We show in this paper that helical hydrodynamic turbulence can magnify perturbation seed eddies and that this can lead to the appearance of large-scale structures. The characteristic size of the structures which appear is much larger than the size of the turbulence containing the energy and it is determined by the ratio of the topological invariant to the turbulence energy. The evolution of the instability found in this paper is accompanied by a transfer of energy from small-scale to large-scale fluid motions.

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1. BASIC EQUATIONS

We shall assume that the initial turbulence is described by the Navier-Stokes equation for the random velocity $V'$ and by the equation of continuity for turbulent pulsations of density $\rho'$. For the sake of simplicity we consider a polytropic gas with polytropic index equal to two:

$$\rho'_{t} + (\rho'V')_{\cdot} = 0,$$

where $\rho'$ is the density and $c_{s}$ the sound velocity for $\rho = \rho_0$.

In that case the initial equations have the form

$$\frac{\partial V'}{\partial t} + V' \cdot \nabla V' = -\nabla p' + \eta \nabla^2 V' + \xi,$$

$$\frac{\partial \rho'}{\partial t} + \rho' \nabla \cdot V' = 0,$$

where $\xi$ is a random part of the velocity.

The appearance of a large-scale vortex structure must mean the existence of an instability leading to the magnification and support of large-scale perturbations at the expense of the small-scale turbulence. We shall understand by large-scale perturbations an average field $\langle V \rangle$ with characteristic size $L \approx 1$.

Let us at $t_0 = 0$ create a small large-scale perturbation $\langle V_0 \rangle$. When $t > t_0$, after $\langle V_0 \rangle$, the average $\langle V(x,t) \rangle$ is changed to $\langle V(x,t) \rangle$ and the random part of the velocity has acquired an addition $\langle V(x,t) \rangle$. This means that the total velocity at any time can identically be written in the form

$$\langle V \rangle = \langle V \rangle + \xi; \quad \langle \xi \rangle = 0.$$

The problem of the origin of structures in non-equilibrium systems has recently attracted a great deal of attention. (See in this connection the monographs 1 to 3 and the literature cited there.) Hydrodynamic turbulence is clearly one of the most widespread distributed non-equilibrium systems in nature. It is well known that turbulence is characterized by two scales: $A_1$, the external scale which contains the main energy and the dissipation scale $A_2$, where the energy is dissipated into heat. A turbulent system is thus an open non-equilibrium system.

The traditional presentation of turbulence is connected with that kind of disordered fluid motion in which any large-scale perturbation transfers its energy through random chaotic motions to ever smaller scales. It would seem that such ideas about turbulence are incompatible with the existence of coherent structures in it. However, recent papers (see, e.g., Refs. 4 to 7) indicate that there may appear in turbulence large-scale vortex formations, the nature of which in final analysis is not explained.

In a homogeneous isotropic turbulence a large-scale perturbation (fluctuation) transfers its energy, apparently, to small-scale pulsations, owing to damping by turbulent viscosity. If, however, the turbulence is helical, meaning the existence of an instability leading to the magnification of small-scale coherent structure, the characteristic size $L$ of the vortex structure is determined by the internal properties of the turbulence. The characteristic size $L$ of the structures found are therefore not "relics" of the average flow (such as the von Karman vortices) but are produced by the helical turbulence itself. Their characteristic size $L$ is thus naturally connected only with the integral characteristics of the turbulence. The result obtained is interesting already for the reason that it is possible to obtain it systematically from the Navier-Stokes equations without having recourse to phenomenological models.

Generally speaking, the transfer of energy to large vortices is characteristic also of two-dimensional turbulence, but in that case there does not appear a well-defined scale of the long-wavelength fluctuations and there occurs a continuous transfer along the spectrum (see, e.g., Ref. 8). We note that the mechanism considered in the present paper for the generation of large-scale vortices is impossible as a matter of principle in two-dimensional turbulence as in that case $V \cdot \nabla V = 0$. 

INTRODUCTION

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Let us at $t_0 = 0$ create a small large-scale perturbation $\langle V_0 \rangle$. When $t > t_0$, after $\langle V_0 \rangle$, the average $\langle V(x,t) \rangle$ is changed to $\langle V(x,t) \rangle$ and the random part of the velocity has acquired an addition $\langle V(x,t) \rangle$. This means that the total velocity at any time can identically be written in the form

$$\langle V \rangle = \langle V \rangle + \xi; \quad \langle \xi \rangle = 0.$$
Similarly we can write the total density \( \rho \) also in the form
\[
\rho = \langle \rho \rangle + \langle \rho' \rangle,
\]
where \( \langle \rho \rangle \) and \( \langle \rho' \rangle \) are the average and fluctuating densities, respectively.

Clearly, the total velocity \( V \) as well as the velocity \( V' \) and \( \rho \) satisfy the Navier-Stokes equation, and the total density the continuity equation. We shall understand by the average values everywhere the result of averaging over an ensemble of realizations of the turbulent pulsations \( V' \) and \( \rho' \). Our problem consists in obtaining closed equations for the average velocity \( \langle V \rangle \) and density \( \langle \rho \rangle \). To do this we use the approach developed in Refs. 9 and 10 and operate on the Novikov-Furutsu formula.

Substituting (3) and (4) into the Navier-Stokes and continuity equations and after that taking the ensemble average we get the equations
\[
\frac{\partial \langle V \rangle}{\partial t} + \nabla \cdot \langle \rho \rangle \langle V \rangle = -\nabla p + \nabla \cdot \Gamma + \langle \rho' \rangle \langle V' \rangle + \langle \rho \rangle \langle \rho' \rangle \langle V \rangle,
\]
\[
\frac{\partial \langle \rho \rangle}{\partial t} + \nabla \cdot \langle \rho \rangle \langle V \rangle = \langle \rho' \rangle \langle V' \rangle + \langle \rho \rangle \langle \rho' \rangle \langle V \rangle - \nabla \cdot \rho \langle \nabla P \rangle/\gamma.
\]
Equations (5), (6) and (7) are the basis of what follows.

2. CLOSURE OF THE AVERAGED EQUATIONS

The basic problem consists in the closure of Eqs. (5) and (6). As we consider the case with a small average velocity against the background of initially homogeneous and isotropic turbulence, i.e.,
\[
\langle \langle V' \rangle \rangle = 0,
\]
we may also assume that the inhomogeneous perturbations of the random field \( V' \) produced by the average velocity \( \langle V \rangle \) are also small:
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\langle \langle V' \rangle \rangle = 0.
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Later we shall indicate the criterion for satisfying inequalities (16). The condition \( \lambda > \lambda_0 \) allows us to separate the slow evolution of \( \langle V \rangle \) from the fast turbulent pulsations. In that case we can assume the turbulent pulsations \( V' \) to be, to a first approximation, delta-function-correlated in time as compared to the slow evolution of the average velocity \( \langle V \rangle \), i.e.,
\[
\langle \langle V' \rangle \rangle = 0.
\]
In the present paper we consider the case which is the simplest one from a formal point of view when the random field of the turbulent fluctuations in the velocity can be considered to be Gaussian. Turbulence consisting mainly of vortices with characteristic size \( \lambda_0 \) can serve as an example of such a situation. The velocity field after the lapse of a sufficiently long time is also Gaussian, if the turbulence source is
switched off or even if we do not consider strong turbulence, but small hydrodynamic fluctuations.

To close Eqs. (5) and (6) we must evaluate the averages

\[ \langle p_k, \frac{\partial V}{\partial x_n} \rangle, \quad \langle p, \frac{\partial V}{\partial x_n} \rangle, \quad \langle \phi' V' \rangle, \quad \langle \phi' \rangle \]

The quantities \( F \) and \( \phi' \) are functionals of the random fields \( V' \) and \( p' \). The way \( F \) and \( \phi' \) depend on \( V' \) and \( p' \) is given by Eqs. (7), (8). In our statement of the problem the random fields are assumed to be given and described by the correlators (9), (10). To evaluate the averages in which we are interested we can use the Novikov-Furutsu formula (11,12) (see also Ref. 13):

\[ \langle F(z), \phi' \rangle = \int \langle F(x), \phi' \rangle \delta(z - x) \, dx \]  
(18)

where \( z \) is a Gaussian random process and \( F(z) \) a functional of this random process.

The application of Eq. (18) gives in our case

\[ \langle F(x, t), V' \rangle = \int \langle F(x, t'), V' \rangle \delta(t - t') \, dt' \]  
(19)

\[ \langle \phi(x, t), V' \rangle = \int \langle \phi(x, t'), V' \rangle \delta(t - t') \, dt' \]  
(20)

\[ \langle \phi'(x, t), V' \rangle = \int \langle \phi'(x, t'), V' \rangle \delta(t - t') \, dt' \]  
(21)

The integrations in Eqs. (19) to (21) are over the whole of space while the integrals over time are taken from zero to \( t \), which reflects the causality principle. (The quantities \( V, \phi' \) can depend on \( V', \phi' \) only at earlier times.) The averages of interest to us which occur in Eq. (5) are found from (19). For instance,

\[ \langle F(x, t), \frac{\partial V}{\partial x_n} \rangle = \lim_{n \to \infty} \langle F(x, t), \frac{\partial V}{\partial x_n} \rangle \]  
(22)

\[ \langle \phi(x, t), \frac{\partial V}{\partial x_n} \rangle = \int \langle \phi(x, t'), \frac{\partial V}{\partial x_n} \rangle \delta(t - t') \, dt' \]  
(23)

\[ \langle p(x, t), \frac{\partial V}{\partial x_n} \rangle = -\frac{1}{2} \frac{\partial }{\partial x_n} \delta(x - y) \]  
(24)

\[ \langle \phi'(x, t), \frac{\partial V}{\partial x_n} \rangle = -\frac{1}{2} \frac{\partial }{\partial x_n} \langle \phi'(x, t') \rangle \delta(x - y) \]  
(25)

Substituting (17) and (22) to (25) into Eqs. (19) to (21) and integrating we get for the averages

\[ \langle F(x, t), \frac{\partial V}{\partial x_n} \rangle + \langle F(x, t), \frac{\partial V}{\partial x_n} \rangle \]  
(26)

\[ \frac{\partial}{\partial x_n} \langle \phi V \rangle = -\frac{1}{4} C(0) \langle \phi \rangle + \frac{1}{4} D(0) \text{div}(V) \]  
(27)

\[ \frac{\partial}{\partial x_n} \langle \phi' V \rangle = -\frac{1}{4} D(0) \text{div}(V) \]  
(28)

We used Eq. (13) in obtaining Eqs. (26) to (28).

Substituting (26) to (28) into Eqs. (5), (6) we finally get

\[ \frac{\partial}{\partial t} + \frac{1}{2 \rho} \langle \phi \rangle = -\frac{1}{2} C(0) \text{div}(V) - \frac{1}{2 \rho} \langle \phi \rangle \]  
(29)

\[ \frac{\partial}{\partial t} + \text{div}(\langle \phi \rangle V) + \frac{1}{2} D(0) \text{div}(V) + \frac{1}{4} C(0) \text{div}(\langle \phi \rangle) \]  
(30)

where \( v = C(0) / 4 \). We note that there occurs in Eq. (30) a term proportional to \( D(0) \text{div}(V) \). The presence of that term leads to a change in the constant part of the average density:

\[ \langle \rho \rangle = \langle \rho \rangle + D(0) / 2 \]  

and, as a consequence of this, to a change in the sound velocity

\[ c = \sqrt{\langle \rho \rangle + D(0) / 2} \]  

for small Mach numbers \( M^2 = (\langle p' \rangle^2 / \rho^2) \leq 1 \) this change in the sound speed is small.

3. LARGE-SCALE INSTABILITY AND APPEARANCE OF STRUCTURE

The solenoidal part of the velocity is split off from the potential part in Eq. (29). As we are interested in the generation of vortex structures we consider the solenoidal part of the velocity. Operating with the curl on Eq. (29) and introducing the notation \( \Omega = \text{curl}(V) \) we get

\[ \frac{\partial}{\partial t} + \frac{1}{2 \rho} \langle \phi \rangle + \frac{1}{2} D(0) \text{div}(V) + \frac{1}{4} C(0) \text{div}(\langle \phi \rangle) \]  

and, as a consequence of this, to a change in the sound velocity

\[ c = \sqrt{\langle \rho \rangle + D(0) / 2} \]  

For small Mach numbers \( M^2 = (\langle p' \rangle^2 / \rho^2) \leq 1 \) this change in the sound speed is small.
supported by turbulence and, hence, lead to the appearance of coherent structures.

Fourier-transforming Eq. (31) leads to

\[ -i\omega \Omega(k) - (-i)g(k \Omega(k)) + i\gamma \omega \Omega(k) = 0. \tag{32} \]

Choosing the system of coordinates such that the \( z \)-axis is directed along the vector \( k \) and using the relation \( k \times \Omega = 0 \) we get the dispersion equation

\[ \omega = \omega(k) = -i \Delta \nu = -i \Delta \nu. \tag{33} \]

From this equation it is clear that \( \omega_{\text{max}} = 0 \). The instability growth rate \( \gamma \) is equal to

\[ \gamma(k) = -i \Delta \nu = -i \Delta \nu. \tag{34} \]

We easily find from (34) the largest value of the growth rate \( \gamma_{\text{max}} \), and the value \( \Delta \nu \) for which the growth rate has its maximum

\[ \gamma_{\text{max}} = k \Delta \nu/\Delta \nu, \tag{35} \]

\[ k = \Delta \nu/\Delta \nu. \tag{36} \]

As a result we get an estimate for the characteristic size of the instability:

\[ L^* = (V \text{ rot } V) / \gamma_{\text{max}} E_r / E_r. \tag{37} \]

Formula (38) is interesting because in it there appears only the ratio of two integrals of motion: the topological invariant and the normal invariant. The topological invariant \( L_r \) characterizes the number of linkages of vortex lines with flow lines. The characteristic size of unstable fluctuations is thus determined by the "natural" characteristics of the turbulence, its invariants, and is an internal property of the turbulence itself.

Since assumed from the outset the instability to be large-scale, we must require that

\[ k \ll \Delta \nu, \tag{39} \]

Inequality (39) is the condition on the correlator parameter of the turbulence velocity field (11):

\[ \Delta \nu \ll (V \text{ rot } V) / E_r. \tag{40} \]

Apart from condition (39) we assumed that the large-scale instability was slow, i.e.,

\[ T - \gamma \gg \tau, \tag{41} \]

\[ \frac{1}{E_r} \gg \frac{\lambda}{E_r} \gg \frac{\lambda}{E_r}, \tag{42} \]

i.e., (42) follows from (40). Thus, a long-wavelength instability is automatically slow, i.e., it can be considered in the delta-function-correlated random process approximation.

One can estimate from Eq. (7) the characteristic magnitude of the velocity \( F_0 \):

\[ \left\langle (V \text{ rot } V) / E_r \right\rangle \Delta \nu. \tag{43} \]

The main terms in (7) give

\[ \frac{d}{dt} \left\langle V \right\rangle \text{ rot } \frac{\partial \left\langle V \right\rangle}{\partial x} - 1; \tag{44} \]

whence we find

\[ \right\langle F_0 \right\rangle \sim \left\langle (V \text{ rot } V) / (V \text{ rot } V) \right\rangle \Delta \nu. \tag{45} \]

The correlator \( \left\langle F_0 \right\langle (x_1 \lambda) \right\langle (x_2 \lambda) \text{ rot } (x_3 \lambda) \right\rangle \right\rangle \) taken in one point \( (x_1 \lambda) \) depends on \( t_1 \) and \( \gamma \gg \tau \) and the time scale of turbulence \( \tau \). Therefore

\[ \left\langle F_0 \right\rangle \sim \left\langle (V \text{ rot } V) / (V \text{ rot } V) \right\rangle \Delta \nu. \tag{46} \]

As \( \tau \ll T \) we have \( F_0 \Delta \nu \) as we assumed earlier. We can rewrite the estimate (44) as

\[ \left\langle F_0 \right\rangle \sim \left\langle (V \text{ rot } V) / (V \text{ rot } V) \right\rangle \Delta \nu. \tag{47} \]

We note that from the estimates (44), (45) it follows that \( \left\langle F_0 \right\rangle \sim (t / \lambda) \right\rangle \) \( \left\langle (V \text{ rot } V) \right\rangle \). Hence, as we consider the case \( \left\langle V \text{ rot } V \right\rangle \), hence

\[ (t / \lambda) \Delta \nu \gg 1. \tag{48} \]

From (44) it follows that \( \tau \ll (t / \lambda) \Delta \nu \) and hence condition (46) is equivalent to \( E_r \Delta \nu \) and is satisfied automatically. Then (45) gives

\[ \left\langle F_0 \right\rangle \sim \left\langle (V \text{ rot } V) / (V \text{ rot } V) \right\rangle \Delta \nu. \tag{49} \]

We can introduce the spectrum of the quantity \( F \);

\[ \left\langle F_0 \right\rangle \sim \int F(k) \Delta \nu. \tag{50} \]

We then get from (44) an estimate for the spectral density:

\[ E(k) \sim (t / \lambda) \Delta \nu. \tag{51} \]

Hence it follows that the characteristic scale of the fluctuations \( F_0 \) is the same as the external scale of turbulence \( \lambda \).

The characteristic time of these fluctuations is of order \( \tau \) because of the transfer of \( \nu \) by the velocity \( V \). In the long-wavelength part of the spectrum we have \( F \sim (t / \lambda) \Delta \nu \text{ rot } (V \text{ rot } V) \) as the energy-containing part of \( F \) is concentrated at a scale \( \lambda \), and satisfies the estimate (47). We note that we can use Eq. (31) to find the actual configuration of the structure in \( x \)-space. To do this it is necessary to assign to it an actual boundary value problem.

CONCLUSION

We note that in magnetohydrodynamics helical turbulence has the capacity to magnify initially small magnetic fields.\(^\text{16,17}\) The equation of the “dynamo” theory which is then obtained has the same structure as Eq. (31). The growth rates of the two instabilities then turn out to be of the same order of magnitude.

The results obtained show that there occurs in helical turbulence a large-scale instability leading to the appearance of vortex structures. It follows from Eq. (31) that these structures themselves are helical and it is natural to call them “g-structures.” The evolution of the instability found here is accompanied by the transfer of energy from small-scale to large-scale sizes. We emphasize that the g-structures are not relics of the average flow such as the von Karman vortices and the instability found in this paper is not a modification of the instability of shear flows. The magnification of a coherent structure occurs merely due to the fact that the turbulence is not invariant under reflection.

It is important that the characteristic size of the struct-
tures which occur is determined by the internal characteristics of the turbulence: the ratio of the topological invariant to the energy of the turbulence. In the present paper we considered the Gaussian case which, apparently, contains the basic physical traits of the process of the appearance of a large-scale structure in turbulence. It is nevertheless interesting that an isotropic velocity field is able to magnify large-scale regular motions.

The case of non-Gaussian velocity field is more complicated from a formal point of view. However, also in that case it is possible to obtain closed averaged equations of motion. The basic features of the large-scale instability are apparently retained. We note merely that the moments \((V_1 V_3 \ldots V_n \text{curl } V')\) which characterize higher order helicity contribute to the growth rate of the instability. We shall consider the case of non-Gaussian turbulence in another paper.

Note added in proof (31 October 1983). In the present paper we considered a model in which the velocity of the turbulent pulsations is assumed to be a random process, -correlated in time. In the case of a small, but finite correlation time instead of \(g(t)\) there occurs in Eq. (31) \(Mg'(t)\) where \(M = \left(\langle \alpha^2 \rangle \right)'/c_0\) is the Mach number, and correspondingly, the growth rate \(\gamma\) (see [35]) is multiplied by \(M\).

Translated by D. ter Haar

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