Phase diagram with respect to the coupling constant and the local limit in quantum electrodynamics

V. A. Miranskiĭ

Institute of Theoretical Physics, Academy of Sciences of the Ukrainian SSR, Kiev

(Submitted 25 October 1984)
Zh. Eksp. Teor. Fiz. **88**, 1514–1526 (May 1985)

The phase diagram with respect to the coupling constant in quantum electrodynamics (QED) and its relationship to spontaneous breakdown of chiral symmetry are discussed. A breakdown mechanism based on the "collapse" phenomenon is considered, and on this basis the results of some recent computer calculations of QED on a lattice are interpreted. The existence problem for a nontrivial local limit in QED is analyzed and the following hypothesis is offered: in the zero charge case (renormalization constant $Z_3 = 0$) local QED for a certain fixed value of the priming coupling constant $\alpha^{(0)} = \alpha_c \sim 1$ involves a non-trivial S-matrix. The physical meaning of the hypothetical local theory is discussed.

§1. INTRODUCTION

The determination of the structure of the phase diagram with respect to the coupling constant is one of the most important problems of quantum field theory. In particular, it is closely connected with the existence problem for a nontrivial local limit (the ultraviolet cutoff parameter $\Lambda \to \infty$) in the asymptotic non-free theories.

Recent computer studies of quantum electrodynamics (QED) on a lattice have obtained results touching on the spontaneous breakdown of chiral symmetry.[1] This paper aims to give a simple physical interpretation of these results and to apply it to the existence problem for a nontrivial local limit in quantum electrodynamics.

In 1954 Gell-Mann and Low[2] showed that a nontrivial local QED can exist only if the bare coupling constant $\alpha^{(0)}$ determined by the ultraviolet-stable zero of the renormalization group β-function,

$$\beta_{QED} = \alpha^{(0)} \frac{\partial}{\partial \ln \mu} Z_{3\mu} \approx -\alpha^{(0)} \frac{\partial}{\partial \ln \Lambda} Z_{3\mu}, \tag{1}$$

where μ is the renormalization group parameter and $Z_{3\mu}$ is the renormalization constant for the photon propagator. The existence problem for such a zero became especially acute after the appearance of papers by Landau and Pomeranchuk[3] and Fradkin[4] arguing that the local limit in QED arises in the free theory, and in particular that in the limit $\Lambda \to \infty$ for arbitrary values of the bare coupling constant vacuum polarization effects lead to the vanishing of the variable coupling constant $\alpha(r)$ at all non-zero distances; $\alpha(r) = 0$ for $r > 0$ and $\alpha(0) = \alpha^{(0)}$ (the zero charge case; the renormalization constant $Z_{3\mu} = 0$). Subsequently the possibility of a nontrivial zero of the function $\beta_{\rm qed}$ was investigated in a program of "finite" QED,[5] but without a definite answer. Recently, new arguments have been put forward, supporting the view that such a zero of the β-function (1) cannot exist.[6]

This problem is closely related to the renormalization of the charge. References 2–5 used the relation

$$\alpha_\mu = Z_{3\mu} \alpha^{(0)}(\Lambda), \tag{2}$$

which is equivalent to (1). It has been proved in every order of perturbation theory and apparently should hold for the exact theory. However, as proved in Ref. 7, in the asymptotic nonfree field theories, for large values of the bare coupling constant the renormalization ratios may change essentially if we take account of the dynamical generation of particle masses. This notion originated in the mechanism of spontaneous breakdown of chiral symmetry in massless QED proposed in Refs. 8 and 9 (reviewed in Ref. 10). For supercritical values $\alpha^{(0)}(\Lambda) > \alpha_c \sim 1$ of the bare coupling constant this mechanism leads to additional mass divergence (for more detail see §2). For renormalization group theory the critical value α_c, separating the massless and massive phases, is the ultraviolet-stable fixed point. But this value is defined, not by the zero of the β-function (1) related to the subcritical phase ($\alpha^{(0)}(\Lambda) < \alpha_c$) but by the zero in the supercritical phase. The value of α_c defines a local theory.

Curiously, a similar phenomenon was observed[11] in $(2 + n^{-1})$-dimensional φ^{4n+2} models ($n \geqslant 1$; in the classical limit these models are scale-invariant).

In this paper we show that the basic results of computations in lattice QED[1] may be easily understood via the dynamic mechanism of chiral symmetry breakdown. Also we discuss the extent to which a departure from the approximation with "frozen" fermions used in Ref. 1 (an approximation neglecting the contribution of the diagrams with fermion loops) may influence the results. This analysis leads us to the following unexpected possibility: the zero charge case ($Z_{3\mu} = 0$) does not mean the theory is necessarily trivial in the local limit; the S-matrix of the local QED with a fixed value of the bare coupling constant can be nontrivial. A characteristic peculiarity of such a local theory is the appearance of a new induced vertex in the Yukawa-type interaction of fermions and antifermions with the constituent pseudoscalar boson.

This paper has the following structure. In §2 we briefly consider the mechanism for dynamic breakdown of chiral symmetry in QED [Refs. 8–10] and the influence of this mechanism on the structure of renormalization. In §3 we discuss certain properties of it that are related to the singularities of the passage to the local limit in the theory. This is important since anomalies of various kinds may affect the character of the breakdown. In particular we solve the well-known Goldstein problem in this way. In §4 we interpret the results of some computer calculations on lattice QED and discuss the existence problem for a nontrivial local limit in quantum electrodynamics.

§2. THE MECHANISM FOR SPONTANEOUS BREAKDOWN OF CHIRAL SYMMETRY IN QED

The mechanism for spontaneous breakdown of chiral symmetry in QED [Refs. 7–10] starts from the analogy between this effect and the generation of electron-positron pairs in a supercritical Coulomb field. We known[12] that for $Z > Z_c \sim 137$ ($\alpha = Ze^2/4\pi r > 1$) the Dirac operator with Coulomb potential $V(r) = -Ze^2/4\pi r$ is poorly defined and needs correction at small distances.[1)] For instance,

$$V(r) = -\frac{Ze^2}{4\pi r} \to \tilde{V}(r) = -\frac{Ze^2}{4\pi}\left[\frac{\theta(r-r_0)}{r} + \frac{\theta(r_0-r)}{r_0}\right]. \quad (3)$$

To show the role of the cutoff parameter $\Lambda = r_0^{-1}$ in this problem we introduce an expression for the energy $\varepsilon^{(n)}$ of the $nS_{1/2}$-levels for a light electron (mass $m \ll |\varepsilon^{(n)}|$)[8]:

$$\varepsilon^{(n)} \approx \varepsilon_0^{(n)} - m/2 - im^2/50|\varepsilon_0^{(n)}|, \quad (4)$$

where the energy $\varepsilon_0^{(n)}$ of a massless electron is

$$\varepsilon_0^{(n)} \approx a\Lambda(\sin\varphi - i\cos\varphi)\exp\{-\pi n/(\alpha^2-1)^{1/2}\}, \quad n=1,2,\ldots;$$
$$a \approx 0.4, \quad \varphi = -\frac{\pi}{2}\operatorname{cth}\pi \approx -\frac{\pi}{2}\cdot 1.004. \quad (5)$$

The generally accepted interpretation of the levels with $\mathrm{Re}\,\varepsilon < 0$ is that the level $\varepsilon^{(n)}$ defines a positron state with energy

$$\varepsilon_p^{(n)} = -\varepsilon^{(n)}; \quad \mathrm{Re}\,\varepsilon_p^{(n)} > 0, \quad \mathrm{Im}\,\varepsilon_p^{(n)} < 0,$$

corresponding to an emitted positron wave.[2)] In the limit the energy $\varepsilon^{(n)}$ diverges, representing the case of a falling in to the center, i.e., collapse.

Reference 8 proposed the hypothesis that a similar phenomenon may occur in QED with a large enough value of the bare coupling constant, but here it leads to spontaneous breakdown of chiral symmetry.[3)] This hypothesis was employed (with the ladder approximation) in Ref. 9; for detailed exposition see the survey in Ref. 10. For convenience we postpone to Appendix I:[4)] (a) the equations for the dynamic mass function of the fermion, $m_d(q^2) = B_d(q^2)/A(q^2)$ (relating to the spontaneous breakdown of chiral symmetry; for the fermion propagator we have $S(q) = [-\hat{q}A(q^2) + B_d(q^2)]^{-1}$); and (b) the equations for the wave function of the Goldstone boson. Here we present the basic results: the chiral group is $SU_L(K) \times SU_R(K)$, where K is the number of fermions. The dynamic mass m_d is given by the equation

$$m_d = \Lambda f(\alpha^{(0)}), \quad (6)$$

where, for values of $\alpha^{(0)}$ near the critical, i.e., for

$$(\alpha_0 - \alpha_c)/\alpha_c \ll 1, \quad \alpha_c = \pi/3,$$

we have

$$f(\alpha^{(0)}) \approx 4\exp(-\pi/2\gamma), \quad \gamma = {}^1/_2(\alpha^{(0)}/\alpha_c - 1)^{1/2}. \quad (7)$$

The Bethe-Salpeter (B-S) wave function for $K^2 - 1$ Goldstone bosons in the Euclidean region has the form

$$\chi^r = C\lambda^r\gamma_5\chi(q^2), \quad \chi(q^2) = (q^2 + m_d^2)^{-1}F({}^1/_2 + i\gamma, {}^1/_2 - i\gamma, 2; -q^2/m_d^2), \quad (8)$$

where λ^r is the $(K^2 - 1)$-matrix of the fundamental representation of the group $SU(K)$, q is the relative momentum of a fermion-antifermion pair becoming a boson, F is the hypergeometric function, and the renormalization constant C can in principle be determined from the normalization of the wave function.

Following Ref. 7, let us now consider the renormalization in this problem. In the local limit, $\Lambda \to \infty$, $\alpha^{(0)} > \alpha_c = \pi/3$, the mass m_d diverges, for the following reason: If we expand the hypergeometric function[14] we find that as $q^2 \to \infty$ the function $\chi(q^2)$ has the form

$$\chi(q^2) \underset{q^2\to\infty}{\approx} \frac{1}{q^2}\left(\frac{q^2}{m_d^2}\right)^{-1/2}\left[\frac{\operatorname{cth}\pi\gamma}{\pi\gamma(\gamma^2 + {}^1/_4)}\right]^{1/2} \times \sin\left[\gamma\ln\frac{q^2}{m_d^2} - \operatorname{arctg}2\gamma + \Sigma(\gamma)\right], \quad (9)$$

where

$$\Sigma(\gamma) = \arg\left[\frac{\Gamma(1+2i\gamma)}{\Gamma^2({}^1/_2 + i\gamma)}\right],$$

and Γ is Euler's gamma function. In the local limit, and for arbitrary mass m_d, the wave function has an infinite number of zeroes. This is a typical symptom of "falling in to the center," i.e., collapse,[13] in which the energy of the ground state is not bounded below and therefore the energy gap (mass gap) is infinite.

To eliminate this divergence we need to renormalize the bare parameters. Taking account of the fact that $m^{(0)} \equiv 0$ for the chiral-invariant Lagrangian bare mass we have a unique such parameter, the bare coupling constant $\alpha^{(0)}$. The relations (6) and (7) imply that the mass m_d remains constant as $\Lambda \to \infty$ if the bare coupling constant is fixed:

$$\alpha^{(0)}(\Lambda) = \alpha_c + \pi^2\alpha_c/\ln^2(4\Lambda/m_d) \underset{\Lambda\to\infty}{\longrightarrow} \alpha_c = \pi/3. \quad (10)$$

For the renormalization group, the ultraviolet-stable fixed point α_c is a critical value separating the massless from the massive phases. The appearance of such a point in the ladder approximation is defined by the nonperturbative interaction dynamics.

Note that the mass divergence (6) differs from the loop divergences in perturbation theory. The latter arise from processes in which the particle number is not conserved, while the divergence of (6) is related to the singularities in the short range behavior of the exchange interaction which conserve particle number. Since such divergence in quantum mechanics (see, for instance, equation (5)) it is natural to call this a quantum-mechanical divergence.

We note also the two following features:

1) Since in the photon propagator the renormalization constant satisfies $Z_{3\mu} = 1$ in the ladder approximation, the

renormalization (10) violates the renormalization relation (2);

2) in the local limit (10) the wave function satisfies

$$\chi(q^2)=(q^2+m_d^2)^{-1}F\left(\frac{1}{2},\frac{1}{2},2;-\frac{q^2}{m_d^2}\right)$$

$$\underset{q^2\to\infty}{\approx}\frac{2}{\pi q^2}\left(\frac{q^2}{m_d^2}\right)^{-1/2}\ln\frac{q^2}{m_d^2},\qquad(11)$$

and therefore the renormalization (10) changes the form of the wave function, and in particular the oscillations vanish [cf. Eq. (9)]. We recall that in standard renormalization theory the perturbations occur in an equation of the form

$$G^{(\mu)}(\{q\},\ \alpha_\mu)=Z(\Lambda/\mu,\ \alpha_\mu)G^{(\Lambda)}(\{q\},\ \alpha^{(0)})$$
$$+\text{Small correction terms.}\qquad(12)$$

($G^{(\mu)}$ and $G^{(\Lambda)}$ are Green's functions, the first renormalized and the second not). Therefore to within small correction terms of the form μ/Λ, q/Λ, etc., the renormalized and non-renormalized Green's functions have the same form as functions of $\{q\}$. The breakdown of this property as a result of the renormalization (10) is easy to understand: assuming the cutoff and keeping the mass m_d finite we eliminate the collapse and so eliminate its consequence, namely the oscillations.

In this approximation the phase diagram with respect to the coupling constant for massless QED has the following form: for all subcritical values $\alpha^{(0)}<\alpha_c=\pi/3$ the ρ-function vanishes (there are no ultraviolet divergences) and all these values of $\alpha^{(0)}$ make a line of fixed points; in the mass phase, with $\alpha^{(0)}>\alpha_c$ we have the renormalized coupling constant, which leads to the ultraviolet-stable fixed point $\alpha^{(0)}=\alpha_c$. In the exact theory the shape of the phase diagram depends on other renormalizations as well. This question will be discussed in §4. Here we make the following observation: It is essential to note that our picture of the reconstruction of a vacuum is related to the collapse phenomenon, whose existence in relativistic quantum mechanics follows immediately from the uncertainty principle[5]. Moreover, collapse for supercritical values of the coupling constant occurs in some exactly soluble two-dimensional models, in particular the sine-Gordon equation[7–10] and as we noted in the Introduction, in $(2+n^{-1})$-dimensional φ^{4n+2} models[1] with $n\geqslant 1$. These observations support the view that the phenomenon we are considering is not an artifact of the ladder approximation.

We shall return to a discussion of these questions in §4, but first we deal in the following section with some properties of the dynamic breakdown of chiral symmetry in QED that are related to the properties of the passage to the local limit in the theory.

§3. THE LOCAL LIMIT AND THE NATURE OF THE BREAKDOWN OF CHIRAL SYMMETRY IN QED

In field theory, anomalies signal a breakdown of symmetry. The most widely known example is the Adler-Bell Jackiw (ABJ) anomaly in a singlet axial vector current. Even before the discovery of this anomaly it was known from "finite" QED (Ref. 5) that in general the vanishing of the bare mass of a fermion does not guarantee the conservation of the axial vector currents in the local theory.[5,15] In the literature this is sometimes referred to as the Johnson-Pagels anomaly. In this section we shall show that it can be eliminated by a suitable procedure for passing to the local limit, as opposed to the ABJ anomaly. We shall also show that in this way we can solve the familiar Goldstein problem.[16]

Consider a QED with K fermions. In the local limit K^2-1 axial vector currents $j^r_{5\mu}=\bar\psi\gamma_\mu\gamma_5\lambda^r\psi$, $r=1,2,...K^2-1$, free of the ABJ anomaly, satisfy the equation

$$\partial^\mu j_{5\mu}{}^r=\lim_{\Lambda\to\infty}m^{(0)}(\Lambda)\,(\bar\psi\gamma_5\lambda^r\psi)_\Lambda,\qquad(13)$$

where $m^{(0)}(\Lambda)$ is the bare mass of the fermion. It is essential that the operator $(\bar\psi\gamma_5\lambda^r\psi)_\Lambda$ be expressible in terms of components and depend on the cutoff parameter Λ:

$$(\bar\psi\gamma_5\lambda^r\psi)_\Lambda=Z_{m\mu}{}^{-1}(\bar\psi\gamma_5\lambda^r\psi)_\mu,\qquad(14)$$

where $(\bar\psi\gamma_5\lambda^r\psi)_\mu$ is the renormalized component operator. We find from (13) and (14) that the axial-vector currents are conserved if

$$\lim_{\Lambda\to\infty}m^{(0)}(\Lambda)Z_{m\mu}{}^{-1}=0.\qquad(15)$$

We show in Appendix II that in the ladder approximation the renormalization constant satisfies the relations

$$Z_{m\mu}\approx(\mu^2/\Lambda^2)^{1/2-\gamma'},\quad \gamma'=i\gamma={}^1/_2(1-3\alpha^{(0)}/\pi)^{1/2}\qquad(16)$$

for subcritical values $\alpha^{(0)}<\pi/3$ and

$$Z_{m\mu}\approx\mu/\Lambda\qquad(17)$$

for supercritical values of $\alpha^{(0)}$. It is clear from this and from (15) that the vanishing of the bare mass in the local limit, $m^{(0)}\equiv\lim_{\Lambda\to\infty}m^{(0)}(\Lambda)=0$ does not guarantee the conservation of the axial-vector currents. A sufficiently sharp decrease (like $0(Z_{m\mu})$) in the excitation mass as $\Lambda\to\infty$ does provide such a guarantee. In particular, this condition is satisfied for $m^{(0)}(\Lambda)\equiv 0$, i.e., if the Lagrangian theory with cutoff is chosen to be chiral-invariant and the local theory is considered as a limit of it (as in §2).

We support this conclusion by an immediate consideration of the equation for the mass function of the fermion [cf. App. II, Eq. (II.3)]:

$$m(q^2)=m^{(0)}(\Lambda)$$
$$+\frac{3\alpha^{(0)}}{4\pi}\int_0^{\Lambda^2}dk^2\frac{m(k^2)}{k^2+m^2}\left[\frac{k^2}{q^2}\theta(q^2-k^2)+\theta(k^2-q^2)\right]\qquad(18)$$

(for $m^{(0)}(\Lambda)=0$, the mass satisfies $m=m_d$). The solution of this equation has the same form as the solution of the equation with $m^{(0)}(\Lambda)=0$:

$$m(q^2)=CF\left(\frac{1}{2}+\gamma',\frac{1}{2}-\gamma',2;\frac{-q^2}{m^2}\right),$$

the normalizing constant is $C=\xi m$, where ξ is a numerical parameter; it must however satisfy another boundary condition for $q^2=\Lambda^2$

$$\left(q^2\frac{dm(q^2)}{dq^2}+m(q^2)\right)\Big|_{q^2=\Lambda^2}=m^{(0)}(\Lambda). \quad (19)$$

Using the asymptotic expansion of the hypergeometric function for $m^2/\Lambda^2 \ll 1$ (Ref. 14), we find from (19) the following equations:

$$\xi m\Gamma(2\gamma')/\Gamma^2(1/2+\gamma')=(\Lambda/m)^{1-2\gamma'}m^{(0)}(\Lambda) \quad (20)$$

for $\alpha^{(0)}<\pi/3$ and

$$\xi m\left[\frac{\operatorname{cth}\pi\gamma}{\pi\gamma}\right]^{1/2}\sin\left[\gamma\ln\frac{\Lambda^2}{m^2}+\Sigma(\gamma)\right]=\frac{\Lambda}{m}m^{(0)}(\Lambda) \quad (21)$$

for the supercritical values $\alpha^{(0)}>\pi/3$.

With the condition (15) in mind, we find from (16) and (20) that for the subcritical values $\alpha^{(0)}<\pi/3$ spontaneous breakdown of chiral symmetry ($m_d\neq 0$) does not occur. On the other hand, we find from (15) and (17) that in the local limit, for $\alpha^{(0)}>\pi/3$ Eq. (21) goes over into the equation with $m^{(0)}(\Lambda)=0$, and we return to the situation with the ultraviolet-stable fixed point $\alpha^{(0)}=\pi/3$, which we considered in §2.

The following observation is essential: for $\Lambda=\infty$ and $m^{(0)}=0$ the function

$$m(q^2)=\xi mF(1/2+\gamma',\ 1/2-\gamma',\ 2;\ -q^2/m^2)$$

satisfies the boundary condition (19) and therefore the equation (18) for all values of $\alpha^{(0)}$. Thus the condition for the conservation of axial-vector currents, which distinguishes the solution corresponding to the spontaneous breakdown of chiral symmetry, acts as a supplementary boundary condition defining the local limit. In particlar, we can in this way obtain the solution of the Goldstein problem[16] Reference 16 considers the B-S equation, the ladder approximation, for a massless parapositron (i.e., for a Goldstone boson). It is not difficult to show (see Appendix II) that the replacement of the B-S wave function $\chi(q^2)$ by $(q^2+m^2)^{-1}m_d(q^2)$ the equation goes over into equation (19) with $\Lambda=\infty$ and $m^{(0)}(\Lambda)|_{\Lambda=\infty}=0$. Since this equation formally has a solution for all $\alpha^{(0)}$ we appear to have a paradox: a massless parapositron exists no matter how small the bare coupling constant. If, however, we take note of the behavior of the passage to the limit $\Lambda\to\infty$ we conclude that in this approximation the massless parapositron exists only for the fixed value $\alpha^{(0)}=\pi/3$ of the coupling constant 6).

§4. THE COUPLING CONSTANT PHASE DIAGRAM IN QED

In this section we discuss the results of some numerical QED calculations on a lattice[1], from the viewpoint of the mechanism we have discussed for the spontaneous breakdown of chiral symmetry. We also consider the form of the phase diagram with respect to the coupling constant in QED and we discuss the possibility that a nontrivial local limit exists.

The computerized calculations in Ref. 1 were made in an approximation with frozen fermions, i.e., they took no account of the contribution from fermion loops. The basic results are the following:

1) In massless QED the ordering parameter $\langle 0|(\bar\psi\psi)_\Lambda|0\rangle$ is different from zero (i.e., spontaneous breakdown of chiral symmetry occurs) for all values of the bare coupling constant that exceed the critical value $\alpha_c\approx 0.3$. The value of the parameter $\langle 0|\bar\psi\psi|0\rangle$ is sensitive to the short-range interaction dynamics.

2) The calculations on an anisotropic lattice show no significant temperature dependence of the dynamics of spontaneous breakdown of chiral symmetry in QED.

The first result is in qualitative agreement with the picture developed in §2 for the dynamics of the spontaneous breakdown of chiral symmetry in QED and the associated appearance of collapse. Since the critical value of the coupling constant (as opposed to the critical index) depends essentially on the shape of the short range regularization, a direct comparison of the critical value of the coupling constant in the lattice theory with its value in the ladder approximation in the theory based on a cutoff in momentum space does not yield an estimate of the contributions from nonladder diagrams. Nevertheless, the qualitative agreement between the results of the computer calculations and those obtained in the ladder approximation does support the view that the latter gives us the characteristic features of the dynamics of spontaneous breakdown in chiral symmetry in QED, and therefore may be held to be a reasonable model for the study of the phenomenon.

The second result of the computations can be easily understood if we adopt the customary view that the critical temperature T_c, at which symmetry is restored, can be expressed via the distance r at which breakdown of chiral symmetry begins, i.e., $T_c\sim r^{-1}$. At collapse, $r\sim\Lambda^{-1}$, and therefore only for large values of the temperature $T\sim\Lambda$ can spontaneous breakdown of chiral symmetry occur.

Let us now discuss the existence problem for the local limit in QED. First we establish the following general assertion: if in the massless QED with cutoff for some fixed value of $\alpha^{(0)}=\alpha_c$ there exists a second order phase transition related to the spontaneous breakdown of chiral symmetry, the local QED with a fixed value of the bare coupling constant has a nontrivial S-matrix,

Our postulate implies that for supercritical values of $\alpha^{(0)}$ the fermion has the dynamic mass m_d. Since the theory allows a unique dimensional parameter Λ, this mass is given by

$$m_d=\Lambda f(\alpha^{(0)}), \quad (22)$$

where $f(\alpha^{(0)})$ is some function or other. The equation

$$f(\alpha^{(0)})=0 \quad (23)$$

has a positive root coinciding with the critical value $\alpha^{(0)}=\alpha_c$ of the coupling constant. In the local limit

$$\lim_{\Lambda\to\infty} m_d/\Lambda=f(\alpha^{(0)})\to 0$$

it defines a local theory with a nontrivial S-matrix: the Bethe-Salpeter wave function for the Goldstone boson, corresponding to the spontaneous breakdown of chiral symmetry, determines the effective vertex of the interaction between this boson and the fermion and antifermion. Therefore there must exist a pole in the fermion/antifermion scattering S-matrix corresponding to the Goldstone boson. The appearance of a sufficiently small bare fermion mass (in the case of partial conservation of the axial vector current) should not essentially influence this picture.

It follows that in proving the existence of a nontrivial local QED is suffices to prove the spontaneous breakdown of chiral symmetry for sufficiently large values of the bare coupling constant $\alpha^{(0)}$ in the theory with cutoff.

The calculations in Ref. 1 assumed that the fermions were frozen. To compute the coupling-constant phase diagram in the exact theory we need to know how strongly the results are influenced by a departure from this approximation, and above all, the effect of polarization of the vacuum. Later we shall argue that spontaneous breakdown of chiral symmetry in QED with a sufficiently large value of the bare coupling constant occurs even when the zero-charge case appears in a polarized vacuum.[3,4]

The analysis in the ladder approximation and with frozen fermions shows that the dynamics of spontaneous breakdown of chiral symmetry in QED with a cutoff occur in the neighborhood where the variable coupling constant has a value $\alpha(r) > \alpha_c \sim 1$. On the other hand, if we accept the argument of Landau and Pomeranchuk,[3] the equation

$$\alpha(r) = \alpha^{(0)}\left[1 + \frac{2K\alpha^{(0)}}{3\pi}\ln(\Lambda r)\right]^{-1}, \tag{24}$$

where K is the number of fermions, is qualitatively valid even for large values of $\alpha^{(0)}$. Then it follows that the vacuum polarization effects decrease the value of the variable coupling constant $\alpha(r)$ to a value near to 1 at distances $r = \rho/\Lambda$, where $\rho > 1$ (it follows from (24) that ρ is not large; $\rho < 10$ for $K = 3$). We can simulate these effects by bringing in a parameter for the infrared cutoff $\delta = \Lambda/\rho$, $\rho > 1$. Since the qualitative picture of the breakdown of chiral symmetry in the ladder approximation with frozen fermions appears to be a general one, we say suppose that the role of the cutoff parameter δ survives in the equations of the ladder approximation. The coupling constant in these equations will be interpreted as some mean value of the varying coupling constant $\alpha(r)$ in the interval $\Lambda^{-1} < r < \rho/\Lambda$.

Reference 19 considers the B-S equation in a ladder approximation with the infrared cutoff parameter δ for Goldstone bosons, although with aims that differ from ours. It follows from the results obtained there that the critical value of the coupling constant is given by the equation

$$\gamma_c \ln\frac{\Lambda}{\delta} + \operatorname{arctg} 2\gamma_c = \frac{\pi}{2}, \quad \gamma_c = \frac{1}{2}\left(\frac{3\alpha_c}{\pi} - 1\right)^{1/2}. \tag{25}$$

For our purposes it is essential that for $\delta = \Lambda/\rho$ the value of $\alpha_c(\rho)$ defined by (25) remain finite for all $\rho > 1$ ($\alpha_c(\rho) \to \infty$ as $\rho \to 1$). Moreover, since for $\delta = \Lambda/\rho$ the parameter Λ disappears from this equation, it follows that without regard to the cuttoff of the interaction in the limit $\Lambda \to \infty$ at all nonzero distances ($r > 0$), ($\Delta r = \lim_{\Lambda\to\infty} \Lambda^{-1}(\rho - 1) = 0$) a spontaneous breakdown of chiral symmetry will occur for $\alpha^{(0)} > \alpha_c(\rho)$ even in the local limit. However (and this is important), the dynamic mass

$$m_d = \Lambda f(\rho, \alpha^{(0)})$$

remains finite in this limit only for a fixed value of the bare coupling constant [cf. Eq. (6)].

This analysis leads us to the following hypothesis. In the zero-charge situation the local QED may offer a nontrivial theory: the residual δ-function interaction ($\alpha(r) = 0$ for $r > 0$ and $\alpha^{(0)} = \alpha(0)$) can lead to the formation of a fermion-antifermion coupled Goldstone state and so to an induced fermion-antifermion-boson peak.

In this case the coupling-constant phase diagram can be explained as follows: in the subcritical phase with $\alpha^{(0)} < \alpha$ there is only the trivial infrared fixed point $\alpha_\mu \equiv \alpha(r)|_{r=\mu^{-1}} = 0$ and therefore in the local limit there arises for all these values of $\alpha^{(0)}$ only the free theory (the standard zero-charge case[3,4]. In the supercritical phase there is an ultraviolet-stable fixed point $\alpha^{(0)} = \alpha_c$ which defines a nontrivial local theory with a Yukawa interaction among the fermions, antifermions, and residual Goldstone bosons. The appearance of a sufficiently small excitation mass in the fermion (the case of conservation of axial currents) should not significantly change the phase diagram.[7)] A numerical test of this hypothesis would be interesting.

§5. CONCLUSIONS

The phenomenon of collapse in quantum field theory can have an effect on the structure of renormalization, i.e., the shape of the phase diagram with respect to the coupling constant (and does so in some two-dimensional models[7,10]).

An analysis of chiral-invariant Lagrangian QED suggests the existence of a critical value of the coupling constant $\alpha_c \sim 1$ that separates the massless and massive phases of the theory. The critical constant α_c is the field-theory analogue of the critical coupling constant $Z_c e^2/4\pi \gtrsim 1$ in the Dirac equation with a Coulomb potential. We have presented arguments in support of the view that this value defines a nontrivial local theory.

This dynamic picture allows us to give a clear interpretation of recent calculations for noncompact lattice QED (Ref. 1). Furthermore, our hypothesis on the existence of a nontrivial local QED, even in the zero-charge case, can in principle be tested soon by computer calculations.

It would be of great interest to test the possibility of realizing the case in which a nontrivial local theory is determined by the zero of the β-function of the supercritical phase, and for other asymptotically nonfree theories. Currently a search is under way for the mechanism of the breakdown of scale symmetry in finite supersymmetric theories.[20] It would be interesting to see whether a breakdown mechanism related to the collapse phenomenon is possible in these theories.

I thank D. V. Volkov, V. P. Gusynin, and P. I. Fomin for their helpful discussions.

APPENDIX I

In the spontaneous breakdown of chiral $SU_L(K) \times SU_R(K)$ symmetry in massless QED the fermion propagator has the form

$$S_{ij}(q) = \delta_{ij}(-\hat{q}A(q^2) + B_d(q^2))^{-1}, \tag{I.1}$$

where $i, j = 1,2,..., K$. In the approximation with the bare photon propagator and with the photon-fermion-antifermion vertex the Schwinger-Dyson equations for the photon propagator in the covariant calibration gauge with the gauge parameter d_l have the following form in the Euclidean region:

$$A(q^2)-1=d_l\frac{\alpha^{(0)}}{4\pi}\int_0^{\Lambda^2}dk^2\frac{A(k^2)}{k^2A^2(k^2)+B_d^2(k^2)}\times\left[\frac{k^4}{q^4}\theta(q^2-k^2)+\theta(k^2-q^2)\right], \quad (I.2)$$

$$B_d(q^2)=(3+d_l)\frac{\alpha^{(0)}}{4\pi}\int_0^{\Lambda^2}dk^2\frac{B_d(k^2)}{k^2A^2(k^2)+B_d^2(k^2)}\times\left[\frac{k^2}{q^2}\theta(q^2-k^2)+\theta(k^2-q^2)\right]. \quad (I.3)$$

In the Landau gauge ($d_l = 0$) we have $A(q^2) = 1$. Moreover, in this gauge equations (I.1) and (I.2) do not change if the vertex $\Gamma_{\mu;ij}$ is written in the form

$$\Gamma_{\mu;ij}(q_2,q_1)=\gamma_\mu\delta_{ij}+\frac{P_\mu}{P^2}\delta_{ij}\Phi, \quad (I.4)$$

where $P_\mu = q_{2\mu} - q_{1\mu}$, and Φ is an arbitrary Lorentz-invariant function. If $\Phi = B_d(q_1^2) - B_d(q_2^2)$ the vertex (I.4) satisfies the Ward identity

$$P^\mu\Gamma_{\mu;ij}(q_2,q_1)=S_{ij}^{-1}(q_1)-S_{ij}^{-1}(q_2). \quad (I.5)$$

Let us now look at this identity for the vertex $\Gamma_{5\mu}^r$ of the axial-vector current:

$$j_{5\mu}{}^r=\bar\psi\gamma_\mu\gamma_5\lambda^r\psi,\quad r=1,2,\ldots,K^2-1:$$
$$P^\mu\Gamma_{5\mu}{}^r(q_2,q_1)=-\gamma_5\lambda^rS^{-1}(q_1)-S^{-1}(q_2)\gamma_5\lambda^r. \quad (I.6)$$

For spontaneous breakdown of chiral symmetry the peak $\Gamma_{5\mu}^r$ has a pole at zero with respect to the variable P^2; the residue at the pole can be calculated from the B-S wave function $\chi^r(P,q)$ for the Goldstone boson ($q = (q_1 + q_2)/2$):

$$\Gamma_{5\mu}{}^r|_{P^2\approx0}\approx\frac{iP_\mu f}{P^2}S^{-1}(q_2)\chi^r(P,q)S^{-1}(q_1), \quad (I.7)$$

where the parameter f is determined by the equation

$$\langle 0|j_{5\mu}{}^{r'}|P,r\rangle=i\delta_{rr'}fP_\mu.$$

Substituting (I.7) in (I.6) and passing to the limit $P_\mu\to0$ we find that

$$\chi^r(q)\equiv\chi^r(P,q)|_{P=0}=\lambda^r\gamma_5\chi(q^2),$$
$$\gamma_5\chi(q^2)=2if^{-1}S(q)\gamma_5B_d(q^2)S(q). \quad (I.8)$$

In the ladder approximation (I.8) leads to the equation

$$\chi(q^2)=2if^{-1}\frac{B_d(q^2)}{q^2+m_d^2}, \quad (I.9)$$

and the B-S equation for $\chi(q^2)$ in the Landau gauge has the form

$$(q^2+m_d^2)\chi(q^2)=\frac{3\alpha^{(0)}}{4\pi}\int_0^{\Lambda^2}dk^2\left[\frac{\theta(q^2-k^2)}{q^2}+\frac{\theta(k^2-q^2)}{k^2}\right]k^2\chi(k^2). \quad (I.10)$$

Comparing (I.10) with (I.3) and taking account of (I.9) we find that with the Landau gauge the ladder approximation corresponds to a linear version of equation (I.3) if $B_d(q^2)$ in the denominator is replaced by m_d^2.

The equation (I.10) can be solved in the following way: We differentiate it with respect to q^2 and find

$$\frac{d}{dq^2}\left\{q^4\frac{d}{dq^2}[(q^2+m_d^2)\chi]\right\}+\frac{3\alpha^{(0)}}{4\pi}q^2\chi=0, \quad (I.11)$$

$$q^4\frac{d}{dq^2}[(q^2+m_d^2)\chi]|_{q^2=0}=0, \quad (I.12)$$

$$\left\{q^2\frac{d}{dq^2}[(q^2+m_d^2)\chi]+(q^2+m_d^2)\chi\right\}\Big|_{q^2=\Lambda^2}=0. \quad (I.13)$$

The solution of (I.11) that satisfies the boundary condition (I.12) has the form

$$B_d(q^2)\equiv\frac{f}{2i}(q^2+m_d^2)\chi(q^2)=CF\left(\frac12+i\gamma,\frac12-i\gamma,2;\frac{-q^2}{m_d^2}\right),$$
$$\gamma=\frac12\left(\frac{3\alpha^{(0)}}{\pi}-1\right)^{1/2}, \quad (I.14)$$

where F is the hypergeometric function. The normalizing constant $C = \mathscr{C}m_d$, where $\mathscr{C}$ is a dimensionless parameter, can in principle be determined from the normalizing constraint on the B-S wave function. The second boundary condition (I.13) determines the mass spectrum. An analytical result can be obtained if $m_d{}^2/\Lambda^2\ll1$. Then by using the connection formulas for the hypergeometric function [cf. Ref. 14] we obtain from (I.13) the equation

$$\sin\left[\gamma\ln\frac{\Lambda^2}{m_d^2}+\Sigma(\gamma)\right]=0,\qquad \Sigma(\gamma)=\arg\left[\frac{\Gamma(1+2i\gamma)}{\Gamma^2(1+i\gamma)}\right]. \quad (I.15)$$

and this yields

$$m_d^{(n)}=\Lambda\exp\left[\frac{-\pi n+\Sigma(\gamma)}{2\gamma}\right]\underset{\gamma\ll1}{\approx}4\Lambda\exp\left(\frac{-\pi n}{2\gamma}\right);$$
$$n=1,2,\ldots. \quad (I.16)$$

It can be shown[10] that only the maximum value $m_d^{(1)}$ yields a stable vacuum.

APPENDIX II

We know [Ref. 21] that the renormalization constant $Z_{m\mu}$ for the state operator $(\bar\psi\gamma_5\lambda^r\psi)_\Lambda$ coincides with the renormalization constant for the state operator $(\bar\psi\psi)_\Lambda$ ($(\bar\psi\psi)_\Lambda = Z_{m\mu}^{-1}(\bar\psi\psi)_\mu$). From the equation

$$m^{(0)}(\Lambda)(\bar\psi\psi)_\Lambda=m_\mu{}^c(\bar\psi\psi)_\mu, \quad (II.1)$$

(cf. Ref. 21) where m_μ^c is the renormalized current mass of the fermion, we obtain

$$Z_{m\mu}=m^{(0)}(\Lambda)/m_\mu{}^c. \quad (II.2)$$

We stress the fact that both the masses $m^{(0)}(\Lambda)$ and m_μ^c are related to the breakdown of chiral symmetry. To determine the ratio $m^{(0)}(\Lambda)/m_\mu^c$ we use the equation for the mass function $m(q^2)=B(q^2)/\Lambda(q^2)$ (the propagator $S=(-\hat{q}A+B)^{-1}$). In the ladder approximation with the Landau gauge ($A(q^2)=1$) this equation takes the form

$$m(q^2)=m^{(0)}(\Lambda)$$

$$+\frac{3\alpha^{(0)}}{4\pi}\int_0^{\Lambda^2}dk^2\frac{m(k^2)}{k^2+m^2}\left[\frac{k^2}{q^2}\theta(q^2-k^2)+\theta(k^2-q^2)\right], \quad \text{(II.3)}$$

(compare with (I.3)) where $m\to m_d$ in the limit as $m^{(0)}(\Lambda)\to 0$. The solution of this equation is the function

$$m(q^2)=\xi mF\left(\frac{1}{2}+i\gamma,\frac{1}{2}-i\gamma,2;-q^2/m^2\right) \quad \text{(II.4)}$$

(cf. (I.14); $\gamma=\frac{1}{2}(3\alpha^{(0)}/\pi-1)$ and ξ is a numerical constant) and it satisfies the boundary condition

$$\left(q^2\frac{dm(q^2)}{dq^2}+m(q^2)\right)\bigg|_{q^2=\Lambda^2}=m^{(0)}(\Lambda) \quad \text{(II.5)}$$

for $q^2=\Lambda^2$.

Next we consider the supercritical phase $\alpha^{(0)}>\pi/3$; the simpler subcritical case would be treated in the same way. Using the connection formulus for the hypergeometric function with $m^2/\Lambda^2\ll 1$, we find from (II.4) and (II.5) that

$$\frac{\xi m^2}{\Lambda}\left(\frac{\operatorname{cth}\pi\gamma}{\pi\gamma}\right)^{1/2}\sin\left[2\gamma\ln\frac{\Lambda}{m}+\Sigma(\gamma)\right]=m^{(0)}(\Lambda);$$

$$\Sigma(\gamma)=\arg\frac{\Gamma(1+2i\gamma)}{\Gamma^2(1/2+i\gamma)}. \quad \text{(II.6)}$$

Since spontaneous breakdown of chiral symmetry occurs for $\alpha^{(0)}>\pi/3$, the mass $m=m(q^2)|_{q^2=m^2}$ has the form $m=m_d+m_c$, where the current mass is given by $m_c\equiv m_\mu^c|_{\mu=m}$. For our purposes we need consider only the case $m^c\ll m_d$ (partial conservation of axial current). Then we find from (II.6) that

$$Z\equiv Z_{m\mu}|_{\mu=m}=\frac{m^{(0)}(\Lambda)}{m^c}$$

$$\approx 2\xi\left(\frac{\gamma\operatorname{cth}\pi\gamma}{\pi}\right)^{1/2}\frac{m_d}{\Lambda}\underset{m_d/\Lambda\ll 1}{\approx}2\xi m_d/\pi\Lambda. \quad \text{(II.7)}$$

Then it follows that for $\mu\gg m_d$ we find

$$Z_{m\mu}\approx\mu/\Lambda \quad \text{(II.8)}$$

for the renormalization constant.

[1] In other words, in the relativistic theory a fall into the center (collapse) appears for such a value of the potential[12,13] and the system has no ground state (is a vacuum).

[2] The appearance of such quasistationary levels is interpreted as the creation of an electron-positron pair from a vacuum[12]. The electron is coupled to the center, which shields it, and the positron recedes to infinity. The process is repeated until the central charge falls to the subcritical level.

[3] The role of the fermion mass in the problem of the supercritical Coulomb center can be discerned in equations (4) and (5). With the growth of m the imaginary part Im $\varepsilon^{(n)}$ decreases and the stability of the system increases. Therefore there are in principle two ways to stabilize such a system: spontaneous shielding of the charge, and generation of fermion mass. In the Coulomb center problem, the formulation of the problem itself prevents any solution but the first. The hypothesis of Ref. 8 stated that in supercritical QED the second way—creation of fermion mass—could occur.

[4] The equations are treated in the Landau gauge. This choice is not random; it is dictated by the following considerations. The very statement of the problem of spontaneous breakdown of a symmetry in a given approximation can be made only if the approximation is compatible with the Ward identity that corresponds to the symmetry in question. As was shown in Appendix I, when this requirement is applied to the ladder approximation the Landau gauge is preferred.

[5] In the relativistic theory the kinetic energy satisfies $E_k=(q^2+m^2)^{1/2}-m\approx q$ as $q\to\infty$. Therefore the energy satisfies $E=m+E_k-\alpha/r\approx(1-\alpha)/r$ as $r\to 0$ (because of the uncertainty principle for the momentum $q\sim r^{-1}$), and collapse occurs for $\alpha>1$.

[6] The behavior in the passage to the limit $\Lambda\to\infty$ occurs in other problems of QED. In Ref. 17 behavior of this kind is encountered in the problem of self-energy of the electron when gravity is taken into account. The need to supplement the local equations is not peculiar to QED; it is characteristic of problems of dynamic breakdown of symmetry in local gauge field theory. In quantum chromodynamics, taking account of the conditions for conservation of axial vector currents enables one to derive the asymptotic ultraviolet mass function for quarks as an immediate consequence of the equations for the Green's function, without using assumptions about the validity of the operator expansion[18].

[7] Real QED, i.e. the phenomenological theory, with an ultraviolet cutoff parameter specifying the low-energy interaction of leptons and photons, appears to relate to the subcritical phase. In reality there exists no candidate for the role of the Goldstone boson (or "almost" Goldstone boson) consisting of leptons. Moreover, if QED is to be part of a grand unified theory the variable electrodynamic coupling constant must be small at all distances (for instance, in $SU(5)$ theory $\alpha(r)\leqslant 0.02$).

[1] J. Bartholomew, J. Kogut, S. H. Shenker *et al.* Nucl. Phys. B. **230**, 222 (1984).
[2] M. Gell-Mann and F. Low, Phys. Rev. **95**, 1300 (1954).
[3] L. D. Landau and I. Ya. Pomeranchuk, Dok. Akad. Nauk. SSSR **102**, 489 (1954).
[4] E. S. Fradkin, Zh. Eksp. Teor. Fiz. **28**, 750 (1955) [Sov. Phys. JETP **1**, 604 (1955).
[5] K. Johnson, M. Baker, and R. Willey, Phys. Rev. B. **136**, 1111 (1964).
[6] N. V. Krasnikov, Phys. Lett. B. **126**, 483 (1983).
[7] V. A. Miransky, Phys. Lett. B. **91**, 421 (1980).
[8] P. I. Fomin and A. V. Miransky, Phys. Lett. B. **64**, 166 (1976).
[9] P. I. Fomin, V. P. Gusynin, and V. A. Miransky, Phys. Lett. B. **78**, 136 (1978).
[10] P. I. Fomin, V. P. Gusynin, V. A. Miransky, and Yu. A. Sitenko, Nuovo Cim. **6**, 1 (1983).
[11] W. A. Bardeen, M. Moshe, and M. Bander, Phys. Rev. Lett. **52**, 1188 (1984); M. B. Halpern Phys. Lett. B. **137**, 382 (1984).
[12] Ya. B. Zel'dovich and V. S. Popov, Usp. Fiz. Nauk **105**, 403 (1971) [Sov. Phys. Uspeki **14**, 673 (1971)].
[13] L. D. Landau and E. M. Lifshitz, Kvantovaya Mekhanika Nauka, Moscow, 1974, §35 (Quantum Mechanics, Pergamon Press, Oxford).
[14] A. Erdelyi, Vysshei transtsendentynye funktsii Nauka, Moscow, 1974 (Higher transcendental functions McGraw-Hill, New York).
[15] Th. A. J. Maris, G. Jacob, and B. Liberman, Nuovo Cim. A. **52**, 116 (1967); H. Pagels, Phys. Rev. D. **7**, 3689 (1967).
[16] S. Goldstein, Phys. Rev. **91**, 1516 (1953).
[17] I. B. Khriplovich, Yad. Fiz., **3**, 575 (1966) [Sov. J. Nucl. Phys. **3**, 415 (1966)].
[18] V. A. Miransky, Yad. Fiz., **38**, 468 (1983) [Sov. J. Nucl. Phys. **38**, 280 (1983)].
[19] V. A. Miransky, V. P. Gusynin, and Yu. A. Sitenko, Phys. Lett. B. **100**, 157 (1981).
[20] S. Mandelstam, Nucl. Phys. B., **213**, 149 (1983).
[21] L. S. Adler and W. A. Bardeen, Phys. Rev. D. **4**, 3045 (1971).

Translated by A. Brown