Nonlocal (parafermion) currents in two-dimensional conformal quantum field theory and self-dual critical points in Z_N-symmetric statistical systems

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A two-dimensional exactly solvable model of a conformal quantum field theory is developed which is self-dual and has Z_N symmetry. The operator algebra, the correlation functions, and the anomalous dimensions of all fields are calculated for this model, which describes self-dual critical points in Z_N-symmetric statistical systems.

INTRODUCTION

There are good grounds for believing that fluctuations of the macroscopic fields ("of the order parameter") at a second-order phase-transition point possess conformal as well as scale invariance.[1] From this viewpoint, the principal task in the theory of second-order phase transitions—the classification of all types of universal critical behavior—can be formulated as the problem of finding the conformally invariant solutions of quantum field theory. Polyakov[2] suggested constructing these solutions by combining the requirement of conformal invariance with the Ansatz that the field operators form an algebra. This problem has recently been found[3] to admit at $d = 2$ an exact solution in many cases; this comes about because the conformal group of two-dimensional space is infinite-dimensional, and the fields that form an operator algebra can be classified in terms of the representations of the Virasoro algebra, for which a well-developed theory exists.[4,5] An infinite set of exactly solvable "minimal" conformal field theory models (related to the strongly degenerate representations of the Virasoro algebra) was found in Ref. 3; in these models, the space of fields forming the operator algebra contains a finite number of irreducible representations. Each of these models is described by two relatively prime numbers p, q and corresponds to a central charge.

$$c=1-6(p-q)^2/pq \tag{1.1}$$

in the Virasoro algebra. It was shown in Ref. 6 that only the "principal series" ($q = p + 1, p \geqslant 3$) of minimal models satisfies the positivity condition.[1)] The mathematical structure of the minimal models in the principal series is by now fairly well understood; in particular the correlation function and structure constants of the operator algebra have been calculated.[7]

The four simplest models in the principal series ($p = 3$, 4, 5, 6) turn out to describe the critical points of the Ising ($p = 3$) (Ref. 3) and Z_3 Potts ($p = 5$) (Ref. 8) models, in addition to the corresponding tricritical models ($p = 4$, 6) (Refs. 6, 9). In Ref. 10, the anomalous dimensions corresponding to all the principal-series minimal models were found to coincide with the exponents at the "ferromagnetic" critical points of the exactly solvable RSOS model recently discovered in Ref. 11.

The minimal models certainly do not exhaust the class of all solutions of conformal field theory for $d = 2$. For instance, a series of exactly solvable models with superconformal symmetry[9,12] and conformally invariant solutions of the Wess-Zumino models[13] are known to exist. In these cases the conformally invariant field theory also possesses a more general type of infinite symmetry which is generated by local currents and is described by the Neveu-Schwarz and Kac-Moody algebras in the first and second cases, respectively. Reference 14 examines some other types of infinite symmetries generated by local currents.

In this paper we consider a conformal field theory with an infinite symmetry generated by nonlocal currents with fractional spins ("parafermions"). Conserved parafermion currents were discovered by one of us (V. F.) in the minimal models with $p = 5$, 6. In general, the parafermion currents form closed operator algebras which are associated in a natural way with commutative groups. We will investigate a series of very simple exactly solvable models corresponding to the cyclic groups Z_N with $N > 2$. These models are conformally invariant and have central charge

$$c=2(N-1)/(N+2) \quad , \tag{1.2}$$

in the Virasoro algebra. In addition to their explicit Z_N symmetry, the operator algebras for these models are also self-dual (they possess "order–disorder" symmetry). Thus, for a given N there exist $N - 1$ fields σ_k, $k = 1, 2, \ldots, N - 1$ (order parameters) with anomalous dimension

$$2d_k=k(N-k)/N(N+2), \tag{1.3}$$

and $N - 1$ "dual" fields μ_k ("disorder parameters") with the same dimensions as in (1.3), and all the correlation functions are invariant under the interchange $\sigma \leftrightarrow \mu$. The operator algebra also contains Z_N-neutral fields $\varepsilon^{(j)}$, $j = 1, 2, \ldots, \leqslant N/2$ of dimension

$$2D_j=2j(j+1)/(N+2). \tag{1.4}$$

We note here that the dimensions given by (1.3), (1.4) coincide exactly with the exponents characterizing the "antiferromagnetic" critical points of the RSOS model.[11,10] Moreover, the solutions that we construct for $N = 2$, 3 also coincide[2)] with the minimal theories $p = 3$, 5 and describe critical points of the Z_2 Ising and Z_3 Potts models, respec-

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tively. We propose that our solutions for $N\geqslant 3$ describe the self-dual critical points of the Z_N Ising models; we will discuss this hypothesis more fully in Sec. 6.

2. SELF-DUAL SYSTEMS WITH Z_N SYMMETRY AND PARAFERMION CURRENTS

We consider a two-dimensional lattice statistical system with cyclic Z_N symmetry (the Z_N Ising model, the simplest example of such a system, is discussed in Sec. 6). We can describe the Z_N degrees of freedom by associating "spin" variables $\sigma_{\mathbf{r}}$ to each node $\mathbf{r}\in L$ in a (square) lattice L; the $\sigma_{\mathbf{r}}$ take the N values ω^q $(q=0, 1, 2, \ldots, N-1)$, where

$$\omega=\exp(2\pi i/N). \tag{2.1}$$

It is helpful to immediately define the $N-1$ (dependent) variables $\sigma_{k,\mathbf{r}}=(\sigma_{\mathbf{r}})^k$, $k=1, 2, \ldots, N-1$, which take the value ω^{kq}; we will write $\sigma_{N-k,\mathbf{r}}=\sigma_{k,\mathbf{r}}^+$. Assume that the system has a critical point at which the correlation radius of the "spins" $\sigma_{k,\mathbf{r}}$ becomes infinite. In such a theory the long-range correlations of the spins σ_k can be described by continuous conformal fields $\sigma_k(x)$, $x\in\mathbf{R}^2$ $(\sigma_{N-k}=\sigma_k^+)$ with anomalous dimensions $2d_k$, where $d_k=d_{N-k}$. The critical theory will then be Z_N-symmetric if and only if the correlations are invariant under the transformation

$$\sigma_k(x)\to\omega^{mk}\sigma_k(x) \tag{2.2}$$

for arbitrary integral m. We will say that the field $\sigma_k(x)$ has Z_N-charge equal to k; of course, the Z_N-charge is only defined modulo N.

Z_N-symmetric systems possess order-disorder duality (Kramers-Wannier symmetry).[15–18] We will assume that the Z_N-system is *self-dual* (i.e., invariant under the Kramers-Wannier transformation) at the critical point in question. This means that in addition to the conformal fields $\sigma_k(x)$ (order parameters), the theory also contains dual conformal fields $\mu_k(x)$, $k=1, 2, \ldots, N-1$ (disorder parameters) with the same anomalous dimensions $2d_k$. The main properties of the dual fields $\mu_k(x)$ are as follows (see, e.g., Refs. 19 and 20 for the details of the definition). The correlation functions (CF)

$$\langle\sigma_{k_1}(x_1)\ldots\sigma_{k_n}(x_n)\mu_{l_1}(y_1)\ldots\mu_{l_m}(y_m)\rangle \tag{2.3}$$

are in general N-valued functions of the coordinates x_i, $y_i\in\mathbf{R}^2$; if we analytically continue the CF (2.3) with respect to x_i, say, along a path enclosing the point y_i counterclockwise as shown in Fig. 1, the CF acquires a phase factor $\omega^{-k_il_j}$. Fields $\sigma_k(x)$ and $\mu_l(x)$ with this property will be called mutually semilocal[3)] with exponent $\gamma_{kl}=-kl/N$. Two fields $A(x)$ and $B(x)$ are said to be mutually local if γ_{AB} is an integer. The field $A(x)$ is semilocal (local) if it is semilocal (local) with respect to itself. For example, the fields $\sigma_k(x)$ and $\sigma_{k'}(x)$ [as well as $\mu_k(x)$ and $\mu_{k'}(x)$] are mutually local.

In self-dual systems all correlations are invariant under the interchange $\sigma_k\leftrightarrow\mu_k$. In addition to Z_N-symmetry (2.2), the theory possesses the dual $\widetilde{Z}_N$-invariance

$$\mu_l(x)\to\omega^{ln}\mu_l(x),\quad n\in\mathbf{Z}. \tag{2.4}$$

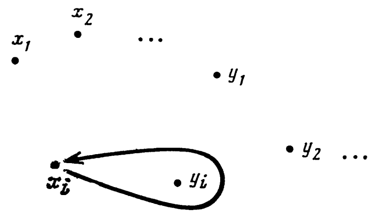

FIG. 1. Path along which the correlation function is continued with respect to the variable x_i.

The fields $\mu_l(x)$ thus have $\widetilde{Z}_N$-charge equal to l. In general, we will say that the field $\phi_{(k,l)}(x)$ has $Z_N\times\widetilde{Z}_N$-charge $\{k, l\}$ if it transforms as

$$\phi_{\{k,l\}}(x)\to\omega^{km+ln}\phi_{\{k,l\}}(x) \tag{2.5}$$

under (2.2) and (2.4). Thus, the fields $\sigma_k(x)$ and $\mu_k(x)$ have charge $\{k, 0\}$ and $\{0, k\}$. We observe that the Z_N-theories $\phi_{(k,l)}(x)$ and $\phi_{(k',l')}(x)$ are mutually semilocal with exponent

$$\gamma_{\{k',l'\}}^{\{k,l\}}=-(kl'+k'l)/N. \tag{2.6}$$

We will assume that our $Z_N\times\widetilde{Z}_N$ theory is C- and P-invariant, and that these inversions transform the fields as

$$\begin{aligned}&\text{C:}\quad \sigma_k\to\sigma_k^+;\quad \mu_k\to\mu_k^+;\\ &\text{P:}\quad \sigma_k\to\sigma_k;\quad \mu_k\to\mu_k^+.\end{aligned} \tag{2.7}$$

For the case of Z_2-symmetry (Ising model), the most singular terms in the operator expansion of the product $\sigma(x)\mu(0)$ (where $\sigma=\sigma_1$, $\mu=\mu_1$) are given by (see, e.g., Ref. 3)

$$\sigma(z,\bar z)\mu(0,0)=\frac{1}{\sqrt{2}}(z\bar z)^{-1/8}[z^{1/2}\psi(0)+\bar z^{1/2}\bar\psi(0)+\ldots], \tag{2.8}$$

where we have introduced the complex coordinates of $\mathbf{R}^2$:

$$z=x_1+ix_2\quad \bar z=x_1-ix_2 \tag{2.9}$$

and $\psi(z)$ and $\psi(\bar z)$ are the "right- and left-hand" components of the free Fermi field[22] satisfying the massless Dirac equation $\partial_{\bar z}\psi=\partial_z\bar\psi=0$. For $N>2$ the most singular terms in the operator expansions of $\sigma_k(x)\mu_k(0)$ and $\sigma_k(x)\mu_k^+(0)$ can be written in the form

$$\begin{aligned}&\sigma_k(z,\bar z)\mu_k(0,0)=z^{\Delta_k-2d_k}\bar z^{\bar\Delta_k-2d_k}[\psi_k(0,0)+\ldots];\\ &\sigma_k(z,\bar z)\mu_k^+(0,0)=z^{\bar\Delta_k-2d_k}\bar z^{\Delta_k-2d_k}[\bar\psi_k(0,0)+\ldots],\end{aligned} \tag{2.10}$$

where we have used the symmetries (2.7). Here ψ_k and $\bar\psi_k$ are certain conformal fields, and the parameters Δ_k and $\bar\Delta_k$ are the nonnegative "dimensions" of these fields. Since the mutual-locality exponent of the fields σ_k and μ_k is equal to $-k^2/N$, we have

$$\Delta_k-\bar\Delta_k=-\frac{k^2}{N}\ (\mathrm{mod}\ \mathbf{Z}).$$

We assume that there exist self-dual critical points of the Z_N-models with $N>2$ for which $\bar\Delta_k=0$ as in (2.8), and that the fields ψ_k and $\bar\psi_k$ satisfy the equations

$$\partial_{\bar z}\psi_k=0;\quad \partial_z\bar\psi_k=0, \tag{2.11}$$

so that we may write $\psi_k = \psi_k(z)$, $\bar{\psi}_k = \bar{\psi}_k(\bar{z})$. The parameters Δ_k in (2.10) are then equal to the spins of the fields $\psi_k(z)$ and are given by

$$\Delta_k = m_k - k^2/N, \qquad (2.12)$$

where the m_k are integers. The fields ψ_k and $\bar{\psi}_k$ are semilocal with $Z_N \times \tilde{Z}_N$-charges equal to $\{k, k\}$ and $\{k, -k\}$, respectively. Unlike local fields, they can have fractional spins and it is natural to refer to $\psi_k(z)$ and $\psi_k(\bar{z})$ as *parafermion currents.*

The simplest expression for Δ_k that satisfies (2.12) and the requirement $\Delta_{N-k} = \Delta_k$ is

$$\Delta_k = k(N-k)/N. \qquad (2.13)$$

We will henceforth assume that the spins Δ_k of the parafermion currents ψ_k are given by (2.13).

3. THE PARAFERMION CURRENT ALGEBRA

We will concentrate on the parafermion currents $\psi_k(z)$ $(k = 1, 2, \ldots, N-1)$ bearing in mind that all of our results are also valid for $\bar{\psi}_k(\bar{z})$. It will be convenient to adjoin the identity operator I to the family $\{\psi_k\}$ (we set $\psi_0 \equiv I$) and write $\psi_k^+ = \psi_{N-k}$.

We consider the operator expansion of the product $\psi_k(z)\psi_{k'}(z')$. Since the mutual locality exponent of ψ_k and $\psi_{k'}$ is equal to $-2kk'/N$ and the spins (2.13) satisfy

$$\Delta_{k+k'} - \Delta_k - \Delta_{k'} = -2kk'/N, \qquad (3.1)$$

we have for this operator expansion

$$\psi_k(z)\psi_{k'}(z')$$
$$= c_{k,k'}(z-z')^{\Delta_{k+k'}-\Delta_k-\Delta_{k'}} \sum_{n=0}^{\infty} (z-z')^n \Psi_{k+k',k}^{(n)}(z'), \qquad (3.2)$$

where the fields $\Psi_{k+k',k}^{(n)}$ have $Z_N \times \tilde{Z}_N$ charge $\{k+k', k+k'\}$. (Here and below, we understand the sums $k+k'$ to be modulo N.) The field $\Psi_{k+k',k}^{(0)}(z)$ has spin $\Delta_{k+k'}$ and coincides with the parafermion current $\psi_{k+k'}(z)$. We call the scalars $c_{k,k'}$ the *structure constants* of the parafermionic current algebra; they depend on how the fields ψ_k are normlaized. We put

$$\langle \psi_k(z)\psi_{k'}^+(0)\rangle = \delta_{kk'} z^{-2\Delta_k}, \qquad (3.3)$$

i.e., $c_{k,N-k} = 1$. The remaining structure constants $c_{k,k'}$ must be found from the associativity condition for the operator algebra (3.2) (see below).

We pause to discuss expansion (3.2) with $k + k' = 0$ (mod N) more fully. In this expansion $\Psi_{0,k}^{(0)}$ is equal to the identity operator I and the field $\Psi_{0,k}(z)$ has spin 1. If such a field existed in the theory then it would generate the symmetry group $U(1)$, which is bigger than Z_N. Since we are interested in the Z_N-symmetric critical theory described in Sec. 2, we require that

$$\Psi_{0,k}^{(1)}(z) = 0. \qquad (3.4)$$

It is natrual to identify the fields $\Psi_{0,k}^{(2)}(z)$ of spin 2 with the corresponding component of the energy-momentum tensor of the theory:

$$\Psi_{0,k}^{(2)}(z) = (2\Delta_k/c)T(z). \qquad (3.5)$$

The scalar factor $2\Delta_k/c$ is necessary for conformal invariance[3]; c is the central charge in the corresponding Virasoro algebra of conformal generators, and is defined by the operator expansion

$$T(z)T(z') = \frac{c}{2(z-z')^4} + \frac{2T(z')}{(z-z')^2} + \frac{1}{z-z'}\partial_{z'}T(z') + O(1). \qquad (3.6)$$

The operator expansions defining the algebra of the parafermion currents ψ_k in the $Z_N \times \tilde{Z}_N$ theory have thus the form

$$\psi_k(z)\psi_{k'}(z') = c_{k,k'}(z-z')^{-2kk'/N}[\psi_{k+k'}(z') + O(z-z')];$$
$$k + k' < N, \qquad (3.7a)$$

$$\psi_k(z)\psi_{k'}^+(z') = c_{k,N-k'}(z-z')^{-2k(N-k')/N}[\psi_{k-k'}(z')$$
$$+ O(z-z')]; \quad k' < k, \qquad (3.7b)$$

$$\psi_k(z)\psi_k^+(z') = (z-z')^{-2k(N-k)/N}[I + (2\Delta_k/c)(z-z')^2 T(z')$$
$$+ O((z-z')^3)]; \qquad (3.7c)$$

$$T(z)\psi_k(z') = \frac{\Delta_k}{(z-z')^2}\psi_k(z') + \frac{1}{z-z'}\partial_{z'}\psi_k(z') + O(1). \qquad (3.8)$$

Equation (3.8) means simply that the fields $\psi_k(z)$ are conformal.[3]

The operator expansions (3.7) can be used to explicitly calculate any correlation function

$$\langle \psi_{k_1}(z_1)\psi_{k_2}(z_2)\ldots\psi_{k_n}(z_n)\rangle. \qquad (3.9)$$

We first consider the $2n$-point function

$$\langle \psi_1(z_1)\psi_1(z_2)\ldots\psi_1(z_n)\psi_1^+(\eta_1)\psi_1^+(\eta_2)\ldots\psi_1^+(\eta_n)\rangle. \qquad (3.10)$$

If we multiply (3.10) by

$$\prod_{i=2}^{n}(z_1-z_i)^{2/N}\prod_{j=1}^{n}(z_1-\eta_j)^{-2/N},$$

we get an analytic function of the variable z_1 which (due to the mutual semilocality of the fields ψ_1, ψ_1^+, see Sec. 2) is single-valued in the entire complex plane and has second-order poles at the points $\eta_1, \eta_2, \ldots, \eta_n$. This function is uniquely determined by its residues at these poles, because the condition for $\psi_k(z)$ to be regular as $z \to \infty$ is[3]

$$\psi_k(z) \sim z^{-2\Delta_k} \quad \text{as} \quad z \to \infty. \qquad (3.11)$$

We can use the expansion (3.7c) to evaluate the residues. We find that

$$\langle\psi_1(z_1)\dots\psi_1(z_n)\psi_1^+(\eta_1)\dots\psi_1^+(\eta_n)\rangle$$
$$=\prod_{i=2}^{n}(z_1-z_i)^{-2/N}\prod_{j=1}^{n}(z_1-\eta_j)^{2/N}\sum_{k=1}^{n}\Big\{\frac{1}{(z_1-\eta_k)^2}+\frac{2/N}{(z_1-\eta_k)}\Big[\sum_{l=2}^{n}\frac{1}{\eta_k-z_l}-\sum_{\substack{m=1\\m\neq k}}^{n}\frac{1}{\eta_k-\eta_m}\Big]\Big\}$$
$$\times\prod_{q=2}^{n}(z_q-\eta_k)^{2/N}\prod_{p=1}^{k-1}(\eta_p-\eta_k)^{-2/N}\prod_{r=k+1}^{n}(\eta_k-\eta_r)^{-2/N}$$
$$\times\langle\psi_1(z_2)\dots\psi_1(z_n)\psi_1^+(\eta_1)\dots\psi_1^+(\eta_{k-1})\psi_1^+(\eta_{k+1})\dots\psi_1^+(\eta_n)\rangle, \tag{3.12}$$

which expresses the $2n$-point function (3.10) in terms of $(2n-2)$-point functions. We can now use (3.12) to evaluate (3.10) by successively decreasing the order of the CF. Of course, any CF (3.9) can then be evaluated from (3.10) by using (3.7). The same CF (3.9) can be evaluated in many different ways; the result will be the same in all cases (i.e., the operator algebra defined by the relations (3.7) will be associative) only if the structure constants $c_{k,k'}$ have the form

$$c_{k,k'}^2=\frac{\Gamma(k+k'+1)\Gamma(N-k+1)\Gamma(N-k'+1)}{\Gamma(k+1)\Gamma(k'+1)\Gamma(N-k-k'+1)\Gamma(N+1)}. \tag{3.13}$$

In addition, the parafermion current algebra (3.7) is consistent with the conformal Ward identities[3] defined by (3.6), (3.8) if the central charge c in (3.6) is equal to

$$c=2(N-1)/(N+2). \tag{3.14}$$

We note that the parafermion current algebra constructed above is not the only possible associative algebra. Appendix A describes an alternative associative Z_N-algebra for which the fields ψ_k have different spins.

4. THE SPACE OF FIELDS IN THE $Z_N\times\tilde{Z}_N$-THEORY AND REPRESENTATIONS OF THE PARAFERMION CURRENT ALGEBRA

The operator algebra of a conformal field theory was formulated in Ref. 3 in terms of a complete set of mutually local fields. For our purposes it will be helpful to enlarge this space by considering a complete set $\{F\}$ of mutually semilocal fields, i.e., one which is closed with respect to the operator algebra. It is natural to assume that the components $T(z)$ and $\overline{T}(\overline{z})$ of the energy-momentum tensor are local with respect to all of the fields $\{F\}$. In this case the classification in Ref. 3 in terms of the representations of the Virasoro algebra remains valid. The operators L_n and $\overline{L}_n$, $n=0,\pm1,\pm2,\dots$, operate on the space $\{F\}$; they are associated with the fields $T(z)$ $\overline{T}(\overline{z})$ and satisfy the commutation relations

$$[L_n,L_m]=(n-m)L_{n+m}+\frac{c}{12}(n^3-n)\delta_{n+m,0} \tag{4.1}$$

(the $\overline{L}_n$ also satisfy (4.1) and commute with L_m). There exist several (possibly infinitely many) semilocal conformal fields $\phi_i\in\{F\}$ which are characterized by two nonnegative numbers $(d_i,\overline{d}_i)$ ("dimensions") and satisfy the equations

$$\begin{gathered}L_n\phi_i=\overline{L}_n\phi_i=0,\quad(n>0),\\ L_0\phi_i=d_i\phi_i,\quad \overline{L}_0\phi_i=\overline{d}_i\phi_i,\\ L_{-1}\phi_i=\partial_z\phi_i,\quad \overline{L}_{-1}\phi_i=\partial_{\overline{z}}\phi_i.\end{gathered} \tag{4.2}$$

Although the sum $d_i+\overline{d}_i$ and difference $d_i-\overline{d}_i$ are equal to the usual anomalous scaling dimension and spin of the field ϕ_i, respectively, the spins of the semilocal fields can in general take arbitrary fractional values. All the other fields in $\{F\}$ can be obtained by repeatedly applying the operators L_n, $\overline{L}_n$ with $n<0$ to the conformal fields ϕ_i.

In the $Z_N\times\tilde{Z}_N$-invariant theory described in Secs. 2 and 3, the field space $\{F\}$ splits naturally into a direct sum of subspaces with specified $Z_N\times\overline{Z}_N$-charges $\{k,l\}$,

$$\{F\}=\frac{1}{2}\bigoplus_{\substack{N\geqslant q,\overline{q}\geqslant 1-N\\ q+\overline{q}\in 2Z}}\{F_{[q,\overline{q}]}\}, \tag{4.3}$$

where we have expressed the charge $\{k,l\}$ in the more convenient form $[q,\overline{q}]=[k+l,\,k-l]$. The right and left charges q and $\overline{q}$ are defined mod $2N$ and take arbitrary integer values such that $q+\overline{q}$ is even. It is convenient to extend the summation in (4.3) over all $l-N<q,\ \overline{q}\leqslant N$; each value of the $Z_N\times\tilde{Z}_N$-charge then occurs twice, which accounts for the factor 1/2 in (4.3). With these conventions the parafermion currents ψ_k ($\overline{\psi}_k$) have charge $[2k,0]$ ($[0,2k]$) and

$$\sigma_k\in\{F_{[k,k]}\},\quad \mu_k\in\{F_{[k,-k]}\}.$$

It is clear that an arbitrary field $\phi_{[k,\overline{k}]}\in\{F_{[k,\overline{k}]}\}$ has spin

$$2s_{[k,\overline{k}]}=(\overline{k}^2-k^2)/2N\,(\mathrm{mod}\,Z), \tag{4.4}$$

and that the mutual locality exponent of any two fields $\phi_{[k,\overline{k}]}$ and $\phi_{[q,\overline{q}]}$ is equal to

$$\gamma_{[k,\overline{k}]}^{[q,\overline{q}]}=-(kq-\overline{k}\overline{q})/2N. \tag{4.5}$$

In particular, all the fields in $\{F_{[k,k]}\}$ and $\{F_{[k,-k]}\}$ are local, and I, $T(z)$, and $\overline{T}(\overline{z})$ belong to $\{F_{[0,0]}\}$.

It turns out that the fields spanning the space (4.3) can be classified by the associated representations of the parafermion current algebra (3.7). Let $\phi_{[k,\overline{k}]}\in\{F_{[k,\overline{k}]}\}$ be a field of dimension $(d,\overline{d})$ and consider the operator expansions

$$\psi_1(z)\,\phi_{[k,\overline{k}]}(0,0)=\sum_{m=-\infty}^{\infty}z^{-k/N+m-1}A_{(1+k)/N-m}\phi_{[k,\overline{k}]}(0,0); \tag{4.6a}$$

$$\psi_1^+(z)\,\phi_{[k,\overline{k}]}(0,0)=\sum_{m=-\infty}^{\infty}z^{k/N+m-1}A^+_{(1-k)/N-m}\phi_{[k,\overline{k}]}(0,0), \tag{4.6b}$$

where the exponents in (4.6) are given by (4.5), and $A\phi$ and $A^+\phi$ are fields in $\{F\}$ with

$$A_{(1+k)/N-m}\phi_{[k,\overline{k}]}\in\{F_{[k+2,\overline{k}]}\},\quad d+m-\frac{1+k}{N},\ \overline{d}, \tag{4.7a}$$

$$A^+_{(1-k)/N-m}\phi_{[k,\overline{k}]}\in\{F_{[k-2,\overline{k}]}\},\quad d+m-\frac{1-k}{N},\ \overline{d}, \tag{4.7b}$$

(the dimensions of the fields are given on the right). The expansions (4.6) define the operators A_ν and A^+_μ with

$$\nu=\frac{1+k}{N}\ (\mathrm{mod}\ \mathbf{Z}),\quad \mu=\frac{1-k}{N}\ (\mathrm{mod}\ \mathbf{Z});$$

they act on the space $\{F\}$ as follows:

$$A_{(1+k)/N+m}:\quad \{F_{[k,\bar{k}]}\}\to\{F_{[k+2,\bar{k}]}\};\tag{4.8a}$$

$$A^{+}_{(1-k)/N+m}:\quad \{F_{[k,\bar{k}]}\}\to\{F_{[k-2,\bar{k}]}\}.\tag{4.8b}$$

The fields in the expansion (4.6) can also be represented conveniently in the form

$$A_{(1+k)/N+m}\phi_{[k,\bar{k}]}(0,0)=\oint\frac{dz}{2\pi i}\psi_1(z)z^{k/N+m}\phi_{[k,\bar{k}]}(0,0),\tag{4.9a}$$

$$A^{+}_{(1-k)/N+m}\phi_{[k,\bar{k}]}(0,0)=\oint\frac{dz}{2\pi i}\psi_1^{+}(z)z^{-k/N+m}\phi_{[k,\bar{k}]}(0,0).\tag{4.9b}$$

Proceeding in the same way with the currents $\bar{\psi}_1(z)$ and $\bar{\psi}_1^{+}(\bar{z})$, we can define operators $\bar{A}_{\bar{\nu}}$ and $\bar{A}^{+}_{\bar{\mu}}$ with

$$\bar{\nu}=\frac{1+\bar{k}}{N}\ (\mathrm{mod}\ \mathbf{Z}),\quad \bar{\mu}=\frac{1-\bar{k}}{N}\ (\mathrm{mod}\ \mathbf{Z});$$

they act on $\{F\}$ but differ from Eqs. (4.6)–(4.9) only in that z and k replaced by $\bar{z}$ and $\bar{k}$. We have considered only the currents ψ_1, ψ_1^{+}, $\bar{\psi}_1$, and $\bar{\psi}_1^{+}$, because according to (3.7) the full algebra of the parafermion currents ψ_k, $\bar{\psi}_k$ is uniquely determined by the algebra generated by the operators, A, A^{+}, $\bar{A}$, $\bar{A}^{+}$.

We find the commutation relations defining the operator algebra A, A^{+} by considering the double integral

$$\oint_{C_1}\frac{dz_1}{2\pi i}\oint_{C_2}\frac{dz_2}{2\pi i}\psi_1(z_1)\psi_1^{+}(z_2)z_1^{k/N+n}\,z_2^{-k/N+m}\times(z_1-z_2)^{-(N+2)/N}\phi_{[k,\bar{k}]}(0,0),\tag{4.10}$$

where the exponents are chosen so that the integrand is single-valued in z_1, z_2. The integration paths C_1 and C_2 are shown in Fig. 2. The integral (4.10) can be evaluated by two different methods. We can first integrate over z_2 and then use Eqs. (4.9) to do the z_1 integration. In the second method we must first deform the contour C_1 in Fig. 2 so that it lies inside C_2. In this case account must be taken of the contribution of the pole $z_1=z_2$ to the z_1-integral; the residue is given by Eq. (3.7c) in terms of the field $T(z)$. Equating the results of both methods, we find that

$$\sum_{l=0}^{\infty}C^{(l)}_{-(N+2)/N}[A_{-(1-k)/N+n-l-1}A^{+}_{(1-k)/N+m+l+1}-A^{+}_{-(k+1)/N+m-l}A_{(1+k)/N+n+l}]=\frac{N+2}{N}L_{n+m}+\frac{1}{2}\left(n+\frac{k}{N}\right)\left(n-1+\frac{k}{N}\right)\delta_{n+m,0},\tag{4.11}$$

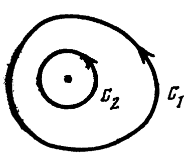

FIG. 2. Paths of integration with respect to z_1 and z_2.

where the binomial coefficients are given by

$$C^{(l)}_{\lambda}=\frac{\Gamma(l-\lambda)}{l!\,\Gamma(-\lambda)}.\tag{4.12}$$

We have used (2.13) and (3.14) for the parameters Δ_1 and c in deriving (4.11). Similarly, we find by considering the integral

$$\oint_{C_1}\frac{dz_1}{2\pi i}\oint_{C_2}\frac{dz_2}{2\pi i}\psi_1(z_1)\psi_1(z_2)z_1^{k/N+n}\,z_2^{k/N+m}\,(z_1-z_2)^{2/N}\phi_{[k,\bar{k}]}(0,0),$$

that

$$\sum_{l=0}^{\infty}C^{(l)}_{2/N}[A_{(3+k)/N+n-l}A_{(1+k)/N+m+l}-A_{(3+k)/N+m-l}A_{(1+k)/N+n+l}]=0.\tag{4.13}$$

The operators A^{+} of course satisfy the same "commutation" relations, with k replaced by $-k$.

We recall that according to (4.7) the fields $A_\nu\phi$ and $A_\mu\phi$ have dimensions $(d-\nu,\bar{d})$ and $(d-\mu,\bar{d})$ if ϕ has dimension $(d,\bar{d})$. The requirement that the dimensions of the fields be bounded from below therefore implies that there exist fields which are annihilated by all operators A_ν, A^{+}_{μ} with $\nu,\mu>0$. We use the equations

$$\begin{aligned}&A_{(1+k)/N+n}\sigma_k=A^{+}_{(1-k)/N+n+1}\sigma_k=0,\\&\bar{A}_{(1+k)/N+n}\sigma_k=\bar{A}^{+}_{(1-k)/N+n+1}\sigma_k=0\end{aligned}\tag{4.14}$$

with $n\geqslant0$ to define N scalar fields (order parameters) $\sigma_k\in\{F_{[k,k]}\}$, $k=0,1,\ldots,N-1$, for the $Z_N\times\tilde{Z}_N$ theory. If we apply Eq. (4.11) with $n=m=0$ to the fields σ_k and use (4.14), we immediately find the dimensions

$$d_k=k(N-k)/2N(N+2).\tag{4.15}$$

Similarly, the dual fields $\mu_k\in\{F_{[k,-k]}\}$ (disorder parameters) can be defined by the equations

$$\begin{aligned}&A_{(1+k)/N+n}\mu_k=A^{+}_{(1-k)/N+n+1}\mu_k=0,\\&\bar{A}_{(1-k)/N+n+1}\mu_k=\bar{A}^{+}_{(1+k)/N+n}\mu_k=0\end{aligned}\tag{4.16}$$

with $n\geqslant0$. Their dimensions are of course also given by (4.15).

All the fields spanning the space $\{F\}$ can be obtained by successively applying the operators A, A^{+}, $\bar{A}$, $\bar{A}^{+}$ to the fields σ_k. The independent fields obtained in this way from σ_k generate an irreducible representation $[\sigma_k]_A$ of the parafermion-current algebra (4.11), (4.13), and the space $\{F\}$ is represented by the direct sum

$$\{F\}=\bigoplus_{k=0}^{N-1}[\sigma_k]_A.\tag{4.17}$$

In general, the number of irreducible representations of the Virasoro algebra in (4.17) is infinite. An exception occurs when $N=2$ or 3; these cases are related to the minimal models in the principal series with $p=3$, 5 (see the Introduction).

The dimensions of the fields generating $\{F\}$ can be described as follows. Each "spin" field σ_k corresponds in a natural way to a series of fields $\phi^{(k)}_{[q,k]}$ ($q=-k,-k+2,\ldots,2N-k-2$) defined by

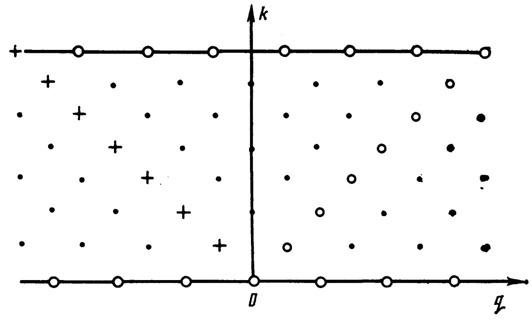

FIG. 3. Diagram describing the system of fields Φ_{kq} for the Z_7 model.

$$\phi^{(k)}_{[k+2l,k]}=A_{(k-1+2l)/N-1}A_{(k-1+2l-2)/N-1}\ldots A_{(1+k)/N-1}\sigma_k;$$
$$l=0,1,\ldots,N-k \quad \left(d_k+\frac{l(N-k-l)}{N},d_k\right), \tag{4.18a}$$

$$\phi^{(k)}_{[k-2l,k]}=A^{+}_{-(1+k-2l)/N}A^{+}_{-(1+k+2-2l)/N}\ldots A^{+}_{(1-k)/N}\sigma_k;$$
$$l=0,1,\ldots,k \quad \left(d_k+\frac{l(k-l)}{N},d_k\right). \tag{4.18b}$$

Here we don't care how the fields are normalized and set $\phi^{(k)}_{[q+2N,k]}=\phi^{(k)}_{[q,k]}$. The expressions in parentheses in (4.18) indicate the dimensions of the fields. The system of fields $\phi^{(k)}_{[q,k]}$ can be represented conveniently as points on the diagram in Fig. 3; the vertical axis gives the values of the "principal quantum number" k while the horizontal axis shows the "right" charge q, and we assume identity $q+2N\equiv q$. Note that the fields $\phi^{(k)}_{[k,k]}$ depicted by the points in Fig. 3 coincide with the "spin" fields σ_k themselves; the dual are fields $\phi^{(k)}_{[-k,k]}=\mu_k$ (crosses in Fig. 3), and the fields $\phi^{(0)}_{[2k,0]}=\phi^{(N)}_{[N+2k,N]}$ (open circles in Fig. 3) coincide with the parafermion currents ψ_k.

Applying the operators $\bar{A}$ and $\bar{A}^{+}$ in analogy with (4.18) we can construct the system of "principal" semilocal fields $\phi^{(k)}_{[q,\bar{q}]}$ ($q,\bar{q}=k \bmod 2Z$) of dimension $(d^{(q)}_k, d^{(\bar{q})}_k)$. According to (4.18), the quantities $d^{(q)}_k$, $k=0,1,2,\ldots,N-1$, are given by

$$d^{(q)}_k=d_k+(k-q)(k+q)/4N, \quad -k\leqslant q\leqslant k;$$
$$d^{(q)}_k=d_k+(q-k)(2N-k-q)/4N, \quad k\leqslant q\leqslant 2N-k. \tag{4.19}$$

We note that all the fields $\phi^{(k)}_{[q,\bar{q}]}$ are mutually local (in particular, they are local with respect to the "spins" σ_k). Of particular interest among them are the $Z_N\times\widetilde{Z}_N$-neutral fields. They exist for even $k=2j$: $\varepsilon^{(j)}=\phi^{(2j)}_{[0,0]}$ and have dimensions (D_j, D_j) where

$$D_j=j(j+1)/(N+2). \tag{4.20}$$

The D_j clearly describe the "thermal" exponents for a self-dual critical point of the Z_N-theory.

Al the other fields in $\{F\}$ have dimensions that differ by integers from (4.19).

5. OPERATOR ALGEBRA AND CORRELATION FUNCTIONS

As in the "minimal models" of Ref. 3, the correlation functions for the "principal" fields in the described $Z_N\times\widetilde{Z}_N$-theory all satisfy linear differential equations of a particular form. In principle these equations can be derived directly by using Eqs. (4.2), (4.11), (4.14), and (4.18). However, it is more convenient to exploit the following remarkable fact. The algebra defined by (3.7), (3.13) for the parafermion currents $\psi_k(z)$ is closely related to the Kac-Moody $\widehat{su}(2)$ algebra (current algebra)[23,4)] in which N plays the role of the central stage. The $Z_N\times\widetilde{Z}_N$-theory can be regarded in the standard way as a "reduction" of an $\widehat{su}(2)\times\widehat{su}(2)$-invariant field theory; the latter theory describes a conformally invariant solution of the two-dimensional $SU(2)\times SU(2)$-chiral Wess-Zumino model[13] and will be called the WZ theory. The field dimensions, operator algebras, and correlation functions for the two theories are related in a very simple way. The fundamental formulas for the WZ theory are given in Appendix B.

We now introduce the free massless Bose field $\widetilde{\phi}(z,\bar{z})$ satisfying the equation $\partial_z\partial_{\bar{z}}\phi=0$, i.e.,

$$\tilde{\varphi}(z,\bar{z})=\varphi(z)+\bar{\varphi}(\bar{z}), \tag{5.1}$$

where the "right" and "left" free fields ϕ and $\bar{\phi}$ are uniquely defined by their two-point functions

$$\langle\varphi(z)\varphi(0)\rangle=-2\log z; \quad \langle\bar{\varphi}(\bar{z})\bar{\varphi}(0)\rangle=-2\log\bar{z};$$
$$\langle\varphi(z)\bar{\varphi}(0)\rangle=0 \ (5.2). \tag{5.2}$$

The components

$$T^{(0)}(z)=(\partial_z\varphi(z))^2, \quad \bar{T}^{(0)}(\bar{z})=(\partial_{\bar{z}}\bar{\varphi}(\bar{z}))^2 \tag{5.3}$$

of the corresponding energy-momentum tensor generate a Virasoro algebra (3.6) with central charge

$$c^{(0)}=1. \tag{5.4}$$

We consider a conformal field theory which splits into the $Z_N\times\widetilde{Z}_N$-theory described in Secs. 2–4 plus a free field (5.1) which does not interact with the $Z_N\times\widetilde{Z}_N$-fields. We define new fields by

$$J^{+}(z)=N^{1/2}\psi_1(z):\exp\left\{\frac{i}{N^{1/2}}\varphi(z)\right\}:,$$
$$J^{-}(z)=N^{1/2}\psi_1^{+}(z):\exp\left\{-\frac{i}{N^{1/2}}\varphi(z)\right\}:, \quad J^{0}(z)=N^{1/2}\partial_z\varphi(z), \tag{5.5}$$

where :: denotes the usual normal ordering. The arguments of the exponentials in (5.5) are chosen so that all three "currents" J^{α} ($\alpha=0,+,-$) have dimensions (1,0). One can readily verify the operator expansions

$$J^{\alpha}(z)J^{\beta}(z')=\frac{Nq^{\alpha\beta}}{(z-z')^2}+\frac{f_{\gamma}^{\alpha\beta}}{(z-z')}J^{\gamma}(z')+O(1), \tag{5.6}$$

where $q^{\alpha\beta}$ and $f^{\alpha\beta}_{\gamma}$ are given by Eqs. (B.2). The total energy-momentum tensor of this "composite" theory is

$$T^{tot}(z)=T(z)+T^{(0)}(z), \tag{5.7}$$

where $T(z)$ is the energy-momentum tensor for the $Z_N\times\widetilde{Z}_N$-theory defined by Eq. (3.7c); T^{tot} satisfies the Virasoro-algebra relation (3.6) with

$$c^{tot}=c+c^{(0)}=3N/(N+2) \tag{5.8}$$

[here we have used (3.14) and (5.4)]. A straightforward verification reveals that T^{tot} is related to the currents (5.5) by

$$(N+2)T^{tot}(z)=q_{\alpha\beta}:J^{\alpha}(z)J^{\beta}(z):. \tag{5.9}$$

Comparison of Eqs. (5.6) and (5.9) with (B.1) and (B.6) shows that the currents (5.5) generate the $su(2)$ Kac-Moody algebra, and the "composite" and WZ theories coincide.

The simple formula

$$\Phi^{(j)}_{m,\bar{m}}(z,\bar{z})=\phi^{(2j)}_{[2m,2\bar{m}]}(z,\bar{z}):\exp\left\{\frac{im}{N^{1/2}}\varphi(z)+\frac{i\bar{m}}{N^{1/2}}\bar{\varphi}(\bar{z})\right\}: \tag{5.10}$$

can be derived without difficulty; it expresses a simple relation between the invariant fields $\Phi^{(j)}_{m,\bar{m}}$ of the WZ theory (see Appendix B) and the "principal" $Z_N\times\tilde{Z}_N$-theory fields $\phi^{(k)}_{[q,\bar{q}]}$ defined in Sec. 4. The dimension (D_j,D_j) of $\Phi^{(j)}$ is the sum of the dimension $(d^{(2m)}_{2j},d^{(2m)}_{2j})$ of the field $\Phi^{(2j)}_{[2m,2\bar{m}]}$ and the dimension of the exponential in (5.10), i.e.,

$$D_j=d^{(2m)}_{2j}+m^2/N, \tag{5.11}$$

in agreement with (4.19) and (B.11). Equation (5.10) implies the formula

$$\langle\Phi^{(j_1)}_{m_1,\bar{m}_1}(z_1,\bar{z}_1)\ldots\Phi^{(j_n)}_{m_n,\bar{m}_n}(z_n,\bar{z}_n)\rangle$$

$$=\prod_{i<i'}^{n}(z_i-z_{i'})^{2m_im_{i'}/N}(\bar{z}_i-\bar{z}_{i'})^{2\bar{m}_i\bar{m}_{i'}/N}$$

$$\times\langle\phi^{(2j_1)}_{[2m_1,2\bar{m}_1]}(z_1,\bar{z}_1)\ldots\phi^{(2j_n)}_{[2m_n,2\bar{m}_n]}(z_n,\bar{z}_n)\rangle, \tag{5.12}$$

which relates the correlation functions for the WZ and $Z_N\times\tilde{Z}_N$-theories. We can use (5.12) to immediately deduce all the structure constants of the operator algebra and some of the correlation functions of the $Z_N\times\tilde{Z}_N$-theory from the corresponding results for the WZ theory (see Appendix B).

The 3-point function of the "spin" fields, normalized by

$$\langle\sigma_k(z,\bar{z})\sigma_{k'}^+(0,0)\rangle=\delta_{kk'}(z\bar{z})^{-2d_k}, \tag{5.13}$$

has the form

$$\langle\sigma_{k_1}(z_1,\bar{z}_1)\sigma_{k_2}(z_2,\bar{z}_2)\sigma^+_{k_1+k_2}(z_3,\bar{z}_3)\rangle$$

$$=C_{k_1,k_2}(z_{12}\bar{z}_{12})^{d_{k_1+k_2}-d_{k_1}-d_{k_2}}(z_{13}\bar{z}_{13})^{d_{k_2}-d_{k_1}-d_{k_1+k_2}}$$

$$\times(z_{23}\bar{z}_{23})^{d_{k_1}-d_{k_2}-d_{k_1+k_2}}, \tag{5.14}$$

where $z_{12}=z_1-z_2$, etc.; the structure constants are given by

$$C^2_{k_1,k_2}=\left[\Gamma\left(\frac{1}{N+2}\right)\Gamma\left(\frac{1+k_1+k_2}{N+2}\right)\times\Gamma\left(\frac{N-k_1+1}{N+2}\right)\Gamma\left(\frac{N-k_2+1}{N+2}\right)\right]$$

$$\times\left[\Gamma\left(\frac{N+1}{N+2}\right)\Gamma\left(\frac{N-k_1-k_2+1}{N+2}\right)\Gamma\left(\frac{k_1+1}{N+2}\right)\Gamma\left(\frac{k_2+1}{N+2}\right)\right]^{-1}. \tag{5.15}$$

The surprising similarity of (5.15) to Eq. (3.13) is noteworthy. The function

$$\langle\sigma_k(z_1,\bar{z}_1)\mu_k(z_2,\bar{z}_2)\psi_k^+(z)\rangle$$

$$=Q_k(z_{12}\bar{z}_{12})^{-2d_k}\left(\frac{(z-z_1)(z-z_2)}{z_1-z_2}\right)^{-\Delta_k}, \tag{5.16}$$

where

$$Q_k^2=k!(N-k)!/N! \tag{5.17}$$

is another interesting 3-point function that includes nonlocal fields.

Here the fields ψ_k and σ_k are assumed normalized by (3.3) and (5.13), respectively; the fields μ_k are normalized by the same formula (5.13). With this normalization, Eq. (2.10) must of course also contain the constant Q_k.

We also state some results for some of the 4-point functions of the theory:

$$\langle\sigma_1(z_1,\bar{z}_1)\sigma_1^+(z_2,\bar{z}_2)\sigma_k(z_3,\bar{z}_3)\sigma_k^+(z_4,\bar{z}_4)\rangle$$

$$=(z_{12}\bar{z}_{12})^{-2d_1}(z_{34},\bar{z}_{34})^{-2d_k}G_{1,k}(x,\bar{x}), \tag{5.18}$$

where

$$x=\frac{z_{13}z_{24}}{z_{14}z_{23}},\quad \bar{x}=\frac{\bar{z}_{13}\bar{z}_{24}}{\bar{z}_{14}\bar{z}_{23}} \tag{5.19}$$

are the standard projective invariants and

$$G_{1,k}(x,\bar{x})=(x,\bar{x})^{-k/N(N+2)}\frac{\Gamma[1/(N+2)]\Gamma[N/(N+2)]}{\Gamma[(N+1)/(N+2)]\Gamma[2/(N+2)]}$$

$$\times\left\{\frac{\Gamma[(k+2)/(N+2)]\Gamma[(N-k+1)/(N+2)]}{\Gamma[(N-k)/(N+2)]\Gamma[(k+1)/(N+2)]}\times F^{(1)}(k,x)F^{(1)}(k,\bar{x})\right.$$

$$+\frac{(x\bar{x})^{(N+1-k)/(N+2)}}{(N+1-k)^2}\frac{\Gamma[1-k/(N+2)]\Gamma[(k+1)/(N+2)]}{\Gamma[k/(N+2)]\Gamma[1-(k+1)/(N+2)]}$$

$$\left.\times F^{(2)}(k,x)F^{(2)}(k,\bar{x})\right\}. \tag{5.20}$$

Here $F^{(1)}$ and $F^{(2)}$ are hypergeometric functions:

$$F^{(1)}(k,x)=F\left(\frac{k}{N+2},-\frac{1}{N+2},\frac{k+1}{N+2},x\right);$$

$$F^{(2)}(k,x)=F\left(\frac{N+1}{N+2},\frac{N-k}{N+2},\frac{2N-k+3}{N+2},x\right). \tag{5.21}$$

The 4-point function describing the mutual correlations of the fields μ_1 and σ_k is given by

$$\langle\mu_1(z_1,\bar{z}_1)\mu_1^+(z_2,\bar{z}_2)\sigma_k(z_3,\bar{z}_3)\sigma_k^+(z_4,\bar{z}_4)\rangle$$

$$=(z_{12}\bar{z}_{12})^{-2d_1}(z_{34}\bar{z}_{34})^{-2d_k}H_{1,k}(x,\bar{x}), \tag{5.22}$$

where

$$H_{1,k}(x,\bar{x})=\bar{x}^{k/N}(x\bar{x})^{-k/N(N+2)}\frac{\Gamma[1+1/(N+2)]\Gamma[N/(N+2)]}{\Gamma[(N+1)/(N+2)]\Gamma[2/(N+2)]}$$

$$\times\left\{\frac{\Gamma[(k+2)/(N+2)]\Gamma[(N-k+1)/(N+2)]}{\Gamma[(N-k)/(N+2)]\Gamma[1+(k+1)/(N+2)]}\times F^{(1)}(k,x)F^{(2)}(N-k,\bar{x})\right.$$

$$+x(x\bar{x})^{-(k+1)/(N+2)}\frac{\Gamma[1-k/(N+2)]\Gamma[(k+1)/(N+2)]}{\Gamma[k/(N+2)]\Gamma[1+(N+1-k)/(N+2)]}$$

$$\left.\times F^{(2)}(k,x)F^{(1)}(N-k,\bar{x})\right\}. \tag{5.23}$$

Finally, we have

$$\langle\sigma_1(z_1,\bar{z}_1)\sigma_1^+(z_3,\bar{z}_3)\varepsilon^{(1)}(z_2,\bar{z}_2)\varepsilon^{(1)}(z_4,\bar{z}_4)\rangle=(z_{13}\bar{z}_{13})^{-2d_1}(z_{24}\bar{z}_{24})^{-2D_1}E(x,\bar{x}), \quad (5.24)$$

where

$$E(x,\bar{x})=[(1-x)(1-\bar{x})]^{-2/N+2}\{F^{(3)}(x)F^{(3)}(\bar{x})+(x\bar{x})^{(N+4)/(N+2)}\rho F^{(4)}(x)F^{(4)}(\bar{x})\}, \quad (5.25)$$

$$F^{(3)}(x)=F\left(-\frac{1}{N+2},-\frac{4}{N+2},-\frac{2}{N+2},x\right),\quad F^{(4)}(x)=F\left(\frac{N+3}{N+2},\frac{N}{N+2},\frac{2(N+3)}{N+2},x\right), \quad (5.26)$$

$$\rho=4\frac{\Gamma^3[1-2/(N+2)]\Gamma^2[1+1/(N+2)]\Gamma[4/(N+2)]}{\Gamma^2[2+2/(N+2)]\Gamma^2[-1/(N+2)]\Gamma[2/(N+2)]\Gamma[1-4/(N+2)]}.$$

Here $\varepsilon^{(1)}(z,\bar{z})$ is the first of the neutral fields (4.20) with standard normalization. The other CFs are expressible generally speaking in terms of the multiple integrals introduced in Ref. 7.

In conclusion, we note that the close relationship to the WZ theory discussed in this section can also be used to prove that the positivity condition[6] is valid for the $Z_N\times\tilde{Z}_N$ theories.

6. DISCUSSION

In this paper we have constructed a conformal field theory which is Z_N-invariant and possesses order-disorder symmetry (self-duality). We now consider the problem of describing the statistical models with this type of critical behavior. As noted in the Introduction, the dimensions (4.15), (4.20) coincide with the exponents characterizing the "antiferromagnetic" critical points of the RSOS models.[11,10] In addition, it seems plausible that our solution might describe the self-dual critical points for the generalizations of the Ising model to Z_N. Although these Z_N-models have been widely discussed in the literature,[16–18,24–26] it will be helpful to examine them here in the context of the above formalism.

We associate with each node $\mathbf{r}$ in a square lattice L a variable $\sigma_\mathbf{r}$ that takes on the values ω^q, $q=0,1,\ldots,N-1$, where ω is defined by Eq. (2.1). If we assume that only nearest neighbors interact, we can write the partition function in the form

$$Z=\sum_{\{\sigma_\mathbf{r}\}}\exp\left\{-\sum_{\mathbf{r},\alpha=1,2}H(\sigma_\mathbf{r},\sigma_{\mathbf{r}+\mathbf{e}_\alpha})\right\}=\sum_{\{\sigma_\mathbf{r}\}}\prod_{\mathbf{r},\alpha}W(\sigma_\mathbf{r},\sigma_{\mathbf{r}+\mathbf{e}_\alpha}), \quad (6.1)$$

where $\mathbf{e}_\alpha=\mathbf{e}_1,\mathbf{e}_2$ are basis vectors of L, and the "pair Hamiltonian" $H(\sigma,\sigma')$ must be chosen so that the theory is Z_N-symmetric and C-invariant:

$$H(\omega\sigma,\omega\sigma')=H(\sigma^+,\sigma^{+\prime})=H(\sigma,\sigma'). \quad (6.2)$$

The function $W(\sigma,\sigma')$ can then be expressed as

$$W(\sigma,\sigma')\equiv\exp\{-H(\sigma,\sigma')\}=\sum_{k=0}^{N-1}w_k(\sigma^+\sigma')^k \quad (6.3)$$

with real nonnegative coefficients satisfying $w_k=w_{N-k}$ (we set $w_0=1$). The interaction in this system (Z_N Ising model) is thus described by real parameters w_k, $k=1,2,\ldots,\leqslant N/2$, whose number int $(N/2)$ stands for the integer part.

The dual transformation for the model (6.1), (6.3) is carried out by the standard technique.[15–18] The "spins" $\sigma_\mathbf{r}$ are replaced by the "dual spins" $\mu_{\tilde{\mathbf{r}}}$ associated with the nodes $\tilde{\mathbf{r}}$ of the dual lattice $\tilde{L}$, and the dual-spin interaction is again described by Eqs. (6.1) and (6.3) with

$$\tilde{w}_k=\left(1+\sum_{q=1}^{N-1}w_q\omega^{kq}\right)\left(1+\sum_{q=1}^{N-1}w_q\right)^{-1}. \quad (6.4)$$

The system is *self-dual* if

$$\tilde{w}_k=w_k,\quad k=1,2,\ldots,N-1. \quad (6.5)$$

One can show that Eqs. (6.5) define a hyperplane of dimensions int $(N/4)$ in the parameter space $\{w_k\}$.

For $N=2$, 3 the model (6.1), (6.3) coincides with the standard Z_2 Ising and Z_3 Potts models, and in this case condition (6.5) defines the critical points. Figure 4a shows a phase diagram of the Z_4 Ising model, which is a special case of the Ashkin-Teller model.[27] The self-dual Z_4-models are described by the line

$$w_2+2w_1=1, \quad (6.6)$$

and the model can be solved exactly for these parameter values.[24,28] All the points on segment AB of the line (6.6) are critical, and the exponents vary continuously along AB. One can show that AB contains a point C, defined by

$$w_1=\frac{\sin(\pi/16)}{\sin(3\pi/16)},\quad w_2=1-2w_1, \quad (6.7)$$

at which the critical theory is described[5)] by $Z_N\times\tilde{Z}_N$-theory with $N=4$. Figure 4b shows a phase diagram for the Z_5 Ising model. The "self-dual line" FB, defined by the equation

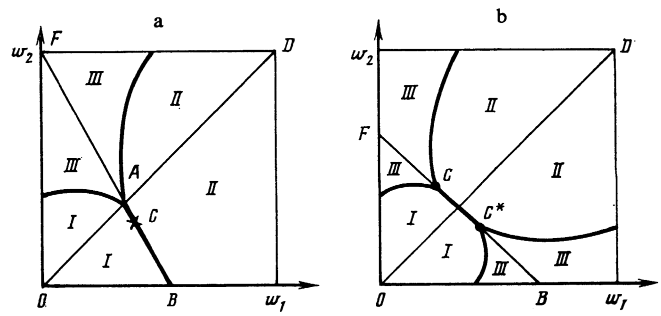

FIG. 4. Phase diagrams for the Z_4 (a) and Z_5 models (b). FB is the self-duality line. Phases I, II, and III correspond to the following values: I) $\langle\sigma\rangle\neq 0$, $\langle\mu\rangle=0$; II) $\langle\sigma\rangle=0$, $\langle\mu\rangle\neq 0$; III) $\langle\sigma\rangle=\langle\mu\rangle=0$.

$$w_1+w_2={}^1/_2(\sqrt{5}-1), \tag{6.8}$$

contains two symmetrically located "bifurcation points" C and C^* which in all probability are critical.[25]

The self-dual Z_N Ising models were shown in Ref. 26 to be completely integrable if

$$w_k=\prod_{l=0}^{k-1}\frac{\sin(\pi l/N+\pi/4N)}{\sin(\pi(l+1)/N-\pi/4N)}. \tag{6.9}$$

Our hypothesis is that the points (6.9) in the phase diagrams of the Z_N-models are critical and described by the conformal $Z_N\times\tilde{Z}_N$-theories constructed in this paper. Of course, this hypothesis requires verification. In particular, it would be interesting to carry out a numerical renormalization-group analysis of the neighborhoods of the points (6.9). If our hypothesis is correct, the corresponding "thermal" exponents should be as given by Eq. (1.4). We note here that the number of "energy" operators $\varepsilon^{(j)}$ in the $Z_N\times\tilde{Z}_N$-theory is equal to the dimension of the phase space for the Z_N Ising model.

APPENDIX A

We consider a conformal theory with parafermion currents $\psi_k(z)$, $k=1, 2, \ldots, N-1$, of dimension $(\Delta_k, 0)$ where

$$\Delta_k=\Delta_{N-k}. \tag{A.1}$$

We assume that the operator algebra for the currents ψ_k is consistent with Z_N-symmetry and is described as in (3.7) by the expansions

$$\psi_k(z)\psi_{k'}(z')=\tilde{c}_{kk'}(z-z')^{\Delta_{k+k'}-\Delta_k-\Delta_{k'}}\times\left[\psi_{k+k'}(z')+(z-z')\frac{\Delta_{k+k'}+\Delta_k-\Delta_{k'}}{2\Delta_{k+k'}}\partial_{z'}\psi_{k+k'}(z')+O((z-z')^2)\right]; \tag{A.2}$$

$$\psi_k(z)\psi_k^+(z')=(z-z')^{-2\Delta_k}\left[I+(z-z')^2\frac{2\Delta_k}{c}T(z')+O((z-z')^3)\right].$$

Here the coefficients $\partial_z\psi$ and $T(z)$ are specified by the conformal symmetry requirement,[3] c is the central charge of the Virasoro algebra, the $\tilde{c}_{k,k'}$ are the structure constants, and $\psi_k^+=\psi_{N-k}$. We see from (A.2) that the mutual locality exponent of the fields ψ_k and $\psi_{k'}$ is equal to

$$\gamma_{k,k'}=\Delta_{k+k'}-\Delta_k-\Delta_{k'} \tag{A.3}$$

which is consistent with the operator algebra (A.2) provided the dimensions Δ_k satisfy

$$\Delta_{k_1+k_2}+\Delta_{k_1+k_3}+\Delta_{k_2+k_3}=\Delta_{k_1+k_2+k_3}+\Delta_{k_1}+\Delta_{k_2}+\Delta_{k_3}(\mathrm{mod}\,Z). \tag{A.4}$$

Using (A.1), we find that the general solution of (A.4) is of the form

$$\Delta_k=pk(N-k)/N+M_k, \tag{A.5}$$

where p and M_k ($k=1,2,\ldots,N-1$) are arbitrary integers subject to $M_{N-k}=M_k$. The simplest solution ($p=1$, $M_k=0$) was examined in the main part of this paper. Here we will present another solution of the associativity algebra (A.2) for which

$$p=2, \quad M_k=0. \tag{A.6}$$

With this choice the solution contains a single free parameter λ; the structure constants are given by

$$\tilde{c}_{k,k'}^2=c_{k,k'}^2\frac{\Gamma(k+k'+\lambda)\Gamma(N+\lambda-k)\Gamma(N+\lambda-k')\Gamma(\lambda)}{\Gamma(N+\lambda-k-k')\Gamma(k+\lambda)\Gamma(k'+\lambda)\Gamma(N+\lambda)}, \tag{A.7}$$

where the $c_{k,k'}$ are defined by (3.13), and the central charge is given by

$$c=\frac{4(N-1)(N+\lambda-1)\lambda}{(N+2\lambda)(N+2\lambda-2)}. \tag{A.8}$$

The study of the representations of this algebra (and in particular, of the positivity condition[6]) lies beyond the scope of this paper. We merely note that for $\lambda=1/2$ the algebra (A.2), (A.5)–(A.7) coincides with the "even" subalgebra of the Z_{2N}-algebra (3.7) ($\tilde{c}_{k,k'}=c_{2k,2k'}$), while for $\lambda=1$ it coincides with the "square" of the Z_N-algebra (3.7) ($\tilde{c}_{k,k'}=c_{k,k'}^2$). Finally, for $N=3$ and $\lambda=1/4$ this algebra arises in the minimal model with $p=6$ ("tricritical Z_3 Potts model").

APPENDIX B

Here we will derive the fundamental formulas for a conformal theory invariant under the $\widehat{su}(2)\times\widehat{su}(2)$ current algebra. Because this theory describes a conformally invariant solution of the two-dimensional Wess-Zumino model,[29,30] we will call it the WZ theory. Some additional details may be found in Ref. 13.

The WZ theory contains local fields ("currents") $J^\alpha(z)$, $\bar{J}^\alpha(\bar{z})$ ($\alpha=0, +, -$), for which the operator expansions

$$J^\alpha(z)J^\beta(z')=\frac{Nq^{\alpha\beta}}{(z-z')^2}+\frac{f_\gamma^{\alpha\beta}}{(z-z')}J^\gamma(z')+O(1) \tag{B.1}$$

hold, where the $f_\gamma^{\alpha\beta}$ are the structure constants of $su(2)$ and the tensor $q^{\alpha\beta}$ is given by the associated Killing form. The nonzero components are given by

$$f^{0+}_{+}=-f^{+0}_{+}=-f^{0-}_{-}=f^{-0}_{-}=1,$$

$$f^{+-}_{0}=-f^{+-}_{0}=2, \qquad q^{00}={}^1/_2q^{+-}={}^1/_2q^{-+}=1. \tag{B.2}$$

The parameter N in (B.1) is called the central charge of the current algebra and takes positive integer values, since otherwise the algebra (B.1) would not have any suitable unitary representations.

Let $\{A\}$ be a complete space of mutually local fields describing the WZ theory. The action of the operators J_n^α, $n=0, \pm1, \pm2, \ldots$, on $\{A\}$ is then determined by

$$J_n{}^\alpha\Phi(0,0)=\oint\frac{dz}{2\pi i}J^\alpha(z)z^n\Phi(0,0), \tag{B.3}$$

where Φ is any field in $\{A\}$ and the integration path encloses the point $z=0$. The expansion (B.1) implies that the J_n^α obey the commutation relation

$$[J_n{}^\alpha, J_m{}^\beta]=f_\gamma{}^{\alpha\beta}J_{n+m}^\gamma+\frac{N}{2}nq^{\alpha\beta}\delta_{n+m,0} \tag{B.4}$$

and therefore form a Kac-Moody algebra. The operators $\bar{J}_n^\alpha$ are defined just as in (B.3) [with $J^\alpha(z)$ replaced by $\bar{J}^\alpha(\bar{z})$] and also satisfy (B.4); moreover, $\bar{J}$ and J commute. The operators J_0^α and $\bar{J}_0^\alpha$ thus generate a global $SU(2)\times SU(2)$-symmetry of the WZ theory; the corresponding Casimir operators are

$$J_0^2=q_{\alpha\beta}J_0^\alpha J_0^\beta, \quad \bar{J}_0^2=q_{\alpha\beta}\bar{J}_0^\alpha\bar{J}_0^\beta, \tag{B.5}$$

where $q_{\alpha\beta}q^{\beta\gamma}=\delta_\gamma^\alpha$ are called the "right" and "left" isotopic spins, respectively.

The component $T^{(WZ)}(z)$ ($\bar{T}^{(WZ)}(\bar{z})$) of the energy-momentum tensor of the WZ theory can be expressed quadratically in terms of the currents $J^\alpha(z)$ ($\bar{J}^\alpha(\bar{z})$):

$$(N+2)T^{(WZ)}(z)=q_{\alpha\beta}:J^\alpha(z)J^\beta(z):, \tag{B.6}$$

where the product of fields on the right-hand side is regularized in the standard way by subtracting the singular term of the operator expansion.[13] The corresponding operators $L_n^{(WZ)}$ are given by

$$(N+2)L_n^{(WZ)}=\sum_{m=-\infty}^{\infty}q_{\alpha\beta}:J_m^\alpha J_{n-m}^\beta: \tag{B.7}$$

and generate a Virasoro algebra (4.1) with central charge

$$c^{(WZ)}=3N/(N+2). \tag{B.8}$$

The symbol :: in (B.7) denotes the standard normal ordering in which operators J_n with $n<0$ appear on the left.

The space $\{A\}$ contains invariant fields $\Phi^{(j)}$, $j=0$, 1/2, 1, 3/2, . . . , satisfying the equations

$$\begin{aligned} &J_n^\alpha\Phi^{(j)}=\bar{J}_n^\alpha\Phi^{(j)}=0, \quad n>0;\\ &J_0^2\Phi^{(j)}=\bar{J}_0^2\Phi^{(j)}=j(j+1)\Phi^{(j)}. \end{aligned} \tag{B.9}$$

Each invariant field $\Phi^{(j)}\in\{A\}$ is an $SU(2)\times SU(2)$ tensor with $(2j+1)^2$ components $\Phi_{m,\bar{m}}^{(j)}$ ($m, \bar{m}=-j, -j+1, \ldots, j$) that can be found from the equations

$$J_0^0\Phi_{m,\bar{m}}^{(j)}=m\Phi_{m,\bar{m}}^{(j)}, \quad \bar{J}_0^0\Phi_{m,\bar{m}}^{(j)}=\bar{m}\Phi_{m,\bar{m}}^{(j)}. \tag{B.10}$$

The fields $\Phi_{m,\bar{m}}^{(j)}$ have the same dimension (D_j, D_j) regardless of m, $\bar{m}$, where

$$D_j=j(j+1)/(N+2). \tag{B.11}$$

The $\Phi^{(j)}$ can be orthonormalized as follows:

$$\begin{aligned} &\langle\Phi_{m_1,\bar{m}_1}^{(j_1)}(z,\bar{z})\Phi_{m_2,\bar{m}_2}^{(j_2)}(0,0)\rangle\\ &=(-)^{2j_1-m_1-\bar{m}_1}\delta^{j_1j_2}\delta_{m_1+m_2,0}\delta_{\bar{m}_1+\bar{m}_2,0}(z\bar{z})^{-2D_{j_1}}. \end{aligned} \tag{B.12}$$

Taken together, the independent fields obtained from a given invariant field $\Phi^{(j)}$ by successive application of the operators $J_n^\alpha, \bar{J}_n^\alpha$ with $n<0$ form a subspace $[\Phi^{(j)}]_J\in\{A\}$ which corresponds to the highest weight of the $\widehat{su}(2)\times\widehat{su}(2)$ current algebra.

The correlation functions of the invariant fields in the WZ theory satisfy the linear differential equations

$$\begin{aligned} &\Big\{(N+2)\frac{\partial}{\partial z_i}-\sum_{\substack{i'=1\\ i'\neq i}}^{n}\frac{q_{\alpha\beta}S_i^\alpha S_{i'}^\beta}{z_i-z_{i'}}\Big\}\\ &\times\langle\Phi^{(j_1)}(z_1,\bar{z}_1)\ldots\Phi^{(j_n)}(z_n,\bar{z}_n)\rangle=0, \end{aligned} \tag{B.13}$$

where the $(2j_k+1)\times(2j_k+1)$ matrices $S_k^\alpha=S^\alpha_{m_k,m_k'}$ act on the "right" tensor indices m_k of the field $\Phi_{m_k,\bar{m}_k}^{(j_k)}$ and correspond to the $su(2)$-algebra representation of spin j_k:

$$[S_k^\alpha, S_k^\beta]=f^\alpha{}_\gamma S_k^\gamma, \quad q_{\alpha\beta}S_k^\alpha S_k^\beta=j_k(j_k+1). \tag{B.14}$$

The equations that follows from (B.13) by replacing z by $\bar{z}$ are of course also valid; in this case the matrices S_k act on the "left" indices $\bar{m}_k$ of the fields $\Phi^{(j_k)}$.

One can show that for a fixed central charge N in (B.1), the space $\{A\}$ contains exactly $N+1$ invariant fields $\Phi^{(j)}$, $j=0$, 1/2, 1, 3/2, . . . , $N/2$, where the field $\Phi^{(0)}$ coincides with the identity operator I. Moreover, $\{A\}$ is the direct sum

$$\{A\}=\bigoplus_{j=0}^{N/2}[\Phi^{(j)}]_J, \tag{B.15}$$

and the fields on the right-hand side generate a closed operator algebra. For example, the product of two invariant fields is expanded in the form

$$\begin{aligned} &\Phi_{m_1,\bar{m}_1}^{(j_1)}(z,\bar{z})\Phi_{m_2,\bar{m}_2}^{(j_2)}(0,0)\\ &=\sum_j\sum_{m,\bar{m}=-j}^{j}C\begin{pmatrix} j, m, \bar{m}\\ j_1, m_1, \bar{m}_1\\ j_2, m_2, \bar{m}_2\end{pmatrix}(z\bar{z})^{D_j-D_{j_1}-D_{j_2}}[\Phi_{m,\bar{m}}^{(j)}]_J(0,0), \end{aligned} \tag{B.16}$$

where at $j_1\geqslant j_2$ the sum over j contains terms with $j=j_1-j_2-k$, $k=0, 1, \ldots, \min(2j_2, N-2j_1)$. The square brackets $[\Phi^{(j)}]_J$ in (B.16) denote the contribution from all fields in the corresponding subspace; the field $\Phi_{m,\bar{m}}^{(j)}$ occurs with multiplicity I, while the coefficients multiplying the other fields in $[\Phi^{(j)}]_J$ are uniquely specified by the $\widehat{su}(2)\times\widehat{su}(2)$ symmetry requirement. The numerical constants **C** are called the structure constants of the operator algebra. With the normalization (B.12), C is symmetric under permutations of all three subscripts (rows) and coincides with the normalization factor in the three-point function:

$$\begin{aligned} &\langle\Phi_{m_1,\bar{m}_1}^{(j_1)}(z_1,\bar{z}_1)\Phi_{m_2,\bar{m}_2}^{(j_2)}(z_2,\bar{z}_2)\Phi_{m_3,\bar{m}_3}^{(j_3)}(z_3,\bar{z}_3)\rangle\\ &=\mathbf{C}\begin{pmatrix} j_1 & m_1 & \bar{m}_1\\ j_2 & m_2 & \bar{m}_2\\ j_3 & m_3 & \bar{m}_3\end{pmatrix}(z_{12}\bar{z}_{12})^{D_{j_3}-D_{j_1}-D_{j_2}}\\ &\times(z_{13}\bar{z}_{13})^{D_{j_2}-D_{j_1}-D_{j_3}}(z_{23}\bar{z}_{23})^{D_{j_1}-D_{j_2}-D_{j_3}}. \end{aligned} \tag{B.17}$$

The structure constants for the WZ theory can be found explicitly (the calculations will be presented elsewhere). The result is

$$\mathbf{C}\begin{pmatrix} j_1 & m_1 & \bar{m}_1\\ j_2 & m_2 & \bar{m}_2\\ j_3 & m_3 & \bar{m}_3\end{pmatrix}=\begin{bmatrix} j_1 & j_2 & j_3\\ m_1 & m_2 & m_3\end{bmatrix}\begin{bmatrix} j_1 & j_2 & j_3\\ \bar{m}_1 & \bar{m}_2 & \bar{m}_3\end{bmatrix}\rho(j_1,j_2,j_3), \tag{B.18}$$

where the first two factors are the Wigner 3j-symbols,[31]

$$\frac{\rho^2(j_1,j_2,j_3)}{(2j_1+1)(2j_2+1)(2j_3+1)}$$

$$=\left[\Gamma\left(\frac{N+3}{N+2}\right)\Gamma\left(1-\frac{2j_1+1}{N+2}\right)\right.$$

$$\left.\times\Gamma\left(1-\frac{2j_2+1}{N+2}\right)\Gamma\left(1-\frac{2j_3+1}{N+2}\right)\right]$$

$$\times\left[\Gamma\left(\frac{N+1}{N+2}\right)\Gamma\left(1+\frac{2j_1+1}{N+2}\right)\right.$$

$$\left.\times\Gamma\left(1+\frac{2j_1+1}{N+2}\right)\Gamma\left(1+\frac{2j_3+1}{N+2}\right)\right]^{-1}$$

$$\times\frac{\Pi^2(j_1+j_2+j_3+1)\,\Pi^2(j_1+j_2-j_3)\,\Pi^2(j_1+j_3-j_2)\,\Pi^2(j_2+j_3-j_1)}{\Pi^2(2j_1)\,\Pi^2(2j_2)\,\Pi^2(2j_3)},\tag{B.19}$$

and Π denotes the function

$$\Pi(j)=\prod_{k=1}^{j}\frac{\Gamma(1+k/(N+2))}{\Gamma(1-k/(N+2))}.\tag{B.20}$$

[1)] In statistical physics the positivity condition must hold for systems describable by a self-adjoint transition matrix. There are interesting models (e.g., random walks without self-crossings) for which the positivity condition fails.

[2)] Actually, our solution with $N=3$ coincides only with the "even" section of the $p=5$ minimal model. The physical significance of the fields comprising the "odd" sector remains unclear.

[3)] Fields with this locality property were used to advantage by Sato, Miwa, and Jimbo in their research into holonomic quantum field theory; see, e.g., Ref. 21.

[4)] The algebra (4.11)–(4.13) has been studied in the mathematical literature[23] in connection with the representations of the Kac-Moody algebra. In the form employed here, this relationship was suggested by A. A. Belavin

[5)] We are grateful to S. V. Pokrovskiĭ for carrying out this verification.

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