A new kind of oscillations in a relativistic plasma

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(Submitted 18 April 1983)
Zh. Eksp. Teor. Fiz. **85**, 1585–1589 (November 1983)

We study the dispersion equation for plasma waves in a one-dimensional and in a three-dimensional relativistic plasma. We show that apart from the Langmuir oscillations in one-dimensional non-isothermal and isothermal plasmas yet another form of plasma oscillations is possible with a dispersion law which is similar to that of ion-sound oscillations.

PACS numbers: 52.35. – g, 52.60. + h

Recently interest has been shown in theoretical studies of high-frequency oscillations in a high-temperature plasma with a temperature T of the order of or larger than the electron rest mass energy $m_e c^2$. This interest is caused, firstly, by the problem of the radio-emission by pulsars, since according to present-day ideas the pulsar atmosphere consists of a relativistic electron-positron plasma[1]; secondly, by the progress in experimental plasma physics. Among the most fundamental papers in this direction we must mention that by Silin[2] in which the dispersion was found for plasma waves with phase velocities larger than the velocity of light, and the paper by Mikhaĭlovskiĭ[3] in which the dispersion and damping rate of plasma waves was studied in the $\omega/k \sim c$ region. Using the results by Silin and Mikhaĭlovskiĭ we show in the present paper that in a relativistic one-dimensional plasma yet another form of plasma oscillations is possible with a dispersion law similar to the dispersion of ion-sound oscillations in a nonrelativistic plasma.[4]

We consider the dispersion equation for longitudinal plasma waves propagating along the x axis in a free plasma (see, e.g., Ref. 1):

$$1+\sum\frac{4\pi e^2}{mc^2}\int\limits_L\frac{u_0u^1}{k_\alpha u^\alpha k_0}\frac{\partial f_0}{\partial u^1}d^3u=0, \tag{1}$$

where k_α is the four-dimensional wave vector, u_α the velocity 4-vector, f_0 the uperturbed velocity distribution function of the plasma particles, and L a Landau contour; the summation is over kinds of particle (electrons and ions).

We shall assume the plasma to be one-dimensional with a thermal velocity spread along the x axis. Such a distribution is realized when there is a strong external magnetic field present; in that case the dispersion equation for longitudinal waves propagating along the field will have the form (1).[5] We choose the velocity distribution for the plasma particles in the form of a relativistic Maxwell distribution so that

$$f_0=n[\exp(-\alpha u_0)/2K_1(\alpha)]\,\delta(u_2)\,\delta(u_3), \tag{2}$$

where $\alpha = mc^2/T$, n the particle number density, and $K_1(\alpha)$ a Macdonald function. Substituting (2) in (1) and introducing a new integration variable u_0 we can easily transform (1) into

$$1+\sum\frac{\alpha a^2\omega_p^2}{2k^2c^2K_1(\alpha)}\left\{\int\limits_1^\infty\left[1+\frac{a^2-1}{2a}\left(\frac{1}{u_0-a}-\frac{1}{u_0+a}\right)\right]u_0e^{-\alpha u_0}\right.$$
$$\left.\times(u_0^2-1)^{-1/2}du_0+i\pi\theta(-\mathrm{Im}\,\omega)\,\theta(-\mathrm{Re}\,\omega+kc)\,(a^2-1)^{1/2}e^{-\alpha a}\right\}$$
$$=0, \tag{3}$$

where $a = kc/(k^2c^2-\omega^2)^{1/2}$, $\theta(x)$ is the Heaviside function, and $\omega_p^2 = 4\pi e^2 n/m$ the plasma frequency.

Furthermore, using the obvious identity

$$\frac{e^{-\alpha u_0}}{u_0\mp a}=e^{\mp\alpha a}\left(\frac{1}{u_0\mp a}-\int\limits_0^\alpha e^{-(u_0\mp a)x}dx\right),$$

we change the order of integration in (3) and taking into account the representation for the zeroth order Macdonald function:

$$K_0(x)=\int\limits_1^\infty e^{-xu_0}(u_0^2-1)^{-1/2}du_0,$$

we find for the dispersion equation

$$1+\sum\frac{\alpha a^2\omega_p^2}{2k^2c^2K_1(\alpha)}\left\{K_1(\alpha)+\frac{a^2-1}{2}[J(a)+J(-a)]\right.$$
$$\left.+i\theta(-\mathrm{Im}\,\omega)\,\theta(-\mathrm{Re}\,\omega+kc)\pi(a^2-1)^{1/2}e^{-\alpha a}\right\}=0, \tag{4}$$

where

$$J(a)=e^{\alpha a}\left(\frac{\ln[(a^2-1)^{1/2}+a]}{(a^2-1)^{1/2}}-\int\limits_0^\alpha K_0(x)e^{-ax}dx\right). \tag{5}$$

In the ultrarelativistic temperature limit ($\alpha \ll 1$) and using the expansion of the Macdonald function for small values of its argument[6]:

$$K_0(x)=-\ln(x/2)-C+\ldots \tag{6}$$

(C is the Euler constant) we find for (5)

$$J(a)=e^{\alpha a}\left(\frac{\ln[(a^2-1)^{1/2}+a]}{(a^2-1)^{1/2}}\right.$$
$$\left.-\frac{1}{a}\left[E_1(\alpha a)+\ln 2a+\left(\ln\frac{\alpha}{2}+C\right)e^{-\alpha a}\right]\right), \tag{7}$$

where $E_1(\alpha a)$ is an integral exponential function of a complex argument.[6]

The dispersion equation for a plasma with a Maxwellian particle velocity distribution admits thus in the ultrarelativistic temperature limit ($T \gg mc^2$) of a representation in terms of the well known special function $E_1(z)$, while in the nonrelativistic temperature limit ($T \ll mc^2$) the dispersion equation can be expressed in terms of the probability integral of complex argument.

We consider now waves with phase velocities close to the velocity of light, for which $|a| \gg 1$. In that case the function (7) takes the form

 0038-5646/83/110922-03$04.00

$$J(a)=-[E_1(z)e^z+\ln(\alpha/2)+C]/a, \quad (8)$$

where $z=\alpha a$. Substituting (8) into (4) we find in the limits $\alpha \ll 1$ and $|a| \gg 1$

$$1+\sum \frac{\omega_p^2 z^2}{4\alpha k^2c^2}\Psi(z)=0 \quad (9)$$

where

$$\Psi(z)=zE_1(-z)e^{-z}-zE_1(z)e^z+2+i2\pi\theta(-\operatorname{Im}\omega)\theta(-\operatorname{Re}\omega+kc)ze^{-z}. \quad (10)$$

We can get from Eq. (9) as special limiting cases the approximate formulae of Silin and Mikhaĭlovskiĭ for a one-dimensional relativistic plasma with $|a| \gg 1$. Indeed, using the expansions of the integral exponential function for large and small values of its argument[6]:

$$E_1(z)=-C-\ln z+\ldots, \quad |z|\ll 1,$$

$$E_1(z)=\frac{1}{z}\left(1-\frac{1}{z}+\frac{2}{z^2}+\ldots\right)e^{-z}, \quad |z|\gg 1,$$

we have for the function (10)

$$\Psi(z)=1+{}^1/_2 i\pi z\theta(-\operatorname{Re}\omega+kc), \quad |z|\ll 1, \quad (11)$$

$$\Psi(z)=-4/z^2+2i\pi\theta(-\operatorname{Im}\omega)\theta(-\operatorname{Re}\omega+kc)ze^{-z}, \quad |z|\gg 1, \quad (12)$$

and substitution of (11) and (12) into (9) leads to the dispersion formulae for a one-dimensional plasma with $\omega \sim kc$ in the Silin and Mikhaĭlovskiĭ approximation.

We now study the question of the existence of plasma waves in the region

$$k^2c^2 \ll (\omega_p^2/\alpha)_{i,e}. \quad (13)$$

In that case we get, neglecting unity in (9), the equation

$$T_i\Psi(z_e)+T_e\Psi(z_i)=0. \quad (14)$$

The question of the existence of plasma waves in the region (13) reduces thus to the problem of the existence of roots of the transcendental equation (14).

We consider a non-isothermal plasma with an electron temperature much higher than the ion temperature: $T_e \gg T_i$. In that case, Eq. (14) takes the form

$$\left[1+i\frac{\pi}{2}\frac{\alpha_e}{\alpha_i}z_i\theta(-\operatorname{Re}\omega+kc)\right]\frac{T_i}{T_e}-\frac{4}{z_i^2}+i2\pi\theta(-\operatorname{Im}\omega)\theta(-\operatorname{Re}\omega+kc)z_i\exp(-z_i)=0, \quad (15)$$

if $|z_e| \ll 1$ and $|z_i| \gg 1$, and if we use (11) and (12). Introducing the notation

$$z_i=x+iy, \quad (16)$$

and make the assumption that $|y| \ll |x|$ we get the solution

$$x=2(T_e/T_i)^{1/2}, \quad (17)$$

$$y=-x\left\{\frac{\pi}{2}\frac{m_e}{m_i}\left(\frac{T_i}{T_e}\right)^{1/2}+2\pi\left(\frac{T_e}{T_i}\right)^{3/2}\exp\left[-2\left(\frac{T_e}{T_i}\right)^{1/2}\right]\right\}. \quad (18)$$

The expression within the braces in (18) must, according to the assumptions we have made, be much less than unity.

Using the notation we introduced for $z_i=\alpha_i kc/(k^2c^2-\omega^2)^{1/2}$ we have for the real frequency $\omega'=\operatorname{Re}\omega$ and the damping rate $\gamma=\operatorname{Im}\omega$ of the plasma waves

$$\omega'=kc\left[1-\frac{\alpha_i^2}{2}\frac{x^2-y^2}{(x^2+y^2)^2}\right], \quad (19)$$

$$\gamma=\alpha_i^2\frac{xy}{(x^2+y^2)^2}kc. \quad (20)$$

Substituting Eqs. (17) and (18) into (19) and (20) we find

$$\omega'=kc\left(1-\frac{\alpha_i^2}{8}\frac{T_i}{T_e}\right), \quad (21)$$

$$\gamma=-\alpha_i^2\frac{T_i}{4T_e}\left\{\frac{\pi}{2}\frac{m_e}{m_i}\left(\frac{T_i}{T_e}\right)^{1/2}+2\pi\left(\frac{T_e}{T_i}\right)^{3/2}\times\exp\left[-2\left(\frac{T_e}{T_i}\right)^{1/2}\right]\right\}kc. \quad (22)$$

The solution obtained satisfies all assumptions made above and exists also in a one-dimensional non-isothermal electron-positron plasma.

An interesting feature of a relativistic one-dimensional plasma is the possibility of the existence in it of plasma oscillations of the form (19) in the region (13) also for the isothermal case when $T_e = T_i$.[7] Indeed, let us consider an electron-positron plasma when $m_e = m_i$. In that case Eq. (14) becomes

$$\Psi(z_e)=0. \quad (23)$$

Using the tables[6] for $E_1(z)$ for a numerical analysis of Eq. (10) we found that Eq. (23) has at least one complex root

$$z_e=x+iy=7{,}467-7{,}092i. \quad (24)$$

Substituting (24) into Eqs. (19) and (20) we have for the frequency and the damping rate of the plasma waves

$$\omega=kc(1-0{,}243\cdot 10^{-3}\alpha_e^2), \quad (25)$$

$$\gamma=-4{,}71\cdot 10^{-3}\alpha_e^2 kc. \quad (26)$$

The solution (25), (26) will be valid also for an isothermal one-dimensional relativistic electron-ion plasma with $m_i \gg m_e$. To check this we note that $z_i = z_e m_i/m_e \gg z_e$ and hence that we can neglect in Eq. (14), according to (12), the ion terms. The dispersion equation for a relativistic electron-ion plasma with heavy ions will thus be the same as (23) in the region (13) and have the solution (25), (26).

We note that the initial Eq. (1) was obtained neglecting completely collisions (correlations) between the particles, the effect of which will be important for small wave vectors.[8]

We consider the propagation of plasma waves in a three-dimensional plasma with a Maxwellian particle velocity distribution:

$$f_0=n\alpha\exp(-\alpha u_0)/4\pi K_2(\alpha). \quad (27)$$

Substituting (27) into (1) and performing the same calculations which led to Eq. (4) we get

$$1+\sum\frac{\omega_p^2\alpha}{k^2c^2}\left\{1+\frac{\alpha}{K_2(\alpha)}\frac{d^2}{d\alpha^2}\left[\frac{1}{\alpha}\frac{a^2-1}{2a}[J(-a)-J(a)]+2\pi\theta(-\operatorname{Im}\omega)\theta(-\operatorname{Re}\omega+kc)e^{-\alpha a}(a^2-1)^{-1/2}\right]\right\}=0. \quad (28)$$

According to (28), at $|a| \lesssim 1$ Eq. (28) is the same as Silin's dispersion equation which does not contain solutions different from the Langmuir branch.[1] For $|a| \gg 1$ we find

$$1+\sum \frac{\omega_p{}^2\alpha}{k^2c^2}\left\{1+\ln\frac{\alpha}{2}+C+\frac{1}{2}g(z)\right.$$

$$\left.+i\pi\theta(-\mathrm{Im}\,\omega)\theta(-\mathrm{Re}\,\omega+kc)(1+z+z^2/2)e^{-z}\right\}=0, \quad (29)$$

where

$$g(z)=(1-z+z^2/2)E_1(z)e^z+(1+z+z^2/2)E_1(-z)e^{-z}. \quad (30)$$

Using the asymptotic formulae for the function $E_1(z)$ one easily checks that in the region $k^2c^2 \ll \omega_p^2\alpha$ does not have any solutions either when $T_e \gg T_I$ or when $T_e = T_i$, i.e., in an isotropic relativistic plasma there cannot exist in the region considered oscillations of the form (19). The vibrational properties of a relativistic plasma depend thus in an essential way on the anisotropy of the distribution function in velocity space.

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Translated by D. ter Haar.