Radiation stationarity length in inhomogeneous media and analog of the Tamm-Mandel'shtam energy-time uncertainty relation

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An analog of the Tamm-Mandel'shtam energy-time uncertainty relation can be used in the theory for the propagation of coherent and partially coherent radiation through slightly inhomogeneous media. The physical meaning of this analog is shown. A useful new longitudinal scale dimension is introduced: the radiation stationarity length, a measure of the distance over which the changes in the average properties of the radiation do not yet exceed their variances. A detailed study is made of the stationarity length of an arbitrary nonparaxial Gaussian beam in a longitudinally homogeneous medium with a parabolic refractive index profile. A detailed study is also made of how various longitudinal inhomogeneities in a medium with a quadratic profile affect the stationarity length for beams. The results can be used to analyze light propagation through waveguides, ultra-long-range propagation of sound in underwater oceanic ducts, and the ultra-long-range propagation of short radio waves in ionospheric waveguiding ducts.

1. INTRODUCTION

Many problems involving the propagation of waves of various types in inhomogeneous media reduce under some approximation or other to the solution of the scalar Helmholtz wave equation. Examples are the propagation of light through multimode graded-index waveguides[1–3]; the ultra-long-range propagation of sound waves in an underwater oceanic acoustic duct which arises from variations of the density, temperature, and salinity over depth[4,5]; and the round-the-world propagation of radio waves in ionospheric ducts.[6] Among the many approximate methods for solving the Helmholtz wave equation, a special place is occupied by the Leontovich-Fock parabolic-equation method[7] (the paraxial approximation), in which the Helmholtz equation is reduced to a paraxial Schrödinger equation. One advantage of this approach is that the sophisticated formalism of quantum mechanics can be used to solve the parabolic wave equation, and either exact solutions of the equation or useful approximations can be derived. In any case, the derived analytic solutions of the problem are exceedingly useful for qualitative and quantitative descriptions of the phenomena under study, and they serve as a convenient zeroth approximation for the derivation of a perturbation theory.[8,9]

A central feature of this approach is the introduction of coordinate and momentum operators $\hat{x}$ and $\hat{p}$, respectively, whose eigenvalues in the Hamiltonian formalism give the position and inclination of a ray with respect to the axis of the medium.[1] (For brevity we will assume that the medium is two-dimensional.) These operators satisfy the ordinary commutation relation $[\hat{x}, \hat{p}] = i/k$, where $k = 2\pi/\lambda$ is the wave number, and λ is the wavelength of the radiation. A consequence of this commutation relation is the Heisenberg uncertainty relation

$$\langle(\Delta\hat{x})^2\rangle\langle(\Delta\hat{p})^2\rangle \geqslant {}^1/_4 k^2, \tag{1}$$

whose role in the geometric-optics quantum-mechanical theory[1] is equally as important as that played by the Heisenberg uncertainty relation in quantum mechanics. For example, relation (1) is a quantitative expression of the wave nature of the radiation; the use of this relation makes it a simple matter to evaluate the wave corrections to the geometric-optics solutions in any problem in which wave effects turn out to be important, i.e., in studies of interference and diffraction phenomena and in the calculations of fields near caustics. The estimates found in this way frequently agree within a coefficient with the exact solutions. In any case, relation (1) yields estimates with an accuracy quite sufficient for practical purposes without resorting to an exact or approximate solution of the wave problem. It is difficult to overestimate the importance of this relation in wave propagation problems.

On the other hand, in quantum mechanics we have a well-known uncertainty relations of another type, namely, the energy-time uncertainty relations. A distinction should be made here between two types of such relations. According to the classification based on Fock's work,[10,11] the Heisenberg-Bohr relation pertaining to a measurement processes belongs to the first type. The relations of the second type break down further into two subtypes: the Tamm-Mandel'shtam relations[12] between the displacement time of a wave packet to the dispersion of the energy in the state under consideration (and generalizations of these relations[13–18]), and the relation between the half-life of a quasistationary state and the level width.[10,19–21]

The validity of the Heisenberg-Bohr uncertainty relation is problematical and the subject of constant debate, since this relation is actually postulated, rather than proved, in quantum mechanics. This debate and the pertinent references are reviewed by Dodonov *et al.*,[22] for example. In contrast, there is no doubt regarding the validity of uncertainty relations of the Tamm-Mandel'shtam type, since they are rigorous consequences of the mathematical formalism of quantum mechanics.

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In this paper we call attention to the existence in waveguide problems of—in addition to the widely used Heisenberg coordinate-momentum uncertainty relation—an analog to the Tamm-Mandel'shtam energy-time uncertainty relation. We determine the physical meaning of this relation in problems involving the propagation of waves through inhomogeneous media; we introduce a new longitudinal scale dimension, the "radiation stationarity length," and describe several examples which illustrate the practical usefulness of this new scale length.

2. THE DENSITY MATRIX AND AN ANALOG OF THE TAMM-MANDEL'SHTAM UNCERTAINTY RELATION FOR A PARAXIAL BEAM IN A SLIGHTLY INHOMOGENEOUS MEDIUM

For simplicity we consider the two-dimensional scalar Helmholtz equation for a monochromatic component of the wave field $E(x, z)$ in the Cartesian coordinate system x, z:

$$\partial^2 E/\partial z^2 + \partial^2 E/\partial x^2 + k^2 n^2(x, z) E = 0, \tag{2}$$

where $k = 2\pi/\lambda$ is the wave number, λ is the wavelength of the radiation in free space, and $n(x, z)$ is the refractive index of the medium. If the wave is propagating at a small angle to the longitudinal (z) axis, Eq. (2) can be written as a Schrödinger equation in the Leontovich-Fock paraxial approximation[7]:

$$\frac{i}{k}\frac{\partial \Psi}{\partial \xi} = \hat{H}\Psi,$$
$$\hat{H} = -\frac{1}{2k^2}\left(\frac{\partial^2}{\partial x^2}\right) + \frac{1}{2}(n_0^2 - n^2), \quad \xi = \int_0^z dz/n_0, \tag{3}$$

for the characteristic field

$$\Psi(x, \xi) = E n_0^{1/2} \exp\left\{-ik\int_0^z n_0\, dz\right\},$$

where $n_0 = n(0, z)$ is the refractive index of the medium on the z axis (Ref. 23, for example).

If the radiation field at $z = 0$, $\Psi(x, 0)$, is known, we can use Hamiltonian $\hat{H}$ and Eq. (3) to calculate the longitudinal evolution of the radiation, i.e., to find the value of the field $\Psi(x, z)$ in any cross section $z = \text{const}$.

If we do not have complete information about the initial field at $z = 0$, because of, for example, random fluctuations of the parameters of the radiation source or of the parameters of the medium in the region $z < 0$, then we know the initial field at $z = 0$ only with a certain probability (the system is in a mixed state). As we have pointed out elsewhere,[8] the radiation can be described in this case by a density matrix $\hat{\rho}$ (Refs. 24–26, for example). A density matrix can be used to calculate expectation values of all the characteristics of the radiation beam in the general case in which we do not have complete information about the radiation field. In Ref. 27, for example, the density-matrix formalism was used to study the intermode pulse dispersion in a longitudinally inhomogeneous medium with a parabolic refractive-index profile, and it was used in Ref. 28 to calculate the coupling of modes in statistically irregular optical fibers. It should be noted that partially coherent radiation beams can also be described conveniently in density-matrix terms. The correlation function of the field and its degree of coherence in this case are specified by corresponding matrix elements of the operator $\hat{\rho}$, and uncertainty relation (1) for a beam described by a density matrix in this case has the meaning of a generalized Heisenberg uncertainty relation for partially coherent radiation, as was studied in Refs. 29, for example.

Denoting by $\hat{\rho}(0)$ the initial value of the density matrix operator, we can describe its longitudinal evolution in the paraxial approximation completely by means of the Liouville equation

$$\dot{\hat{\rho}}(\xi) = ik[\hat{\rho}(\xi), \hat{H}(\xi)], \tag{4}$$

where the dot means differentiation with respect to the longitudinal variable ξ, and the Hamiltonian is given by (3).

Mandel'shtam and Tamm[12] introduced a class of uncertainty times for cases in which it is possible to associate with any operator representing a physical quantity a corresponding uncertainty time which is a direct consequence of the corresponding Heisenberg equation of motion for the operator. Eberly and Singh[13] used this idea to introduce a stationarity time for a stationary quantum system whose density matrix satisfies a Liouville equation. That stationarity time can serve as the time in the energy-time uncertainty relation. In Refs. 14 and 15 this parameter was generalized to the case of nonstationary quantum systems. By virtue of the analogy between Eq. (4) and the quantum Liouville equation, parameters of this sort can also be introduced in a description of the propagation of radiation.

Following Malkin and Man'ko,[15] we introduce the parameters $Z_n(\xi)$ ($n = 1,2,3, \ldots$), which depend on the longitudinal variable ξ, for a radiation beam in an inhomogeneous medium, which can be described in the general case by a density matrix $\hat{\rho}$ and a time-dependent Hamiltonian $\hat{H}$:

$$\frac{1}{Z_n^2(\xi)} = \mathrm{Sp}(\hat{\rho}^n(\xi)\dot{\hat{\rho}}^2(\xi)) - \mathrm{Sp}^2(\hat{\rho}^n(\xi)\dot{\hat{\rho}}(\xi))$$
$$\equiv \mathrm{Sp}(\hat{\rho}^n(\xi)\dot{\hat{\rho}}^2(\xi)). \tag{5}$$

Here we have used the equality $\mathrm{Sp}(\hat{\rho}^n\dot{\hat{\rho}}) = 0$, whose validity is verified by the Liouville equation (4) and by the known identity

$$\mathrm{Sp}[\hat{A}, \hat{B}]\hat{C} = \mathrm{Sp}[\hat{C}, \hat{A}]\hat{B}. \tag{6}$$

Using identity (6) and Liouville equation (4) in a procedure similar to that of Ref. 15, one can show (see the Appendix) that for stationarity lengths $Z_n(\xi)$ introduced in this manner and for the quantity

$$\langle(\Delta H)^2\rangle = \mathrm{Sp}(\hat{\rho}\hat{H}^2) - \mathrm{Sp}^2(\hat{\rho}\hat{H}) \tag{7}$$

an uncertainty relation of the nature of the Tamm-Mandel'shtam energy-time relation holds under the assumption $\mathrm{Sp}\hat{\rho}\dot{\hat{\rho}}^2 \neq 0$:

$$\langle(\Delta\hat{H})^2\rangle Z_n^2(\xi) \geqslant 1/k^2, \quad n = 1, 2, 3, \ldots. \tag{8}$$

From the inequality

$$\mathrm{Sp}\hat{\rho}^n\hat{A} \leqslant \mathrm{Sp}\hat{\rho}\hat{A}, \quad n = 1, 2, 3, \ldots, \tag{9}$$

which holds for any positive semi-definite operator $\hat{A}$, it follows that the stationarity lengths $Z_n(\xi)$ form an increasing sequence $Z_1(\xi)\leqslant Z_2(\xi)\leqslant Z_3(\xi),\ldots$. From the physical standpoint, the stationarity length of greatest interest is undoubtedly the minimum stationarity length:

$$Z_1(\xi)=(\mathrm{Sp}\hat{\rho}\dot{\hat{\rho}}^2)^{-2}. \tag{10}$$

Using Liouville equation (4), we can also show without difficulty that if we do have complete information about the radiation, and if the density matrix corresponds to a pure state ($\hat{\rho}^2=\hat{\rho}$, $Sp\hat{\rho}^2=\hat{S}\hat{p}\rho=1$), inequality (8) becomes an equality, and all the stationarity lengths become equal: $Z_1(\xi)=Z_2(\xi)=\ldots Z_n(\xi)$.

We point out that one could, following Ref. 15, introduce an even smaller stationarity length

$$Z_0(\xi)=(\mathrm{Sp}\dot{\hat{\rho}}^2)^{-2}<Z_n(\xi), \tag{11}$$

for which an uncertainty relation

$$\langle(\Delta\hat{H})^2\rangle Z_0^2(\xi)\geqslant {}^1/_4k^2 \tag{12}$$

holds. However, it is advisable to use this length Z_0. The reason is that the product on the left side of (12) reaches its minimum for pure states, in which, as is easily shown, the following relation holds:

$$(Z_0)_{min}=\frac{1}{\sqrt{2}}(Z_1)_{min}.$$

For this reason, inequality (12) should be written in the following form in a more rigorous approach incorporating (8):

$$\langle(\Delta\hat{H})^2\rangle Z_0^2\geqslant\langle(\Delta\hat{H})^2\rangle_{min}(Z_0^2)_{min}$$
$$=\langle(\Delta\hat{H})^2\rangle_{min}{}^1/_2(Z_1^2)_{min}={}^1/_2k^2; \tag{13}$$

i.e., the number on the right side of (12) is simply too small. In other words, instead of inequality (12) for Z_0 we should use (13), which is essentially the same as relation (8).

3. PHYSICAL MEANING OF THE ANALOG OF THE TAMM-MANDEL'SHTAM UNCERTAINTY RELATION AND EXAMPLES OF ITS USE

To explain the physical meaning of uncertainty relation (8) we begin with the case in which the refractive index of the medium remains constant in the longitudinal direction. A medium of this type is of further interest because for it uncertainty relation (8) can generalized to the case of a nonparaxial radiation beam.

When the refractive index does not vary in the longitudinal direction, Helmholtz equation (2) reduces to an equivalent stationary Schrödinger equation for the field $\Psi(x)$ (Refs. 1-3):

$$\hat{H}\Psi(x)=\varepsilon\Psi(x). \tag{14}$$

Here ε and $\Psi(x)$ are the eigenvalue and eigenfunction of Hamiltonian (3), which in this case does not depend on z. The longitudinal evolution of the wave field component $E(x,z)$ is specified in this case by the propagation constant β_ε:

$$E(x,z)=\exp(i\beta_\varepsilon z)\Psi(x),\qquad \beta_\varepsilon=kn_0(1-2\varepsilon/n_0^2)^{1/2}. \tag{15}$$

It is not difficult to show that the evolution of the density matrix of a nonparaxial radiation beam is described by

$$\dot{\hat{\rho}}(z)=i[\hat{\beta},\hat{\rho}], \tag{16}$$

which generalizes Liouville equation (4) to the nonparaxial case for a medium which is homogeneous in the longitudinal direction. Here the dot means differentiation with respect to the longitudinal coordinate z, and the eigenvalues of the propagation-constant operator

$$\hat{\beta}=kn_0(1-2\hat{H}/n_0^2)^{1/2}\equiv kn_0\left(1-\frac{\hat{H}}{n_0^2}-\frac{\hat{H}^2}{2n_0^4}-\frac{\hat{H}^3}{2n_0^6}-\ldots\right) \tag{17}$$

give the propagation constants in (15). In the paraxial approximation we should retain only the first term in an expansion of this operator in a series in $\hat{H}/n_0^2$; this approach corresponds to replacing $\hat{\beta}$ by $\hat{\beta}_0=kn_0(1-\hat{H}/n_0^2)$. It should be noted that the dot in Eq. (16) means a differentiation with respect to z, not with respect to ξ, as in (4).

From Eq. (16) we find the Tamm-Mandel'shtam uncertainty relation for a nonparaxial radiation beam in a longitudinally homogeneous medium:

$$\langle(\Delta\hat{\beta})^2\rangle Z_n^2\geqslant 1,\quad n=1,2,3,\ldots, \tag{18}$$

where

$$\langle(\Delta\hat{\beta})^2\rangle=\mathrm{Sp}\hat{\rho}\hat{\beta}^2-\mathrm{Sp}^2\hat{\rho}\hat{\beta},\quad Z_n^{-2}=\mathrm{Sp}(\hat{\rho}^n(z)\dot{\hat{\rho}}^2(z)).$$

The physical meaning of uncertainty relation (18) is thus as follows: It relates the spread $\langle(\Delta\hat{\beta})^2\rangle$ of the propagation constants of a radiation beam described by the density matrix $\hat{\rho}$ (e.g., the spread of the propagation constants for a beam of partially coherent radiation) to the stationarity length Z_1, which gives the maximum longitudinal-axis distance over which the parameters of this radiation do not change substantially. Using expression (17), we can easily derive the following expansion for the dispersion of the propagation constant—an expansion useful in the construction of a perturbation theory:

$$\langle(\Delta\hat{\beta})^2\rangle$$
$$=k^2n_0^2\left\{\frac{\mathrm{Sp}\,\hat{\rho}\hat{H}^2-\mathrm{Sp}^2\,\hat{\rho}\hat{H}}{n_0^4}+\frac{\mathrm{Sp}\,\hat{\rho}\hat{H}^3-\mathrm{Sp}\,\hat{\rho}\hat{H}\,\mathrm{Sp}\,\hat{\rho}\hat{H}^2}{n_0^6}+\ldots\right\}. \tag{19}$$

The first term in the curly brackets corresponds to the dispersion of the propagation constant in the paraxial approximation, while the second is the first correction for the deviation from a paraxial situation.

We wish to call attention to the fact that in the derivation of (8) and (18) we have used the condition $\mathrm{Sp}\hat{\rho}\dot{\hat{\rho}}^2\neq0$; i.e., the stationarity length Z_1 has been assumed to be bounded. We now consider conditions under which Z_1 becomes infinite. It follows from (16) that for nonparaxial propagation of light through a longitudinally homogeneous medium the stationarity length Z_1 is determined exclusively from the initial density matrix $\hat{\rho}(0)$, i.e., depends only on the excitation conditions at $z=0$:

$$Z_1^{-2}=\mathrm{Sp}([\hat\rho(0),\hat\beta][\hat\beta,\hat\rho(0)]\hat\rho(0)). \quad (20)$$

It is not difficult to see from this expression that the stationarity length becomes infinite if the initial density matrix $\hat\rho(0)$ describes either a mode of the medium [a stationary state that is an eigenfunction of Hamiltonian (3)] or a mixture of modes [for such a mixture we would generally have $\langle(\Delta\hat\beta)^2\rangle>0$]. It is also obvious from (20) that the stationarity length Z_1 becomes infinite when the initial density matrix is specified as an arbitrary analytic function of the propagation-constant operator $\hat\beta$:

$$\hat\rho(0)=(\mathrm{Sp}f(\hat\beta))^{-1}f(\hat\beta).$$

Let us examine some specific examples.

Example 1. Stationarity length of a nonparaxial Gaussian beam in a longitudinally homogeneous medium with a parabolic refractive-index profile

We assume that the medium is homogeneous in the longitudinal direction, while in the transverse direction its refractive index varies in accordance with the parabolic law

$$n^2(x)=n^2(0)-\omega^2x^2, \quad (21)$$

where the gradient parameter ω specifies the refractive index profile, and where we are assuming that the medium described by (21) is excited at $z=0$ by a coherent Gaussian beam corresponding to a pure state. We wish to find the stationarity length and the spread in propagation constants for a Gaussian beam in the medium described by (21). It was shown in Ref. 30 that an arbitrary Gaussian beam having a spherical wavefront with a radius of curvature r and a width $\sigma_x'=\langle(\Delta\hat x)^2\rangle^{1/2}$, whose center is at the coordinate $x_0=\langle\hat x\rangle$ and which has a momentum $p_0=\langle\hat p\rangle$ that sets the inclination (θ) to the z axis ($p=n\sin\theta$), can be described by a correlated coherent state

$$|\alpha\rangle=\left(\frac{k\cos\chi}{\pi\mu}\right)^{1/4}\times\exp\left\{-e^{-i\chi}\left[\left(\frac{k}{2\mu}\right)^{1/2}x-(\cos\chi)^{1/2}\alpha\right]^2+\frac{\alpha^2}{2}-\frac{|\alpha|^2}{2}\right\}, \quad (22)$$

where

$$\mu=\frac{2\sigma_x r}{(4\sigma_x^2+r^2/k^2)^{1/2}},\quad \sin\chi=\frac{-2\sigma_x}{(4\sigma_x^2+r^2/k^2)^{1/2}}, \quad (23)$$

and the complex parameter of the correlated coherent state,

$$\alpha=(k/2\mu\cos\chi)^{1/2}(\langle\hat x\rangle+i\mu e^{i\chi}\langle\hat p\rangle) \quad (24)$$

is specified by the coordinate of the center and by the slope of the Gaussian beam. We assume that the correlated coherent state (22) is an eigenstate of the non-Hermitian annihilation operator $\hat a(\mu,\chi)$:

$$\hat a(\mu,\chi)|\alpha\rangle=\alpha|\alpha\rangle, \quad (25)$$

where

$$\hat a(\mu,\chi)=(k/2\mu\cos\chi)^{1/2}(\hat x+i\mu e^{i\chi}\hat p),\quad [\hat a,\hat a^+]=1. \quad (26)$$

The correlated coherent state in (22) is convenient for describing Gaussian beams because it makes it possible to calculate by simple algebra all the expectation values which characterize the beam. Hamiltonian (3) for the medium described by (21) can be expressed in terms of the operators $\hat a$ and $\hat a^+$ in (26) in the following way:

$$\hat H=\frac{\omega}{k}\left[\hat a^+\hat a(|u|^2+|v|^2)-\hat a^{+2}uv-\hat a^2u^*v^*+|v|^2+\frac{1}{2}\right], \quad (27)$$

where

$$\left\{\begin{matrix}u\\v\end{matrix}\right\}=\frac{1}{2}(\mu\cos\chi)^{-1/2}\left[\frac{1}{\omega^{1/2}}\{\pm\}\mu e^{i\chi}\omega^{1/2}\right],\quad |u|^2-|v|^2=1, \quad (28)$$

and the density matrix operator for the Gaussian beam corresponding to the pure correlated coherent state (22) is $\hat\rho=|\alpha\rangle\langle\alpha|$.

Accordingly, knowing the effect of the operator $\hat a$ on the state $|\alpha\rangle$ [described by (25)], and using (19), we can easily calculate the spread in propagation constants:

$$\begin{aligned}\langle(\Delta\hat\beta)^2\rangle_\alpha&=k^2\Big\{\frac{1}{n_0^2}(\langle\alpha|\hat H^2|\alpha\rangle-\langle\alpha|\hat H|\alpha\rangle^2)+\frac{1}{n_0^4}\\&\times(\langle\alpha|\hat H^3|\alpha\rangle-\langle\alpha|\hat H|\alpha\rangle\langle\alpha|\hat H^2|\alpha\rangle)+\ldots\Big\}\\&=\frac{\omega^2}{n_0^2}[|\alpha|^2(|u|^2+|v|^2)\\&-2(|u|^2+|v|^2)(\alpha^2u^*v^*+\alpha^{*2}uv)+|u|^2|v|^22(2|\alpha|^2+1)]\\&+\frac{\omega^3}{kn_0^4}[(|u|^2+|v|^2)^3\\&\times|\alpha|^2(1+2|\alpha|^2)+2(|u|^2+|v|^2)^2(|v|^2+1/2)|\alpha|^2\\&+4|u|^2|v|^2(|u|^2+|v|^2)|\alpha|^2(1+|\alpha|^2)+4|u|^2|v|^2\cdot\\&\times(|u|^2+|v|^2)^2(1+5|\alpha|^2+3|\alpha|^4)\\&+4|u|^2|v|^2(|v|^2+1/2)(1+2|\alpha|^2)\\&-(|u|^2+|v|^2)^2uv(4\alpha^{*2}+6\alpha^{*3}\alpha)\\&-(|u|^2+|v|^2)u^*v^*(4\alpha^2+6\alpha^*\alpha^3)\\&+4u^{*2}v^{*2}(|u|^2+|v|^2)\alpha^4\\&+4u^2v^2(|u|^2+|v|^2)\alpha^{*4}-2|u|^2|v|^2u^*v^*\\&\times(6\alpha^2+4\alpha^*\alpha^3)-2|u|^2|v|^2uv(6\alpha^{*2}+4\alpha\alpha^{*3})\\&-4uv(|u|^2+|v|^2)(|v|^2+1/2)\alpha^{*2}\\&-4u^*v^*(|u|^2+|v|^2)(|v|^2+1/2)\alpha^2]+\ldots.\end{aligned} \quad (29)$$

The first term in square brackets in (29) corresponds to the spread of the propagation constants in the paraxial approximation, while the second gives the first correction for the deviation from the paraxial situation.

Since the Gaussian beam of coherent radiation which we are considering here is described by a pure state, uncertainty relation (18) becomes an equality in this case, and the stationarity length of such a beam is

$$(Z_1)_\alpha=\langle(\Delta\hat\beta)^2\rangle_\alpha^{-1/2}. \quad (30)$$

Expression (29) thus actually specifies the stationarity length of an arbitrary Gaussian beam in the paraxial approximation and the first correction for a deviation from a paraxial situation.

It can be seen from (29) that as the parameter α increases, i.e., as the displacement of the beam center from the axis and the beam angle with the axis increase, the spread in propagation constants increases, while the stationarity length decreases. The stationarity length of a Gaussian beam propagating along the axis is

$$(Z_1)_0 \approx \left\{ \frac{\omega^2}{n_0^2} 2|u|^2|v|^2 + \frac{\omega^3}{kn_0^4} 4|u|^2|v|^2 \times \left[(|u|^2+|v|^2)^2 + \left(|v|^2 + \frac{1}{2} \right) \right] \right\}^{-1/2}. \quad (31)$$

Using (23) and (28), we easily see that the expressions (29) and (31) specify the stationarity lengths of arbitrary and axial Gaussian beams, respectively, as functions of the beam width $\sigma_x^{1/2}$ and of the radius of the wavefront r.

Ordinary coherent states specifying the paths and widths of rays in a medium with a quadratic refractive-index profile correspond to Gaussian beams with a plane phase front and with a width equal to the width of the fundamental mode.[8] For such coherent states, we should set $|u|^2 = 1$ and $|v|^2 = 0$ in the case of a medium described by (21). In this case we immediately obtain from (29) an expression for the stationarity length of the rays:

$$(Z_1)_\alpha = \left\{ \frac{\omega^2}{n_0^2} |\alpha|^2 + \frac{\omega^3}{kn_0^4} 2|\alpha|^2 (1+|\alpha|^2) \right\}^{-1/2}. \quad (32)$$

The stationarity length of an axial ray with a parameter $\alpha = 0$ becomes infinite (both in the paraxial approximation and when the deviation from a paraxial situation is taken into account). The reason is that an axial ray coincides with the fundamental mode, which is a stationary state. As the parameter α increases, i.e., as the deviation of the ray from the axis increases, the stationarity length decreases. Since a ray in the medium described by (21) is a Gaussian packet whose angular and spatial widths remain constant in the course of the propagation (the packet width is matched with the gradient parameter of the medium, ω), and whose center oscillates about the axis,[8] the stationarity length (32) of the ray is a measure of the maximum distance along the axis over which the displacement of the center of the packet does not yet exceed its width. To verify this, we consider a paraxial ray, described by a coherent state, in the complex plane of the parameter α which takes the following form when we take into account the z dependence of the expectation values $\langle \hat{x} \rangle$ and $\langle \hat{p} \rangle$ in (24):

$$\alpha = \left(\frac{k}{2} \right)^{1/2} \left(\omega^{1/2} x_0 + \frac{ip_0}{\omega^{1/2}} \right) \exp\left\{ -\frac{i\omega z}{n_0} \right\}, \quad (33)$$

where x_0 and p_0 are the initial coordinate (i.e., that at $z = 0$) of the center of the ray and its initial inclination. The coordinates x_0 and p_0 determine the initial position of the ray in the phase plane, while the region in which the ray is localized in the phase plane is specified by the variances

$$\langle (\Delta \hat{x})^2 \rangle_\alpha^{1/2} = 1/(2k\omega)^{1/2}, \quad \langle (\Delta \hat{p})^2 \rangle_\alpha^{1/2} = (\omega/2k)^{1/2}.$$

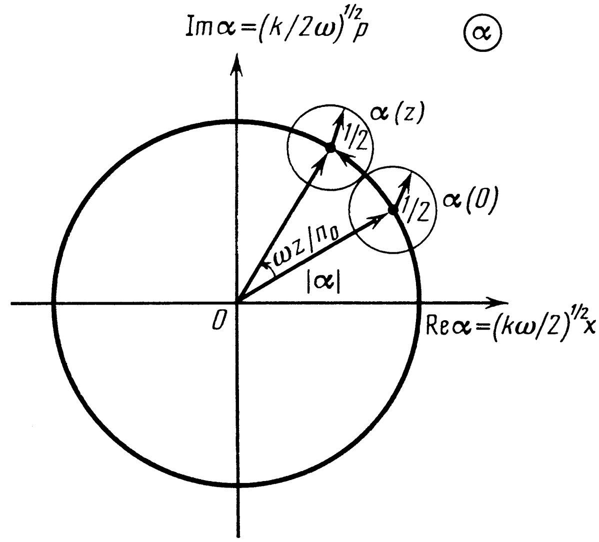

FIG. 1. Path traced out by a paraxial ray in the phase plane of the complex parameter α for a longitudinally homogeneous medium with a parabolic refractive-index profile.

For rays, the Heisenberg uncertainty relation (1) becomes an equality. It is easy to see that in the complex plane of the parameter α, (33), the center of the ray moves along a circle of radius $|\alpha|$, and the localization region of the ray field corresponds to a circle of radius $\Delta\ \mathrm{Im}\alpha = \Delta\ \mathrm{Re}\alpha = 1/2$ (Fig. 1).

If, by analogy with the Rayleigh resolution criterion in spectroscopy, we assume that two packets are distinguishable only if the distance between their centers exceeds their width, then we easily find from Fig. 1 that a ray can be assumed to have undergone a substantial displacement only if $|\Delta\alpha| = |\alpha(z) - \alpha(0)| \geqslant 1$. Assuming $|\alpha| \geqslant 1$, we can also determine that longitudinal distance $(Z_1)_\alpha$ over which the center of the ray is displaced a distance equal to its width in the complex α plane:

$$(Z_1)_\alpha = n_0/(\omega|\alpha|). \quad (34)$$

It is not difficult to see that this distance is in fact the same as the stationarity length (32) in the paraxial approximation.

Since the oscillation period of paraxial rays in the medium described by (21) is $T = 2\pi n_0/\omega$, the paraxial stationarity length of a ray, (34), can conveniently be written

$$(Z_1)_\alpha = T/(2\pi|\alpha|). \quad (35)$$

The stationarity length of an axial ray with $\alpha = 0$ is thus infinite; the stationarity length of a ray with $|\alpha| = 1$ is on the order of one-sixth of its oscillation period. As the deviation of the ray from the axis increases, the paraxial stationarity length of the ray falls off in inverse proportion to $|\alpha|$.

Example 2. Stationarity length of an arbitrary paraxial beam in a longitudinally inhomogeneous medium

In many cases of practical importance it is necessary to study how longitudinal irregularities of a medium affect such characteristics of a radiation beam as its width, the path of its center, its mode composition, and the intermode pulse dispersion. It is important here to have a simple way to estimate which longitudinal inhomogeneities can be ignored.

To resolve this question we consider an arbitrary beam in a medium which is longitudinally homogeneous at $\xi \leqslant 0$ but longitudinally inhomogeneous at $\xi > 0$. For such a medium, Hamiltonian (3) can be written as

$$\hat{H} = \hat{H}_0(x) + \Delta n^2(x, \xi), \tag{36}$$

where $\hat{H}_0$ is the Hamiltonian corresponding to the initial, longitudinally homogeneous, part of the medium at $\xi \leqslant 0$, and $\Delta n(x, \xi)$ is an arbitrary perturbation of the refractive index. We have $\Delta n(x, \xi) = 0$ at $\xi \leqslant 0$ and $\lim \Delta n(x, \xi) = \text{const}$ as $\xi \to \infty$.

We assume that in the initial part of the medium, $\xi \leqslant 0$, the beam is described by the density matrix $\hat{\rho}_0$, and the expectation value of its propagation constants and their standard deviation in the paraxial approximation are β_0 and $\langle (\Delta \hat{\beta}_0)^2 \rangle^{1/2}$, respectively. The stationarity length in the initial part of the medium thus satisfies uncertainty relation (18). For any $\xi > 0$ the beam is described by the density matrix $\rho(\xi)$ which is found from Liouville equation (4) with Hamiltonian (36). We assume that the expectation value of the propagation constants and their standard deviation at $\xi > 0$ are $\beta(\xi)$ and $\langle (\Delta \hat{\beta}(\xi))^2 \rangle^{1/2}$, respectively. Uncertainty relation (8) can then be written in the form

$$\langle (\Delta \hat{\beta}(\xi))^2 \rangle^{1/2} Z_n(\xi) \geqslant 1, \quad n = 1, 2, 3, \ldots, \tag{37}$$

where the stationarity length $Z_n(\xi)$ at any distance $\xi > 0$ is given by (5) and has the meaning of the stationarity length which the packet would have if it had begun to propagate through a longitudinally homogeneous medium at the time ξ. It is natural to assume that the longitudinal inhomogeneities substantially affect the beam characteristics if they lead to a change in the expectation value of the propagation constants, $\delta\beta(\xi) = \beta(\xi) - \beta_0$, by an amount exceeding their total variance, i.e., if

$$\delta\beta(\xi) \geqslant \langle (\Delta \hat{\beta}_0)^2 \rangle^{1/2} + \langle (\Delta \hat{\beta}(\xi))^2 \rangle^{1/2} \geqslant \langle (\Delta \hat{\beta}(\xi))^2 \rangle^{1/2}. \tag{38}$$

Alternatively, using (37), we can write

$$\delta\beta(\xi) \geqslant 1/Z_1(\xi). \tag{39}$$

Inequality (39) thus implicitly specifies the length $(\tilde{\xi})$ of that part of the longitudinally inhomogeneous part of the medium over which the change in the expectation value of the propagation constants does not yet exceed the variances of these constants. If the length of the inhomogeneous part of the medium exceeds $\tilde{\xi}$, this part of the medium will have a substantial effect on the characteristics of the radiation, but if the length of the inhomogeneous part is instead smaller than $\tilde{\xi}$ we can ignore the effect of the inhomogeneities. The longitudinal scale length $\tilde{\xi}$ and the stationarity length $Z_1(\xi)$ are thus useful parameters in a variety of calculations of practical importance.

Example 3. Stationarity length of a paraxial ray in a longitudinally inhomogeneous medium with a quadratic refractive-index profile

Our final example is a two-dimensional medium with a parabolic refractive index profile,

$$n^2(x, \xi) = n^2(0, \xi) - \omega^2(\xi) x^2 + 2f(\xi) x, \tag{40}$$

where the gradient parameter $\omega(\xi)$, which specifies the transverse distribution of the refractive index, and the function $f(\xi)$, which describes the curvature of the axis, vary in an arbitrary way in the longitudinal direction, taking on constant values at infinity: $\omega(\pm\infty) = \omega_\pm$, $f(\pm\infty) = 0$. [If we consider the medium at $\xi \geqslant 0$, we have $\omega(-\infty) = \omega(0) = \omega_-$ and $f(-\infty) = f(0) = 0$.]

We assume that at $\xi \leqslant 0$ a ray $|\alpha\rangle = \Psi_\alpha(0)$ propagates in the medium described by (40). As was shown in Ref. 8, all the characteristics of the beam at an arbitrary $\xi > 0$ can then be found by means of the invariant annihilation operator (an integral of motion)

$$\hat{a}(\xi) = i\left(\frac{k}{2}\right)^{1/2} [\varepsilon(\xi)\hat{p} - \dot{\varepsilon}(\xi)\hat{x}] - d(\xi). \tag{41}$$

In particular, in the course of the evolution in the medium described by (40) the ray $\Psi_\alpha(0)$ is described by the coherent state $\Psi_\alpha(\xi)$ which is constructed as a eigenstate of this operator:

$$\hat{a}(\xi)\Psi_\alpha(\xi) = \alpha\Psi_\alpha(\xi). \tag{42}$$

Here $\varepsilon(\xi)$ is a solution, selected in a definite way, for the equation of a classical oscillator with a variable "frequency" $\omega(\xi)$,

$$\ddot{\varepsilon}(\xi) + \omega^2(\xi)\varepsilon(\xi) = 0, \tag{43}$$

and the complex quantity $d(\xi)$ is given by

$$d(\xi) = i\left(\frac{k}{2}\right)^{1/2} \int_{-\infty}^{\xi} \varepsilon(\xi') f(\xi') d\xi'. \tag{44}$$

It is not difficult to show that the state $\Psi_\alpha(\xi)$ into which the initial ray $\Psi_\alpha(0)$ transforms in the course of the evolution is a correlated coherent state (22) of the type $|\alpha + d(\xi)\rangle$. The parameters μ and χ of this correlated coherent state are given by (28), in which u and v are replaced by

$$\tilde{u} = \frac{1}{2}\left(\omega_+^{1/2}\varepsilon - \frac{i\dot{\varepsilon}}{\omega_+^{1/2}}\right), \quad \tilde{v} = -\frac{1}{2}\left(\omega_+^{1/2}\varepsilon + \frac{i\dot{\varepsilon}}{\omega_+^{1/2}}\right), \tag{45}$$

and ω is replaced by ω_+. We also note that the displacement of the center of a Gaussian beam in the complex plane of the parameter α is specified by the function $d(\xi)$.

Since a ray transforms into a correlated coherent state in the medium described by (40), its stationarity length is described over any distance ξ by expression (30), where the standard deviation of the propagation constants is specified by the paraxial part of expression (29), in which in turn we replace u and v by $\tilde{u}$ and $\tilde{v}$ from (45), α by $\alpha + d(\xi)$, and ω by $\omega(\xi)$.

To see how longitudinal inhomogeneities influence the stationarity length of a ray over a finite longitudinally homogeneous part of a medium, we should examine the behavior of the solution of Eq. (43) in the limit $\xi \to +\infty$. Here it is convenient to use some parameters introduced in Ref. 8. These are the numerical parameters $0 \leqslant R < 1$ and $0 \leqslant \nu < \infty$ and the phases δ_1, δ_2, and β, and describe completely the behavior of a radiation beam in the medium (40) in the limit $\xi \to \infty$. The parameter R is formally the same as the quantum-mechanical coefficient for above-barrier reflection from a barrier whose shape is described by the function $\omega^2(\xi)$, while δ_1 and δ_2 are the same as the phases of the transmitted and reflected waves. The complex quantity

$$d=\lim_{\xi\to\infty} d(\xi)=\nu^{1/2}e^{i\beta}$$

specifies the beam-center displacement, caused by a longitudinal inhomogeneity, in the phase plane. Consequently, by simply varying the numerical values of these parameters we can describe the entire class of regular longitudinal inhomogeneities of the medium (40).

For brevity we write the explicit expression for the paraxial stationarity length in a finite longitudinally homogeneous part of the medium. Noting that in the limit $\xi\to\infty$ we have[8]

$$\tilde{u}=\frac{1}{(1-R)^{1/2}}\exp(i\delta_1+i\omega_+\xi),$$
$$\tilde{v}=\left(\frac{R}{1-R}\right)^{1/2}\exp(i\delta_2-i\omega_+\xi), \tag{46}$$

we find from (29)

$$(Z_1(+\infty))_\alpha=\frac{n_0(1-R)}{\omega_+}\{|\alpha+d|^2(1-R^2)+2R(2|\alpha+d|^2+1) -2R^{1/2}(1+R)[(\alpha+d)^2\exp\{-i(\delta_1+\delta_2)\} +(\alpha^*+d^*)^2\exp\{i(\delta_1+\delta_2)\}]\}^{-1/2}. \tag{47}$$

It can be seen from (47) that longitudinal inhomogeneities generally decrease the stationarity length, because of the packet spreading, studied in detail in Ref. 8. In the case of matched longitudinal inhomogeneities,[8] with $R=0$, the stationarity length is determined exclusively by the curvature of the axis and is

$$(Z_1(\infty))_\alpha=n_0/(\omega_+|\alpha+d|). \tag{48}$$

If, furthermore, there is no curvature of the axis, we have $d=0$, and over a finite region the stationarity length is equal, apart from replacement of ω_+ by ω_-, to a stationarity length (34) of the ray in the initial part of the medium, at $\xi\leqslant 0$.

Finally, in the case of mismatched inhomogeneities, for which we have $R\to 1$ or $\nu\to\infty$, and in which case the radiation escapes from the medium,[8] stationarity length (47) vanishes.

4. CONCLUSION

In summary, by using the density matrix formalism and an analog of the Tamm-Mandel'shtam energy-time uncertainty relation, we have been able to introduce a new longitudinal scale dimension—the radiation stationarity length—in a theoretical study of the propagation of arbitrary beams of coherent or partially coherent radiation in inhomogeneous media. The stationarity length is a measure of the distance over which the changes in the expectation values of the characteristics of the radiation do not yet exceed their variances. We find that the longitudinal inhomogeneities of a medium with a quadratic refractive index profile generally lead to a decrease in the stationarity length. We have studied in detail the stationarity lengths of rays in such media, and we have shown that the maximum stationarity length of an off-axis ray is on the order of $T/2\pi$, where T is the oscillation period of the paraxial rays.

It should be noted that the typical oscillation periods of rays in thick graded-index rods of the "Selfoc" type used in optics are on the order of a few centimeters. In the multimode graded-index fibers used in fiber-optics communications, these typical periods are on the order of a few millimeters; for propagation of sound in an underwater acoustic duct in the ocean the typical periods range from 6 to 60 km; and for ultra-long-range propagation of short radio waves in ionospheric ducts the periods range from 200 to several thousand kilometers.

The stationarity length thus proves to be extremely important in these problems, and it is a useful longitudinal scale dimension, which makes it possible, in particular, to estimate the length of those longitudinal inhomogeneities of a medium which cause only insignificant changes in the characteristics of radiation and which can therefore be ignored.

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APPENDIX

To prove uncertainty relation (8), we make use of the well-known fact that the following uncertainty relation holds for two Hermitian operators $\hat{A}$ and $\hat{B}$ and their commutator $[\hat{A},\hat{B}]=i\hat{C}$:

$$\langle(\Delta\hat{A})^2\rangle\langle(\Delta\hat{B})^2\rangle\geqslant{}^1/_4|\langle\hat{C}\rangle|^2, \tag{A1}$$

where

$$\langle(\Delta\hat{A})^2\rangle=\mathrm{Sp}\,\hat{\rho}\hat{A}^2-\mathrm{Sp}^2\hat{\rho}\hat{A},\quad \langle(\Delta\hat{B})^2\rangle=\mathrm{Sp}\,\hat{\rho}\hat{B}^2-\mathrm{Sp}^2\hat{\rho}\hat{B},$$
$$\langle\hat{C}\rangle=\mathrm{Sp}\,\hat{\rho}\hat{C}.$$

Assuming $\hat{A}=\hat{H}$, $\hat{B}=\dot{\hat{\rho}}/k$, we have $\hat{C}=[\hat{H},[\hat{\rho},\hat{H}]]$ by virtue of (4), and relation (A1) becomes

$$\langle(\Delta\hat{H})^2\rangle\frac{1}{k^2}\mathrm{Sp}\hat{\rho}\dot{\hat{\rho}}^2\geqslant\frac{1}{4}\mathrm{Sp}^2([\hat{H},[\hat{\rho},\hat{H}]]\hat{\rho}). \tag{A2}$$

Furthermore, using $\hat{\rho}^2\leqslant\hat{\rho}$, Liouville equation (4), and identity (6), we find

$$\frac{1}{4}\mathrm{Sp}^2([\hat{H},[\hat{\rho},\hat{H}]]\hat{\rho})\geqslant\frac{1}{4}\mathrm{Sp}^2([\hat{H},[\hat{\rho},\hat{H}]]\hat{\rho}^2) =\frac{1}{4}\mathrm{Sp}^2([\hat{\rho}^2,\hat{H}][\hat{\rho},\hat{H}])=\frac{1}{4k^4}\mathrm{Sp}^2(\dot{\hat{\rho}}^2)\dot{\hat{\rho}}=\frac{1}{k^4}\mathrm{Sp}^2\hat{\rho}\dot{\hat{\rho}}^2. \tag{A3}$$

Consequently, from relation (A1) we find the following inequality, assuming $\mathrm{Sp}\hat{\rho}\dot{\hat{\rho}}^2\neq 0$ and using inequality (9):

$$\langle(\Delta\hat{H})^2\rangle\geqslant\frac{1}{k^2}\mathrm{Sp}\,\hat{\rho}\dot{\hat{\rho}}^2\geqslant\frac{1}{k^2}\mathrm{Sp}\,\hat{\rho}^n\dot{\hat{\rho}}^2. \tag{A4}$$

This is the same as uncertainty relation (8).

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