m** in an unbounded isotropic medium; in general, these modes have different phase and group velocities v_α and s_α, and their polarization vectors $\mathbf{A}_\alpha$ ($\alpha = 1,2,3$) are mutually orthogonal. In the linear theory of elasticity,[1] the squares of the phase velocities and the polarization vectors of the three isonormal waves are the eigenvalues and eigenvectors of the Christoffel tensor $\hat{\mathbf{A}}(\mathbf{m}) = \mathbf{m}\,\hat{c}\mathbf{m}/\rho$, where $\hat{c}$ is the elastic modulus tensor and ρ is the density of the crystal. As **m** varies over the unit sphere $\mathbf{m}^2 = 1$, the functions $\mathbf{v}_\alpha = \mathbf{m}\,v_\alpha(\mathbf{m})$ and $s_\alpha(\mathbf{m})$ describe surfaces which consist of three sheets and are called the phase and group velocity surfaces, respectively (or the velocity and ray surfaces).[1] It is often convenient to consider another characteristic surface—the refraction surface $\mathbf{w}_\alpha = \mathbf{m}/v_\alpha(\mathbf{m})$—in addition to the velocity surface.

The acoustic axes are defined as the directions $\mathbf{m}_0$ along which the phase velocities $v_{0\alpha} \equiv v_\alpha(\mathbf{m}_0)$ coincide (are degenerate) for at least two of the modes. Clearly, the velocity sheets must be in contact with one another along these directions. We will henceforth assume that the degeneracies are two-fold (except where explicitly stated in the concluding section of this paper) and will use the subscript $\alpha = 3$ to describe the nondegenerate branch: $v_{01} = v_{02} \neq v_{03}$, moreover, we take $v_1 \leqslant v_2$ for every **m**. Acoustic axes corresponding to triple degeneracy can exist only if the elastic moduli of the crystal satisfy special conditions; we will show below that they are unstable, i.e., vanish under small perturbations of the tensor $\hat{c}$.

Because of the phase velocity degeneracy, elastic waves propagating along or near the acoustic axis have some unusual properties. For instance, the polarization vectors for degenerate waves can be arbitrary in the plane normal to the polarization of the third nondegenerate wave. The vector-valued functions $\mathbf{A}_{1,2}(\mathbf{m})$ are generally singular near the degenerate points. For certain velocity sheet configurations, a wave normal along an acoustic axis may correspond to several different group velocity vectors, so that internal conical refraction occurs.[1–3]

Degeneracy is also important in the phonon kinetics. Under suitable conditions, transitions between states near degenerate points for the constant-frequency surfaces of the phonon spectrum may give the dominant contribution to such processes as phonon absorption of sound, phonon drag on electrons in semiconductors, and dielectric relaxation.[4–8] The temperature behavior and other physical properties of these effects depend strongly on the geometry of the contact region of the constant-frequency surfaces. Several types of contact geometry were first examined in Ref. 9 for the constant-energy surfaces of electrons and phonons in crystals.

Much work[1–3,10–15] has been done on the properties of elastic waves propagating near acoustic axes. The latter may differ both in the contact geometry of the velocity sheets and in the nature of the singularities in $\mathbf{A}_{1,2}(\mathbf{m})$ near a degenerate point. In this paper we classify the acoustic axes in crystals and find an invariant method for determining the local geometry of the velocity sheets near a degeneracy and the index of the polarization field at the singular point in terms of the tensor $\hat{c}$ and acoustic axis $\mathbf{m}_0$. We also discuss the stability of the degeneracies under perturbations in $\hat{c}$.

LOCAL GEOMETRY OF DEGENERATE VELOCITY AND REFRACTION SHEETS

The directions $\mathbf{m}_0$ of the acoustic axes can be found in the standard way from a knowledge of the tensor $\hat{c}$ (Refs. 10, 12). The invariant degeneracy condition stated in Ref. 12 is equivalent to the requirement that the 7-component vector ξ should vanish:

$$\xi(\mathbf{m},\ \hat{c}) = 0. \tag{1}$$

The degenerate velocity sheets can touch in various ways (at a point or along a curve), may be joined along a cone, or may intersect along a curve (Fig. 1). Any method for determining the contact geometry of the degenerate sheets without solving the wave equation will be of fundamental interest.

It will be more convenient to analyze the local geometry of the refraction surface, which is obtained from the velocity sheet by replacing $v_\alpha(\mathbf{m})$ by $1/v_\alpha(\mathbf{m})$. The normal to the refraction sheet $\mathbf{w}_\alpha(\mathbf{m})$ is parallel to the group velocity vector $\mathbf{s}_\alpha(\mathbf{m})$ everywhere[1]:

$$\mathbf{s}_\alpha(\mathbf{m}) = \mathbf{A}_\alpha(\mathbf{m})\,\hat{c}\mathbf{A}_\alpha(\mathbf{m})\,\mathbf{m}/\rho v_\alpha \tag{2}$$

 0038-5646/85/090531-09$04.00

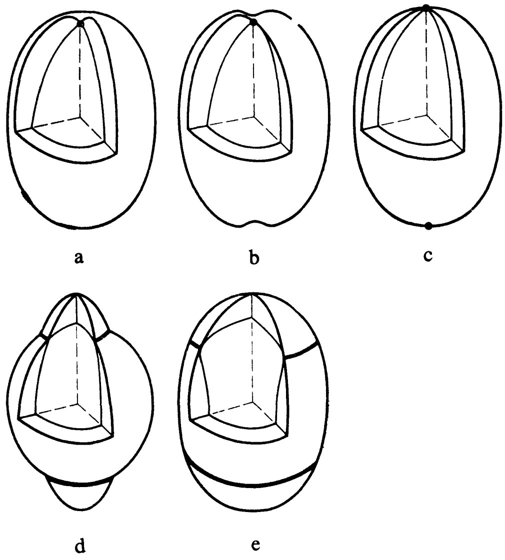

FIG. 1. Contact geometry: conical (a), local wedge (b), and tangential (c) degeneracies at a point; wedge (d) and tangential (e) degeneracies along a line. The degenerate points and lines are indicated. In case b one of the two cross sections shown is defined by the condition that $\Delta\mathbf{m}$ be parallel to $\mathbf{p}\times\mathbf{m}_0$; in this section the curves $\mathbf{w}_{1,2}$ are smooth.

except possibly at the degenerate points, at which the normal need not be defined. Equation (2) relates the sheet geometry to the polarization vector field.

We consider a neighborhood of a degenerate point $\mathbf{m}_0$ at which the normal to the refraction surface is uniquely defined. Since the direction of the polarization vector of the nondegenerate branch is nearly constant near $\mathbf{m}_0$: $\mathbf{A}_3(\mathbf{m}) \approx \mathbf{A}_3(\mathbf{m}_0) \equiv \mathbf{A}_{03}$, for $|\Delta\mathbf{m}| \ll 1$ ($\Delta\mathbf{m} = \mathbf{m} - \mathbf{m}_0$) we can approximate $\mathbf{A}_{1,2}(\mathbf{m})$ by their projections $\mathbf{a}_{1,2}(\mathbf{m})$ on the plane normal to $\mathbf{A}_{03}$:

$$\begin{aligned}\mathbf{a}_1(\mathbf{m}) &= \mathbf{A}_{01}\cos\Phi + \mathbf{A}_{02}\sin\Phi,\\ \mathbf{a}_2(\mathbf{m}) &= -\mathbf{A}_{01}\sin\Phi + \mathbf{A}_{02}\cos\Phi.\end{aligned} \tag{3}$$

Here $\mathbf{A}_{01}$, $\mathbf{A}_{02}$ is an arbitrary pair of unit vectors forming a right-handed orthogonal frame with $\mathbf{A}_{03}$. The angle Φ in (3) depends on $\mathbf{m}$, and the rotation of the polarization field near the degenerate point is determined by the change $2\pi n$ in Φ when we follow a small closed path Γ around the point $\mathbf{m}_0$ on the sphere $\mathbf{m}^2 = 1$ (the integer n is called the Poincaré index of the singularity). The following definition will be useful: the quantity $\gamma(\varphi) = [\Phi(\varphi) - \Phi(0)]/2\pi$ is called the incomplete rotation of the field a_i when the vector $\Delta\mathbf{m}$ rotates by the angle φ along the path Γ. In this terminology, $n = \gamma(2\pi)$.

Strictly speaking, the functions $\mathbf{A}_\alpha(\mathbf{m})$ specify fields of undirected segments (called "line fields" in mathematics) rather than vectors. In fact, polarization vectors that differ only in sign are physically equivalent,[1)] just as in the case, e.g., of the orientation of the director in nematic liquid crystals. When we follow Γ around the point $\mathbf{m}$, a line pointing along $\mathbf{A}_\alpha(\mathbf{m})$ will return to its original position but may point in the opposite direction (it will flip by an integral number of half-rotations). The index n of a singular point of the polarization field may therefore be half-integral, in contrast to classical vector fields, for which n is always an integer.

Substituting the vectors $\mathbf{a}_i$ (3) for $\mathbf{A}_\alpha$ (2), we find that

$$\mathbf{s}_{1,2} = \mathbf{s}_0 \pm (\mathbf{p}\cos 2\Phi + \mathbf{q}\sin 2\Phi) \tag{4}$$

to lowest order in $\Delta\mathbf{m}$, where we have written

$$\mathbf{s}_0 = \hat{P}_+\mathbf{m}_0, \quad \mathbf{p} = \hat{P}_-\mathbf{m}_0, \quad \mathbf{q} = \hat{Q}_{12}\mathbf{m}_0; \tag{5}$$

$$\begin{aligned}&\hat{P}_\pm = \hat{S}_{11} \pm \hat{S}_{22}, \quad \hat{Q}_{\alpha\beta} = \hat{S}_{\alpha\beta} + \hat{S}_{\beta\alpha},\\ &\hat{S}_{\alpha\beta} = \mathbf{A}_{0\alpha}\hat{c}\mathbf{A}_{0\beta}/2\rho v_{01}.\end{aligned} \tag{6}$$

In general the vectors $\mathbf{p}$ and $\mathbf{q}$ are linearly independent, i.e.,

$$\varkappa = [\mathbf{p}\mathbf{q}] \neq 0, \tag{7}$$

where $[\mathbf{p}\mathbf{q}]$ denotes the vector product. The vectors $\mathbf{s}_i(\mathbf{m})$ in (4) are parallel to the generators of the internal refraction cone,[1,2] whose elliptical base is orthogonal to $\mathbf{m}_0$ ($\mathbf{p}\mathbf{m}_0 = \mathbf{q}\mathbf{m}_0 = 0$) with semiaxes $u_{1,2}$:

$$2\mathbf{u}_{1,2}^2 = \mathbf{p}^2 + \mathbf{q}^2 \pm [(\mathbf{p}^2+\mathbf{q}^2)^2 - 4\varkappa^2]^{1/2}. \tag{8}$$

We will show below that when (7) holds, the angle Φ in (3) takes on values from 0 to $\pm\pi$, i.e., the index of the degenerate point is $n = \pm 1/2$. The vectors $\mathbf{s}_i$ thus make a complete circuit around a surface that coincides with the internal refraction cone in the limit $|\Delta\mathbf{m}| \to 0$, i.e., as Γ shrinks to the point $\mathbf{m}_0$.

The local contact geometry of the degenerate sheets is determined by the geometric locus of points formed by the family of tangents $\mathbf{l}_i$ to all sections of the refraction sheets cut by planes passing through the acoustic axis $\mathbf{m}_0$ (these tangents pass through the point $\mathbf{m}_0$ in the limit $|\Delta\mathbf{m}| \to 0$):

$$\mathbf{l}_i \| [\mathbf{s}_i, \mathbf{m}_0, \Delta\mathbf{m}] = \mathbf{m}_0\Delta v_i - \Delta\mathbf{m}\, v_{01}, \tag{9}$$

where $\mathbf{s}_i\,\Delta\mathbf{m} = \Delta v_i$, $\mathbf{s}_0\mathbf{m}_0 = v_{01}$. The corresponding surface $\{\mathbf{l}_i\}$ swept out by the tangents $\mathbf{l}_i$ as $\Delta\mathbf{m}$ makes a complete circuit around the path Γ approximates the degenerate sheets to first order in $\Delta\mathbf{m}$.

One can show [see Eq. (20)] that near the acoustic axis (7)

$$\Delta v_{1,2} \approx \mathbf{s}_0\Delta\mathbf{m} \mp [(\mathbf{p}\Delta\mathbf{m})^2 + (\mathbf{q}\Delta\mathbf{m})^2]^{1/2}. \tag{10}$$

Substituting (10) into (9), we find that in this case the surface $\{\mathbf{l}_i\}$ is a cone which is straightforwardly related to the internal refraction cone. A degenerate point $\mathbf{m}_0$ for which $\varkappa \neq 0$ will therefore be said to be of the conical type (Fig. 1a).

We next consider the case when $\varkappa = 0$ but $\mathbf{p}$ and $\mathbf{q}$ do not vanish simultaneously. For definiteness we choose the vectors $\mathbf{A}_{01}$, $\mathbf{A}_{02}$ so that $\mathbf{p}$ in (5) does not vanish; then

$$\mathbf{p} \neq 0, \quad \mathbf{q} = \eta\mathbf{p}. \tag{11}$$

The vectors $\mathbf{s}_i$ defined by (4) are then coplanar as Φ varies and lie in a plane whose normal $\mathbf{N}$ is parallel to $\mathbf{s}_0 \times \mathbf{p}$. If $\mathbf{m}_0$ is an isolated degenerate point, then the two-dimensional fan generated by the vectors $\mathbf{s}_i$ (4) corresponds to a limiting configuration of the field of normals; as $\mathbf{m}$ varies along Γ, this field forms a cone which becomes flattened along one of the semiaxes as $\mathbf{m}$ tends to $\mathbf{m}_0$. Substituting (11) into (10), we find

$$\Delta v_{1,2} \approx \mathbf{s}_0\Delta\mathbf{m} \mp |\mathbf{p}\Delta\mathbf{m}|(1+\eta^2)^{1/2}. \tag{12}$$

The sheaf of planes passing through the acoustic axes $\mathbf{m}_0$ cuts out various cross sections of the degenerate sheets; Eq. (12) implies that the section defined by the vector

$$\Delta\mathbf{m}\|[\mathbf{m}_0\mathbf{p}], \tag{13}$$

is the only one for which the tangents (9) satisfy

$$\mathbf{l}_i(\Delta\mathbf{m})=-\mathbf{l}_i(-\Delta\mathbf{m})\|[\mathbf{s}_0\mathbf{p}]. \tag{14}$$

This means that the curves $\mathbf{w}_{1,2}$ in section (13) are smooth at $\mathbf{m}_0$ (in the other sections, the tangents to $\mathbf{w}_{1,2}$ are discontinuous at $\mathbf{m}_0$), and the surface $\{\mathbf{l}_i\}$ is a wedge whose edge $\mathbf{N}$ is parallel to $\mathbf{s}_i\times\mathbf{p}$ and orthogonal to the plane of the fan $\mathbf{s}_i$ (4). An isolated degeneracy of this type will be called a wedge (Fig. 1b).

When (11) holds, the smooth curves $\mathbf{w}_{1,2}$ in the section (13) (which of course meet at the point $\mathbf{m}_0$) may actually coincide completely, so that the refraction sheets meet along a curve as shown in Fig. 1d. We call this type of degeneracy a wedge degeneracy along a line. It can be regarded as the intersection of two sheets that are smooth near $\mathbf{m}_0$; the normals $\mathbf{s}_i$ to these sheets and the polarization vectors vary continuously across the degeneracy line. On the other hand, the normals to the outer and inner sheets and the polarization vectors of the corresponding wave branches are of course discontinuous. Because the isonormal wave modes are orthogonal, the orientation of the polarization vectors in each of the branches $v_1(\mathbf{m})\leqslant v_2(\mathbf{m})$ differ by $\pi/2$ on the two edges of the wedge degeneracy line[2] (see Fig. 2k below). At points right on the degeneracy curve, the possible directions of the degenerate-wave polarization vectors are determined by the two-dimensional fan of the vectors $\mathbf{s}_i(\mathbf{m}_0)$, which is orthogonal to $\mathbf{N}$ and bounded by the geometric normals to the smooth refraction sheets.

Finally, both $\mathbf{p}$ and $\mathbf{q}$ may vanish,

$$\mathbf{p}=\mathbf{q}=0. \tag{15}$$

According to (4), the field of normals $\mathbf{s}_i(\mathbf{m})$ then tends to the single orientations $\mathbf{s}_1(\mathbf{m}_0)=\mathbf{s}_2(\mathbf{m}_0)=\mathbf{s}_0$ as $\mathbf{m}\to\mathbf{m}_0$ which is independent of the direction of $\Delta\mathbf{m}$. This implies that for arbitrary rotations of the vectors $\mathbf{A}_\alpha$ near a degenerate point $\mathbf{m}_0$ at which $\mathbf{p}$ and $\mathbf{q}$ both vanish, the vector $\mathbf{s}_0$ given by Eq. (2) must be invariant, i.e., it must be independent of how $\mathbf{m}$ approaches $\mathbf{m}_0$. The degenerate sheets at $\mathbf{m}_0$ thus intersect each other smoothly, i.e., the degeneracy is of the tangential type (either isolated or along a curve, cf. Fig. 1c, e).

The degeneracy condition (1) and (7), (11), (15) thus provide an invariant method for determining the directions of the acoustic axes $\mathbf{m}_0$ and the contact geometry of degenerate refraction or velocity sheets (conical, wedge-point or wedge-line, tangent-point or tangent-line[3]). We stress that the calculations require only a knowledge of the tensor $\hat{c}$—it is not necessary to solve the Christoffel equation. The orientations of the polarizations $\mathbf{A}_{0\alpha}$ needed to find $\mathbf{p}$ and $\mathbf{q}$ are given by the familiar equations (see, e.g., Ref. 1).

SINGULARITIES OF POLARIZATION FIELDS NEAR VARIOUS TYPES OF DEGENERATE POINTS

The above classification of degeneracies in terms of the contact geometry of the refraction (or velocity) sheets is clearly exhaustive. However, classifications based on criteria other than the contact geometry are also possible. In particular, a more diverse classification is obtained by analyzing the rotation of the polarization fields around degenerate points. We will see that each geometric type corresponds to

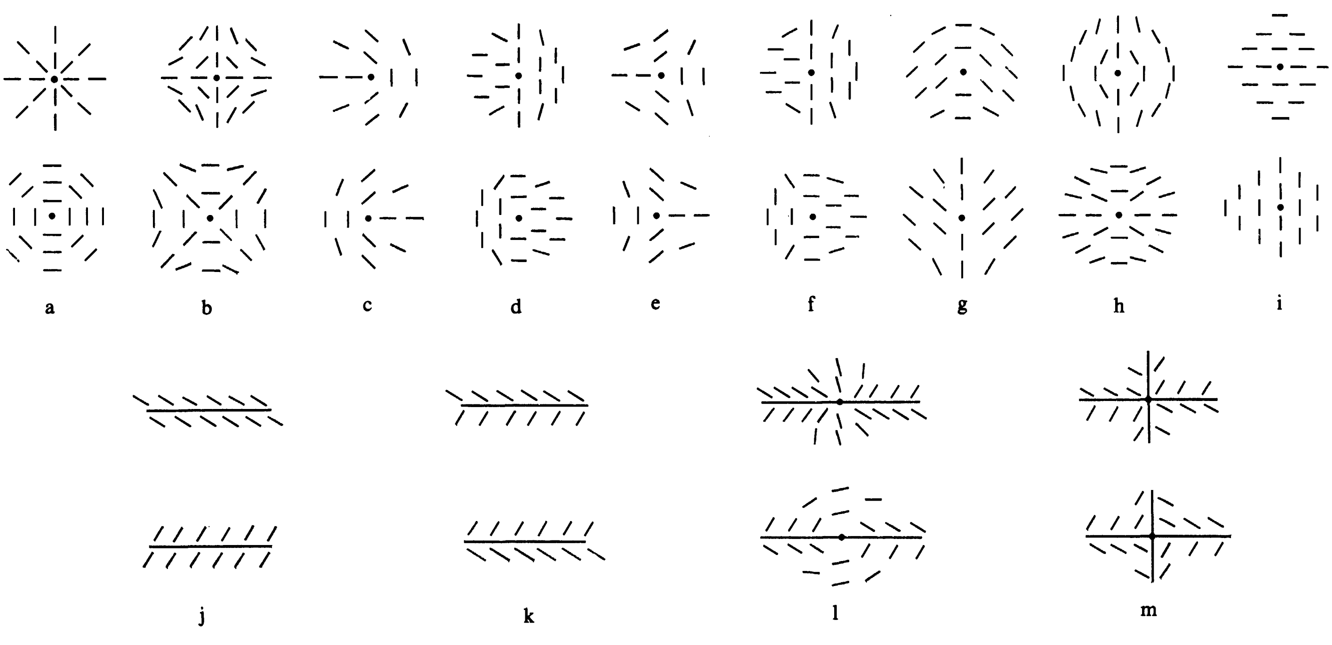

FIG. 2. The polarization fields $\mathbf{A}_i$ ($i=1,2$) near various degeneracies: $a-i$), isolated degeneracies with index a) $n=1$; b) $n=-1$; c, d) $n=1/2$; f, e) $n=-1/2$; g, h) $n=0$, $\gamma(\varphi)\not\equiv0$; i) $n=0$, $\gamma(\varphi)\equiv0$; j) line of tangential degeneracy; k) line of wedge degeneracy; l), superposition of a tangential degeneracy point on a wedge degeneracy line; m) tangential degeneracy at the intersection of two lines of wedge degeneracy.

several types of singular behavior of the polarization field near a singular point.

We have already observed that the index of a singular point of the polarization field may be integral or half-integral. We take the index to be positive (negative) if the direction of rotation of the vectors $\mathbf{a}_i$ is the same (opposite to) the direction along which the path Γ is traversed. The direction of the rotation is determined by observing the rotation planes of $\mathbf{a}_i$ and $\Delta\mathbf{m}_0$ as seen from the ends of the vectors $\mathbf{A}_{03}$ and $\mathbf{m}_0$. For definiteness, we take $\mathbf{A}_{03}$ to be oriented so that it makes an acute angle with $\mathbf{m}_0$. Clearly, the index can be defined uniquely in all cases except when the nondegenerate wave is purely transverse ($\mathbf{A}_{03}\perp\mathbf{m}_0$).

The Christoffel equation near a degenerate point $\mathbf{m}_0$ has the form[1]

$$\hat{\Lambda}\mathbf{A}=v^2\mathbf{A}. \qquad (16)$$

To second order in perturbation theory, the components of the vectors $\mathbf{a}_i$ (3) and the phase velocity are determined by the system of equations

$$M_{\alpha\beta}a_{i\beta}=\Delta(v_i^2)a_{i\alpha};\qquad \alpha,\ \beta=1,\ 2, \qquad (17)$$

where repeated Greek subscripts are understood to be summed, $\Delta(v_i^2)=v_i^2(\mathbf{m})-v_{01}^2$,

$$M_{\alpha\beta}=2v_{01}\left\{\left[\mathbf{m}_0\hat{Q}_{\alpha\beta}\Delta\mathbf{m}+\Delta\mathbf{m}\left(\hat{S}_{\alpha\beta}+\frac{2v_{01}}{v_{01}{}^2-v_{03}{}^2}\mathbf{q}_\alpha\cdot\mathbf{q}_\beta\right)\Delta\mathbf{m}\right]\right\}, \qquad (18)$$

$$\mathbf{q}_\alpha=\hat{Q}_{\alpha3}\mathbf{m}_0, \qquad (19)$$

and $\mathbf{q}_\alpha\cdot\mathbf{q}_\beta$ denotes a dyadic product. System (17) has the solution

$$2\Delta(v_{1,2}^2)=M_{11}+M_{22}\mp[(M_{11}-M_{22})^2+4M_{12}{}^2]^{1/2}, \qquad (20)$$

$$\mathbf{a}_{1,2}\|\{2M_{12},\ M_{22}-M_{11}\mp[(M_{11}-M_{22})^2+4M_{12}{}^2]^{1/2}\}. \qquad (21)$$

[Recall that Eq. (20) implies (10) to first order in $\Delta\mathbf{m}$.] In order to calculate the index n we must find the rotation γ of the $\mathbf{a}_i$ (21) along a path Γ around the point $\mathbf{m}_0$ (it is obvious that the rotations of the mutually perpendicular vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ coincide). However, it is easier first to reduce the problem to calculating the rotation of another vector which can be expressed more easily in terms of the components $M_{\alpha\beta}$, for instance the vector

$$\boldsymbol{\mu}\|(M_{11}-M_{22},\ 2M_{12}). \qquad (22)$$

It is easy to show that the rotation of μ is twice that for $\mathbf{a}_i$ (21):

$$2d\Phi=d\psi, \qquad (23)$$

where Φ and Ψ are the angles between $\mathbf{A}_{01}$ and $\mathbf{a}_1$, $\boldsymbol{\mu}$, respectively. Thus, n is equal to one-half the index of the singular point $\mathbf{m}_0$ for the vector field

$$\boldsymbol{\mu}=(\mu_1,\ \mu_2)\|(\mathbf{p}\Delta\mathbf{m}+\Delta\mathbf{m}\hat{F}\Delta\mathbf{m}/2,\ \mathbf{q}\Delta\mathbf{m}+\Delta\mathbf{m}\hat{G}\Delta\mathbf{m}/2), \qquad (24)$$

where

$$\hat{F}=\hat{P}_-+2v_{01}(\mathbf{q}_1\cdot\mathbf{q}_1-\mathbf{q}_2\cdot\mathbf{q}_2)/(v_{01}{}^2-v_{03}{}^2),$$
$$\hat{G}=\hat{Q}_{12}+2v_{01}(\mathbf{q}_1\cdot\mathbf{q}_2+\mathbf{q}_2\cdot\mathbf{q}_1)/(v_{01}{}^2-v_{03}{}^2). \qquad (25)$$

The rotation $\gamma_\mu(\varphi)$ of the vector field (24) can be calculated from the Poincaré formula

$$\gamma_\mu(\varphi)=\frac{1}{2\pi}\int_0^\varphi\frac{\mu_1(\varphi')d\mu_2(\varphi')-\mu_2(\varphi')d\mu_1(\varphi')}{\mu_1{}^2+\mu_2{}^2}. \qquad (26)$$

Alternatively, one can follow the prescriptions and methods formulated in Ref. 17. We omit the tedious analysis and simply state the results.

A. $\varkappa\equiv[\mathbf{pq}]\neq0$

We have seen that when this condition holds, $\mathbf{m}_0$ is a singularity of the conical type. The index of $\mathbf{m}_0$ in the corresponding polarization field $\mathbf{a}_i$ can only take the two values $n=\pm1/2$ (Fig. 2c, e), where the sign of n is determined by the formula[4)]

$$n={}^1/_2\ \mathrm{sign}\ (\varkappa\mathbf{m}_0). \qquad (27)$$

It is also easy to see that near a conical degenerate point we have

$$\mathbf{a}_i(\Delta\mathbf{m})\perp\mathbf{a}_i(-\Delta\mathbf{m}) \qquad (28)$$

to first order in $\Delta\mathbf{m}$.

B. $\varkappa=0,\quad \mathbf{p}\neq0,\quad \mathbf{q}=\eta\mathbf{p}$

In this case we have a wedge-point or wedge-line degeneracy. An analysis of the polarization field singularities near $\mathbf{m}_0$ shows that three different cases can occur.

a) $g\neq\eta f$, where

$$f=\mathbf{L}\hat{F}\mathbf{L},\quad g=\mathbf{L}\hat{G}\mathbf{L},\quad \mathbf{L}\|[\mathbf{pm}_0]. \qquad (29)$$

The degeneracy is then locally a wedge with index

$$n=0. \qquad (30)$$

One can show that the incomplete rotation does not vanish identically: $\gamma(\varphi)\not\equiv0$, i.e., the field is singular at $\mathbf{m}_0$ (Fig. 2g).

b) $g=\eta f, f\neq0$. The degeneracy is again a local wedge, but with index $\pm1/2$ (Fig. 2d, e) in accordance with the formula

$$n={}^1/_2\,\mathrm{sign}[f\mathbf{p}(\hat{G}-\eta\hat{F})\mathbf{L}]. \qquad (31)$$

Let $\mathbf{A}'$ and $\mathbf{A}''$ be the pair of orthogonal vectors defined by the condition $\mathbf{q}=0$: $(\mathbf{A}'\hat{c}\mathbf{A}''+\mathbf{A}''\hat{c}\mathbf{A}')\mathbf{m}_0=0$. Also, let φ be the angle between the vectors $\Delta\mathbf{m}$ and $\mathbf{L}$. In both cases (30) and (31), the vectors $\mathbf{a}_i$ coincide with $\mathbf{A}'$, $\mathbf{A}''$ to first order in $\Delta\mathbf{m}$ for arbitrary $\varphi\gg|\Delta\mathbf{m}|$. They therefore change direction discontinuously by $\pi/2$ upon passage through the point $\mathbf{m}_0$ (Fig. 2d, f, g). The rotation of the polarizations $\mathbf{a}_i$ thus occurs in a small neighborhood of $\mathbf{L}$ defined by $\varphi\lesssim|\Delta\mathbf{m}|\ll1$. Through second order in $\Delta\mathbf{m}$ we then have

$$\mathbf{a}_i(\Delta\mathbf{m})\|\mathbf{a}_i(-\Delta\mathbf{m}) \qquad (32)$$

in the particular section for which $\Delta\mathbf{m}\|\mathbf{L}$; moreover, $g=0$, i.e., $\eta=0$, with respect to the basis consisting of the vectors $\mathbf{a}_i|_\mathbf{L}=\mathbf{A}_{0i}$. We note that the orientations of $\mathbf{A}'$, $\mathbf{A}''$, and $\mathbf{a}_i|_\mathbf{L}$ coincide if $n=\pm1/2$ (Fig. 2d, f).

c) $f=g=0$. According to (20), in this case the functions $v_{1,2}(\mathbf{m}_0+\Delta\mathbf{m})$ coincide to second order in $\Delta\mathbf{m}$ along the curve χ whose tangent $\Delta\mathbf{m}$ at $\mathbf{m}_0$ is parallel to $\mathbf{L}$. In other words, the requirement $\mathbf{q}=\eta\,\mathbf{p}$ in condition B above is necessary for a wedge-line degeneracy.[5)] We have already found that the polarization vectors change direction discontinuously by $\pi/2$ when the wedge degeneracy curve is crossed (Fig. 2k).

C. $\varkappa=0, \quad \mathbf{p}=\mathbf{q}=0$

According to the results in the previous section, in this case the velocity sheets are degenerate either at an isolated point or along a curve. As in the case of wedge degeneracy, additional information can be gained by examining the singularities of the polarization fields. We consider the matrices $\hat{F}$ and $\hat{G}$ in an arbitrary basis with X_3 axis parallel to $\mathbf{m}_0$ and write

$$\mathbf{F}=(F_{11}, F_{12}, F_{22}), \quad \mathbf{G}=(G_{11}, G_{12}, G_{22}), \tag{33}$$

$$\delta=[\mathbf{FG}];$$

$$D_1=\begin{vmatrix} F_{11} & F_{12} \\ F_{12} & F_{22} \end{vmatrix}, \quad D_2=\begin{vmatrix} F_{11} & F_{12} \\ G_{11} & G_{12} \end{vmatrix}, \tag{34}$$

$$D_3=\begin{vmatrix} F_{12} & F_{22} \\ G_{12} & G_{22} \end{vmatrix}, \quad D_4=\begin{vmatrix} F_{11} & F_{22} \\ G_{11} & G_{22} \end{vmatrix};$$

$$D=4D_2D_3-D_4^2. \tag{35}$$

The following six cases need to be distinguished for condition C.

a) $\delta\neq0, D>0$. The degeneracy is then isolated (a true tangent vector exists[12]) and the index $n=\pm1$ (Fig. 2a, b) is given by

$$n=\operatorname{sign} D_2; \tag{36}$$

b) $\delta\neq0, D<0$. In this case we have an isolated degeneracy (hybrid tangent[14]) with index

$$n=0, \quad \gamma(\varphi)\not\equiv0. \tag{37}$$

As in condition B ($\varkappa=0$), the incomplete rotation $\gamma(\varphi)$ does not vanish identically, so that the field $\mathbf{a}_i$ is singular near $\mathbf{m}_0$ (Fig. 2h).

c) $\delta\neq0, D=0$. This is a necessary condition for a tangential degeneracy to lie on a line of wedge degeneracy. By examining the behavior of the polarization field along a small path Γ around the point $\mathbf{m}_0$ on the sphere $\mathbf{m}^2=1$ which crosses a line of degeneracy χ, one can show that the vectors $\mathbf{a}_i$ rotate continuously by $\pi/2$ on the arc of Γ joining points on the same edge of the curve χ. (If the rotation angle φ of the vectors $\Delta\mathbf{m}$ along Γ is measured from one of the edges of the degeneracy line, we find that $\lim_{\varepsilon\to0}\gamma(\pi-\varepsilon)=\pi/2$ for the incomplete rotation of the polarization field.) As already noted, there is an additional jump of $\pi/2$ in the orientation of $\mathbf{a}_i$ when a line of wedge degeneracy is crossed (Fig. 2l).

d) $\delta=0, D_1>0$. In this case the degeneracy is isolated (sporadic tangent[12]) and

$$n=0, \quad \gamma(\varphi)\equiv0. \tag{38}$$

The associated polarization field $\mathbf{a}_i(\mathbf{m})$ is nonsingular near $\mathbf{m}_0$ (Fig. 2i).

e) $\delta=0, D_1=0$. This condition is necessary for degeneracy along a line passing through the point $\mathbf{m}_0$ for sheets that just graze each other. The polarization fields are nonsingular along this line, (cf. Fig. 2j).

f) $\delta=0, D_1<0$. This is a necessary condition for: 1) tangential degeneracy along one or two lines passing through $\mathbf{m}_0$; 2) wedge degeneracy along two lines that intersect at $\mathbf{m}_0$, where the two sheets are tangent at $\mathbf{m}_0$ (Fig. 2m).

For all cases of tangential degeneracy at $\mathbf{m}_0$, the orientation of the vectors $\mathbf{a}_i$ near $\mathbf{m}_0$ satisfies condition (32) for all sufficiently small $\Delta\mathbf{m}$. We may summarize our results in Table I, which lists the types of acoustic axes.

STABILITY OF THE ACOUSTIC AXES

If the elastic modulus tensor $\hat{c}$ is slightly perturbed by an amount $\Delta\hat{c}$, a degeneracy may move, split into several degeneracies, or disappear. The above results can be used to predict which situation will arise for each type of degeneracy.

We first argue simply as follows. Without loss of generality we may assume that the relative magnitude $|\Delta\hat{c}/\hat{c}|$ of the perturbation is small compared to the average radius $R_\Gamma\approx|\Delta\mathbf{m}|$ of the path Γ on the sphere $\mathbf{m}^2=1$, so that $|\Delta\hat{c}/\hat{c}|\ll R_\Gamma\ll1$. Such a perturbation clearly will not appreciably alter the distribution of the polarization field $\mathbf{a}_i$ along Γ and

TABLE I. Classification of acoustic axes in terms of the contact geometry of the velocity sheets and the singularities of the polarization field near the degeneracy.

Dimension and coordinates of singularity	Geometric type	Algebraic conditions		Index and rotation of the polarization field
Degenerate at the point $\mathbf{m}_0$, $\xi(\mathbf{m}_0)=0$	Conical	$\varkappa\neq0$		$n={}^1/_2\operatorname{sign}(\varkappa\mathbf{m}_0)$
	Local wedge	$\varkappa=0, \mathbf{p}\neq0, \mathbf{q}=\eta\mathbf{p}$	$g\neq\eta f$	$n=0, \gamma(\varphi)\not\equiv0$
			$g=\eta f, f\neq0$	$n={}^1/_2\operatorname{sign}\{f\mathbf{p}(\hat{G}-\eta\hat{F})[\mathbf{pm}_0]\}$
	Tangential	$\varkappa=0, \mathbf{p}=\mathbf{q}=0$	$\delta\neq0, D>0$	$n=\operatorname{sign} D_2$
			$\delta\neq0, D<0$	$n=0, \gamma(\varphi)\not\equiv0$
			$\delta=0, D_1>0$	$n=0, \gamma(\varphi)\equiv0$
Degenerate along a line χ passing through $\mathbf{m}_0$, $\xi\equiv0$ on χ	Wedge line	$\varkappa=0;\ \mathbf{p}\neq0, \mathbf{q}=\eta\mathbf{p};\ f=g=0$		Direction of the polarization vectors changes abruptly by $\pi/2$ upon crossing the line
	Tangent line	$\varkappa=0, \mathbf{p}=\mathbf{q}=0$	$\delta=0, D_1=0$	No singularities, $\lim\gamma(\pi-\varepsilon)=\pi/2$
	Sheets tangent at a point $\mathbf{m}_0$ lying on a wedge line		$\delta\neq0, D=0$	
	One or two tangent lines		$\delta=0, D_1<0$	No singularities
	Tangential degeneracy $\mathbf{m}_0$ at the point where two wedge lines intersect			Direction of the polarization vectors changes abruptly by $\pi/2$ upon crossing the line

will therefore leave the index n unchanged (recall that n takes discrete half-integral values and can only change by a multiple of $\pm 1/2$). It follows that if the degeneracy splits, the sum of the indices of the new degeneracies must be equal to the index of the initial degeneracy, and a degeneracy can vanish only if $n = 0$.

We now consider the stability problem in more detail by analyzing the equation

$$\xi(\mathbf{m}_0+\Delta\mathbf{m},\quad \hat{c}+\Delta\hat{c})=0, \tag{39}$$

which follows from (1) and determines the coordinates of the perturbed acoustic axes. If the basis vectors $\mathbf{A}_{0\alpha}$ are chosen so that the matrix $\hat{\Lambda}(\mathbf{m}_0) = \mathbf{m}_0\,\hat{c}\mathbf{m}_0/\rho$ is diagonal, then to first order in $\Delta\mathbf{m}$ and $\Delta\hat{c}$ condition (39) reduces to the system

$$2\Delta\mathbf{m}\mathbf{p}+d_1=0,\quad 2\Delta\mathbf{m}\mathbf{q}+d_2=0, \tag{40}$$

where

$$d_1=\mathbf{m}_0(\mathbf{A}_{01}\Delta\hat{c}\mathbf{A}_{01}-\mathbf{A}_{02}\Delta\hat{c}\mathbf{A}_{02})\mathbf{m}_0/2\rho v_{01};$$

$$d_2=\mathbf{m}_0\mathbf{A}_{01}\Delta\hat{c}\mathbf{A}_{02}\mathbf{m}_0/\rho v_{01}. \tag{41}$$

We have assumed in (41) that the matrix components $\lambda_{\alpha\beta} = \mathbf{m}_0\mathbf{A}_{0\alpha}\,\Delta\hat{c}\mathbf{A}_{0\beta}\mathbf{m}_0/\rho\, v_{01}^2$ are nonzero and that $|\lambda_{\alpha\beta}| \gg \lambda_{\alpha\beta}^2$. The perturbations $\Delta\hat{c}$ considered below will be assumed to satisfy this condition. In the special cases when this condition fails, the form of Eqs. (40) remains the same but the expressions (41) for $d_{1,2}$ may no longer be correct.

Assume first that the acoustic axis is of the conical type, i.e., that $\varkappa\neq 0$. Then the vectors $\mathbf{p}$ and $\mathbf{q}$ are noncollinear and system (40) has one and only one solution, which determines the displacement $\Delta\mathbf{m}$ of the acoustic axis. To lowest order,

$$\Delta m_1=(d_2p_2-d_1q_2)/2\varkappa\mathbf{m}_0,$$
$$\Delta m_2=(d_1q_1-d_2p_1)/2\varkappa\mathbf{m}_0,\quad \Delta m_3=-(\Delta m_1^2+\Delta m_2^2)/2, \tag{42}$$

where Δm_3 is the displacement along the direction $\mathbf{m}_0$. This implies that a conical acoustic axis cannot split (and therefore of course cannot vanish)—for any sufficiently small perturbation $\Delta\hat{c}$, only the position of a conical degeneracy point $\mathbf{m}_0$ can change. Such degeneracies are said to be stable. The above discussion provides a quantitative basis for the qualitative interpretation in Ref. 12 of conical degeneracy points as points where the two lines defined by Eq. (1) intersect on the sphere $\mathbf{m}^2 = 1$.

Now let the unperturbed acoustic axis be a local wedge of index $n = 0$. Then $\mathbf{q} = \eta\,\mathbf{p}$ but $g\neq\eta f$, and we must retain terms quadratic in $\Delta\mathbf{m}$ in the left-hand side of (39). We get

$$2\Delta\mathbf{m}\mathbf{p}+\Delta\mathbf{m}\hat{F}\Delta\mathbf{m}+d_1=0,\quad 2\Delta\mathbf{m}\mathbf{q}+\Delta\mathbf{m}\hat{G}\Delta\mathbf{m}+d_2=0. \tag{43}$$

If we decompose $\Delta\mathbf{m}$ in terms of its projections on the orthogonal unit vectors $\mathbf{L}$, $\mathbf{M}$, and $\mathbf{m}_0$:

$$\Delta\mathbf{m}=\Delta m_1\mathbf{L}+\Delta m_2\mathbf{M}+\Delta m_3\mathbf{m}_0, \tag{44}$$

where $\mathbf{L} = \mathbf{p}\times\mathbf{m}_0/|\mathbf{p}|$ and $\mathbf{M} = \mathbf{p}/|\mathbf{p}|$, Eqs. (43) and (44) with (11) yield

$$(\Delta m_1)^2(g-\eta f)=\eta d_1-d_2. \tag{45}$$

Clearly, (45) has no solution if

$$(\eta d_1-d_2)/(g-\eta f)<0 \tag{46}$$

while for

$$(\eta d_1-d_2)/(g-\eta f)>0 \tag{47}$$

there are two solutions:

$$\Delta m_1=\pm[(\eta d_1-d_2)/(g-\eta f)]^{1/2}. \tag{48}$$

In the latter case, we readily find the expression

$$\Delta m_2=(d_2f-d_1g)/2|\mathbf{p}|(g-\eta f) \tag{49}$$

for the second coordinate of $\Delta\mathbf{m}$ from (43), (45).

If the unperturbed local-wedge acoustic axis has index $n = \pm 1/2$ then $\mathbf{q} = \eta\,\mathbf{p}$ and $g = \eta f$, and we must consider terms $\sim|\Delta\mathbf{m}|^3$ in (39). Using (44), we obtain

$$\begin{aligned}&2\Delta m_2|\mathbf{p}|+(\Delta m_1)^2f+2\Delta m_1\Delta m_2\mathbf{L}\hat{F}\mathbf{M}\\&\qquad+K_1(\Delta m_1^3)+d_1=0,\\&2\Delta m_2|\mathbf{q}|+(\Delta m_1)^2g+2\Delta m_1\Delta m_2\mathbf{L}\hat{G}\mathbf{M}\\&\qquad+K_2(\Delta m_1^3)+d_2=0,\end{aligned} \tag{50}$$

where we omit the elaborate explicit expressions for K_1 and K_2. Equations (50) lead to the cubic equation

$$K(\Delta m_1)^3-2\mathbf{L}(\hat{G}-\eta\hat{F})\mathbf{M}d_1\Delta m_1/|\mathbf{p}|+d_2-\eta d_1=0 \tag{51}$$

for Δm_1, where $K\equiv K_2-\eta\,K_1$. It is easy to see that for a general perturbation, for which

$$d_2-\eta d_1\sim d_1, \tag{52}$$

Eq. (51) has the unique solution

$$\Delta m_1=[(\eta d_1-d_2)/K]^{1/3}. \tag{53}$$

Moreover, (50) implies that

$$\Delta m_2=-f(\Delta m_1)^2/2|\mathbf{p}|. \tag{54}$$

However, if the perturbation does not satisfy (52) (specifically, if $|\eta\, d_1 - d_2| \lesssim d_1^{3/2}$) then Eq. (51) may have either two or three roots (in the former case, one of the roots is unstable).

The behavior of local wedge degeneracies under a perturbation thus depends on the index n. If $n = 0$, the degeneracy will disappear if (46) holds and will split into two degeneracies if (47) is satisfied. Since in general the vectors $\mathbf{p}$ and $\mathbf{q}$ corresponding to these degeneracies are noncollinear, the index conservation rule implies that the splitting must produce a pair of conical acoustic axes with indices $+1/2$ and $-1/2$. On the other hand, a local wedge degeneracy with $n = \pm 1/2$ is in general stable and remains conical. However, for special types of perturbations $\Delta\hat{c}$, such a degeneracy may split into three conical degeneracies or into a pair of degeneracies, one of which is unstable. If the tensor $\hat{c}$ is perturbed further, the degeneracy may split into a conical pair or else disappear, and consequently its index $n = 0$.

We next examine the behavior of tangential acoustic axes when the tensor $\hat{c}$ is perturbed. In this case $\mathbf{p} = \mathbf{q} = 0$ and condition (39) is approximated by the system

$$\Delta\mathbf{m}\hat{F}\Delta\mathbf{m}+d_1=0,\quad \Delta\mathbf{m}\hat{G}\Delta\mathbf{m}+d_2=0. \tag{55}$$

We take an arbitrary basis with X_3 parallel to $\mathbf{m}_0$ and consider the matrices $\hat{F}$ and $\hat{G}$; in addition to (34), we will write

$$D_{11}=\begin{vmatrix}F_{11} & G_{11}\\ d_1 & d_2\end{vmatrix},\quad D_{12}=\begin{vmatrix}F_{12} & G_{12}\\ d_1 & d_2\end{vmatrix},\quad D_{22}=\begin{vmatrix}F_{22} & G_{22}\\ d_1 & d_2\end{vmatrix}. \tag{56}$$

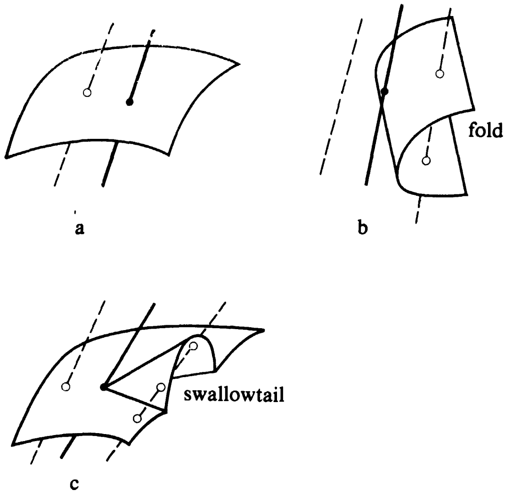

FIG. 3. Intersections of the line LE_2 with the surface Ω for conical (a) and local-wedge degeneracies ($n = 0$ and $\pm 1/2$ for b and c, respectively). The heavy lines and points show the initial position, while the dashed lines and open circles show the positions after the perturbation.

The system (55) has the solutions

$$\Delta m_2/\Delta m_1 = \operatorname{tg} \varphi = [-D_{12} \pm (D_{12}{}^2 - D_{11} D_{22})^{1/2}]/D_{22},$$
$$|\Delta \mathbf{m}|^2 = d_1 (1 + \operatorname{tg}^2 \varphi)/(F_{11} + 2F_{12} \operatorname{tg} \varphi + F_{22} \operatorname{tg}^2 \varphi). \quad (57)$$

The number of real roots of (55) determines the behavior of the various types of tangential degeneracies under perturbations of $\hat{c}$ and can be found by further analysis. A true tangential degeneracy ($n = \pm 1$) splits into a conical pair with equal indices $n = \pm 1/2$. A hybrid tangential degeneracy ($n = 0$, $\gamma(\varphi) \not\equiv 0$) disappears if

$$\operatorname{sign} D_{11} = -\operatorname{sign} D_4, \quad \operatorname{sign} D_{22} = \operatorname{sign} D_4 \quad (58)$$

and splits into four conical degeneracies if

$$\operatorname{sign} D_{11} = \operatorname{sign} D_4, \quad \operatorname{sign} D_{22} = -\operatorname{sign} D_4. \quad (59)$$

Finally, if

$$\operatorname{sign} D_{11} = \operatorname{sign} D_{22} \quad (60)$$

then the degeneracy will either disappear or else split up into four conical degeneracies, depending on the specific nature of $\Delta \hat{c}$. When (59) is satisfied, the degeneracy may also split into a pair of tangential degeneracies (for which we must have $n = 0$, since when $\hat{c}$ is further perturbed they either disappear or else split simultaneously into a pair of conical degeneracies). A generic perturbation will cause a sporadic degeneracy ($n = 0$, $\gamma(\varphi) \equiv 0$) to disappear; however, splitting may also occur for special perturbations $\Delta \hat{c}$.

Finally, one can show that lines of degeneracy are unstable. Indeed, according to Ref. 12 at least five of the seven equations specified by (1) are satisfied identically along every acoustic axis. The degenerate points are thus determined by a system of two equations and can be regarded as the points of intersection of two lines χ_1, χ_2 on the sphere $\mathbf{m}^2 = 1$. A degeneracy line χ will result if χ_1 and χ_2 happen to coincide. Such a line χ is clearly unstable, and a generic perturbation $\Delta \hat{c}$ will remove all degenerate points on a line of wedge or tangential degeneracies with the exception of a few isolated points on or near the line. One sees readily that the same behavior occurs if one of the equations holds identically for all $\mathbf{m}$, i.e., for all points on the sphere $\mathbf{m}^2 = 1$ rather than merely on a line χ_1.

If a tangential acoustic axis $\mathbf{m}_0$ lies on a wedge line, there will be no degenerate points near $\mathbf{m}_0$ if (58) is satisfied, whereas the degeneracy will split into a conical pair if (59) holds [either of these cases can occur if (60) is satisfied]. We note that because the index is conserved by continuous perturbations only if the degeneracy is isolated, the sign of n for conical degeneracies may be arbitrary.

The stability of the acoustic axes can also be treated from the viewpoint of the theory of singularities of smooth mappings.[18] In this case, the problem for two-fold degeneracies reduces to analyzing a certain mapping Ψ of the sphere $\mathbf{m}^2 = 1$ into the three-dimensional space $SM(2)$ of symmetric 2×2 matrices. The image of Ψ is a two-dimensional surface Ω in $SM(2)$; each degeneracy is mapped to a point on Ω which lies on the line LE_2 consisting of matrices of the form $\sigma \hat{I}$, where $-\infty < \sigma < \infty$ and $\hat{I}$ is the unit 2×2 matrix. One can show that an acoustic axis $\mathbf{m}_0$ will be stable if the mapping φ is transverse to LE_2, which is equivalent to requiring that the point $\mathbf{m}_0$ be regular for the composite mapping $\pi \circ \Psi$, where π is the projection on the plane normal to LE_2. Conical degeneracy points have this property (Fig. 3a). On the other hand, local-wedge degeneracies with $n = 0$ correspond to "fold" singularities of $\pi \circ \Psi$ (Fig. 3b), while for $n = \pm 1/2$ Ω has a "swallowtail" degeneracy (Fig. 3c).[18,19] The analysis of the singularities of these mappings for perturbations $\hat{c} \to \hat{c} + \Delta \hat{c}$ reduces to solving equations of the form (45) and (51), respectively. Finally, tangential degeneracies can be studied by analyzing the Porteus invariants[19] of the pair of quadratic forms $\Delta \mathbf{m} \hat{F} \Delta \mathbf{m}$, $\Delta \mathbf{m} \hat{G} \Delta \mathbf{m}$.

CONCLUSIONS

Acoustic axes may be present in crystals of all symmetry classes. Furthermore, no crystals without acoustic axes have been discovered to date, although in principle they could exist (degeneracy of elastic waves would not occur in such crystals).[11,12,16] The maximum number of degeneracies possible for tensors $\hat{c}$ of different symmetry was considered in Ref. 10. The formulas presented in Table I above make it easy to determine which types of acoustic axes can occur along axes of various symmetry. Tangential degeneracies, which always occur along C_∞ or C_4 axes (Refs. 3, 4, 10, 11), correspond to indices $n = 1$ and $n = \pm 1$, respectively, and in the latter case

$$\operatorname{sign} n = \operatorname{sign}[(\Delta_{16}\Delta_{34} - d_{13}{}^2)(d_{12}\Delta_{34} - d_{13}{}^2) - 4c_{16}{}^2 \Delta_{34}{}^2],$$

where

$$\Delta_{\alpha\beta} \equiv c_{\alpha\alpha} - c_{\beta\beta}, \quad d_{12} \equiv c_{12} + c_{66}, \quad d_{13} = c_{13} + c_{55}$$

(clearly, sign n = sign d_{12} for cubic crystals). Conical degeneracies with $n = -1/2$ occur along a third-order axis.[3,4,10,11] Both tangential degeneracies and (if the longitudinal and transverse branches are degenerate) local-wedge degeneracies can occur along a two-fold axis. The symmetry of acoustic axes lying in a plane of symmetry may be arbitrary, except that conical degeneracy of the quasitransverse and quasilongitudinal branches is ruled out. The index of conical degeneracies $\mathbf{m}_0$ in a symmetry plane is given by

$$n=\tfrac{1}{2}\,\mathrm{sign}\{[c_{16}{}^3+\Delta_{15}{}^2c_{26}+d_{12}c_{16}\Delta_{15}-c_{45}(c_{16}{}^2+\Delta_{15}{}^3)]\times[c_{16}d_{13}-\Delta_{15}(c_{36}+c_{45})]\}$$

in a coordinate system with X_1 parallel to $\mathbf{m}_0$.

In triclinic crystals with a generic tensor $\hat{c}$, the acoustic axes must clearly be of the conical type in general. However, since the symmetry decreases the number of independent components of $\hat{c}$, unstable types of degeneracies inevitably occur along certain high-order axes. In addition to the tangential degeneracies along C_4 and C_∞ axes noted above, a line of wedge degeneracy may be present in hexagonal crystals if[1,5,10]

$$0<\Delta_{46}\Delta_{16}/[\Delta_{16}(\Delta_{46}-\Delta_{34})+d_{13}{}^2]<1$$

(this occurs, e.g., in ice, magnesium, and quartz crystals). Although such acoustic axes are unstable, they disappear or split only for perturbations $\Delta\hat{c}$ that violate the symmetry of the degenerate axis $\mathbf{m}_0$ in a specific way.

In other cases, for which the symmetry does not permit the existence of unstable degeneracies, such degeneracies can still arise if some of the elastic constants happen to coincide, as may occur, e.g., due to the critical behavior of the elastic moduli near a phase transition or when driving external fields are present.[6)] More commonly, the elastic constants may be nearly but not exactly equal. In this case, stable acoustic axes (or if they are absent, the close proximity of the velocity sheets) may be regarded as resulting from a small perturbation $\Delta\hat{c}$ of an unstable degeneracy at a point $\mathbf{m}_0$. We stress that in this case the distribution of the polarization vectors at distances $\Delta\mathbf{m}$ from $\mathbf{m}_0$ ($\Delta\hat{c}/\hat{c}_0\ll|\Delta\mathbf{m}|\ll 1$) is almost identical to the configuration of the polarization field for the "unperturbed" unstable degeneracy.

We now pause briefly to discuss the case of three-fold acoustic axes along which the phase velocities of all three of the isonormal modes are equal. In this case all three of the functions $\mathbf{A}_\alpha(\hat{m})$ are singular near $\mathbf{m}_0$. The polarization fields of the degenerate branches are therefore triads rather than dyads[12,16] (i.e., we have a 3-tuple of orthogonal vectors $\mathbf{A}_\alpha$ at each point $\mathbf{m}$), and their singularities thus cannot be described by the Poincaré index formalism. Triple degeneracies are easily shown to be unstable. Indeed, the Christoffel tensor is obviously diagonal along a triple acoustic axis, $\Lambda_{ij}=v_0^2\,\delta_{ij}$, where δ_{ij} is the Kronecker symbol. The condition for a perturbation $\Delta\hat{c}$ to preserve a triple degeneracy near $\mathbf{m}_0$ is that

$$\Delta\Lambda_{ij}=(\mathbf{m}_0\Delta\hat{c}\mathbf{m}_0+\mathbf{m}_0\hat{c}\Delta\mathbf{m}+\Delta\mathbf{m}\hat{c}\mathbf{m}_0)_{ij}/\rho=[v_0{}^2+\Delta(v^2)]\delta_{ij}, \qquad (61)$$

which gives a system of six equations for the three unknowns $\Delta\mathbf{m}$ and $\Delta(v^2)$. It is also easy to see that regardless of the symmetry of the axis, such degeneracies can occur only if the components of the tensor $\hat{c}$ satisfy additional constraints. This explains why triple acoustic axes do not occur.

The approach developed in the previous section for analyzing the stability of acoustic axes is based on the degeneracy condition (1) and can be used to find the coordinates of perturbed degeneracies for matrices $\Delta\hat{c}$ of arbitrary structure. Moreover, our conclusions are independent of the specific nature of the thermodynamic effect that alters the elastic properties of the crystal, and they are valid for all perturbations that can be described formally in the wave equation (16) by adding a real symmetric matrix (not necessarily of the form $\Delta\hat{\Lambda}=\mathbf{m}\Delta\hat{c}\mathbf{m}/\rho$) to the Christoffel tensor. Such perturbations include phase transitions, elastoelectric and piezoelectric effects, electrostriction, etc.[20,21] However, other phenomena cannot be treated in this manner (acoustogyration, for example, because the corresponding term $\Delta\hat{\Lambda}$ responsible for this effect is not real-valued). One can show that if acoustogyration is allowed for by adding a term proportional to the gradient of the deformation[11,15] to the elastic energy, the acoustic axis either remains fixed or else disappears (there can be no splitting or displacement). On the other hand, if we describe elastic wave absorption phenomenologically by adding an imaginary term to the tensor $\hat{c}$, then the effective Christoffel tensor $\hat{\Lambda}+\Delta\hat{\Lambda}$ is nonhermitian, in contrast to the acoustogyration tensor, which is Hermitian. Under such a perturbation, the acoustic axes (including conical ones) may split or disappear.

[1)]However, the mathematical approach in Refs. 12 and 16, in which the polarization fields are assumed in addition to be directed, has proved fruitful in many cases.

[2)]We will see below that a line of wedge degeneracy is unstable, i.e., a small generic perturbation of the tensor $\hat{c}$ will eliminate contact between the outer and inner sheets at all but at most finitely many points on or near the line. Our labeling of the degenerate wave branches by the rule $v_1(\mathbf{m})\leqslant v_2(\mathbf{m})$ is therefore natural in this case also.

[3)]Similar types of contact geometry were considered in Ref. 4 for the constant-frequency surfaces of the phonon and electron spectra.

[4)]According to Refs. 1 and 2, the sign of $\varkappa\,\mathbf{m}_0$ also determines the direction of rotation of the vectors $\mathbf{s}_i(\mathbf{m}_0)$ along the internal refraction cone as the polarization vector of the degenerate wave rotates.

[5)]For $\Delta\mathbf{m}\neq 0$ the equality $v_1=v_2$ breaks down at higher order in perturbation theory. According to (1), the vanishing of the vector ξ on the line χ is necessary and sufficient for an arbitrary degeneracy to occur along χ.

[6)]For example, one can use the temperature dependence of the elastic moduli of Hg_2Cl_2 crystals found in Ref. 22 to show that lines of wedge degeneracy form near the tetragonal→rhombohedral phase transition in Hg_2Cl_2, even though this crystal is not transversely isotropic. These lines cross the sphere $\mathbf{m}^2=1$ at points corresponding to two- and four-fold axes, so that a tangential degeneracy is superposed on a line of wedge degeneracy at these points (for the case of a four-fold axis, at the point where the two lines intersect). According to Ref. 14, the formation and subsequent removal of the degeneracy lines changes the sign of n for a tangential degeneracy along the four-fold axis. An unstable tangential degeneracy not due to symmetry occurs along the two-fold axis (parallel to X_3) in the tetragonal Hg_2Cl_2 phase, because the equality $c_{44}=c_{55}$ (Ref. 22) continues to hold at low temperatures. We may also mention $BaTiO_3$ ceramic, for which the data in Ref. 23 show that a line of tangential degeneracy is present if the ceramic is placed in an external electric field of suitable magnitude.

[1]F. I. Fedorov, Teoriya Uprugikh Voln v Kristallakh (Theory of Elastic Waves in Crystals), Nauka, Moscow (1965).

[2]A. G. Khatkevich, Kristallogr. **7**, 916 (1962) [Sov. Phys. Crystallogr. **7**, 742 (1962)].

[3]K. S. Aleksandrov, in: Problemy Sovremmenoĭ Kristallografi (Topics in Modern Crystallography), Nauka, Moscow (1975), p. 327.

[4]C. Herring, Phys. Rev. **95**, 954 (1954).

[5]V. L. Gurevich and Yu. A. Firsov, Fiz. Tverd. Tela (Leningrad) **4**, 530 (1962) [Sov. Phys. Solid State **4**, 385 (1962)].

[6]B. Ya. Balagurov, V. G. Vaks, and B. I. Shklovskiĭ, Fiz. Tverd. Tela (Leningrad) **12**, 89 (1970) [Sov. Phys. Solid State **12**, 70 (1970)].

[7] V. L. Gurevich, Fiz. Tverd. Tela (Leningrad) **21**, 3453 (1979) [Sov. Phys. Solid State **21**, 1993 (1979)].
[8] V. L. Gurevich, Kinetika Fononnykh Sistem (Kinetics of Phonon Systems), Nauka, Moscow (1980).
[9] C. Herring, Phys. Rev. **52**, 365 (1937).
[10] A. G. Khatkevich, Kristallogr. 7, 742 (1962) [Sov. Phys. Crystallogr. **7**, 601 (1962)]; **9**, 690 (1964) [**9**, 579 (1964)]; **22**, 1232 (1977) [**22**, 701 (1977)].
[11] Yu. I. Sirotin and M. P. Shaskol'skaya, Osnovy Kristallofiziki (Principles of Crystal Physics), Nauka, Moscow (1979).
[12] V. I. Al'shits and E. Lote, Kristallogr. **24**, 672, 683 (1979) [Sov. Phys. Crystallogr. **24**, 387, 393 (1979)].
[13] S. Crampin, Wave Motion **3**, 343 (1981).
[14] V. I. Al'shits and A. L. Shuvalov, Kristallograf. **29**, 629 (1094) [Sov. Phys. Crystallogr. **29**, 373 (1984)].
[15] V. E. Lyamov, Polyarizatsionnye Éffekty i Anisotropiya Vzaimodeĭstviya Akusticheskikh Voln v Kristallakh (Polarization Effects and Anisotropic Interaction of Acoustic Waves in Crystals), Izd. Mosk. Gos. Univ. (1983).
[16] P. Holm and J. Lothe, Proc. Roy. Soc. Lond. **A370**, 331 (1980).
[17] M. A. Krasnosel'skiĭ, A. I. Perov, A. I. Povolotskiĭ, and P. P. Zabreĭko, Vektornye Polya na Ploskosti (Vector Fields in the Plane), Fizmatgiz, Moscow (1963).
[18] M. Golubitsky and V. Guillemin, Stable Mappings and their Singularities, Springer Verlag, New York (1973).
[19] T. H. Brocker and L. Lander, Differential Germs and Catastrophes, Cambridge Univ. Press (1975).
[20] V. I. Alshits and A. L. Shuvalov, Ferroelectrics Lett. **1**, 151 (1984).
[21] K. S. Aleksandrov, B. P. Sorokin, Yu. I. Kokorin, N. A. Chatvergov, and T. I. Grekova, Ferroelectrics **41**, 27 (1982).
[22] Cao Xuan An, G. Hauret, and J. P. Chappelle, Solid State Commun. **24**, 443 (1977).
[23] V. N. Belyĭ and B. B. Sevruk, Kristallogr. **28**, 925 (1983) [Sov. Phys. Crystallogr. **28**, 548 (1983)].

Translated by A. Mason