Integrable models in the problem for particle motion in a two-dimensional potential well

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A class of multi-parametric potentials is constructed for which the motion problem for a particle in a two-dimensional potential well or the behavior of a system of two coupled nonlinear oscillators is completely integrable. The integrable models can be used to study the behavior of a system in continuous transition from a potential with a single (global) minimum to a potential with several minima. It is demonstrated that nonintegrable physical models can be approximated by integrable models. It is noted that integrable models provide new methods for the analysis of interphase boundaries and phase transitions in media with two order parameters, as well as problems which require determining "trap"-type potentials.

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1. One trend in the theory of nonlinear phenomena is based on the construction of completely integrable models that admit of a complete and exact analysis of such essentially nonlinear structures as solitons or domain (interphase) walls. In order to analyze such phenomena as phase transitions, it is essential that the corresponding integrable models be expressed as a function of several structural parameters. Using as an example the Landau–Lifshitz equations, it has been shown[1] that multi-parametric integrable potentials may be constructed and cases of which substantial reconstruction of domain walls were studied. That a complete analysis of the reconstruction of domain walls in magnetically ordered media is possible is clear from general expressions[1] for the anisotropic energy. These expressions demonstrate the integrability of dynamic equations for a single order-parameter vector (that is, a single magnetic-moment vector).

In the present article, a previously proposed[1] method of constructing completely integrable models is extended to the motion of a particle in a two-dimensional potential well. The need for such an extension is clear, for it makes it possible to substantially increase the number of physical and applied problems in which an exact analysis of essentially nonlinear phenomena can be undertaken through the construction of integrable models. For example, we may associate with an integrable-motion problem for a particle in a two-dimensional potential well an interphase-boundary problem in ferroelectric-type media characterized by two order parameters. It is essential that the general class of integrable potentials identified below may depend on an arbitrary number of structure parameters. Thus, a class of integrable potentials is found which can be continuously transformed from the case of a two-dimensional potential well with a single global minimum to that of a potential well with several local minima and maxima by varying one or more structure parameters. Note that such a class of essentially two-dimensional integrable potentials corresponds to the characteristic potentials of the Landau theory of phase transitions with two order parameters, thereby demonstrating the possibility of developing models of phase transitions substantially linked with the "soliton" states of a medium characterized by two order parameters.

By interpreting the integrable potentials found below in terms of two coupled nonlinear oscillators, new ways are found of using integrable models in the analysis of problems related to the motion of a star in the gravitational field of the galaxy, nonlinear oscillations of atoms in planar triatomic molecules, and, finally, the motion of particles in accelerators.[2–4] The typical potential for such problems has the form

$$U=\tfrac{1}{2}\omega_1^2X_1^2+\tfrac{1}{2}\omega_2^2X_2^2+\Lambda(X_1^3+\beta X_1X_2^2)+\delta U. \tag{1.1}$$

Here ω_1 and ω_2 are the eigenfrequencies of the oscillators in a linear approximation; Λ and β are the basic nonlinear parameters; and δU defines the contribution of higher-order nonlinearities. Here the main problem is to determine conditions under which a potential of the form (1.1) leads to stable localization of a moving particle in a bounded (in particular, a designated) region of the configuration space X_1, X_2. In other words, the potential must be a "trap" for the particles. One important feature of integrable trap models is stability of the particle motion relative to the initial conditions and the possibility of determining explicitly (based on the first two integrals) the conditions under which the particle may be found in the trap. However, it is necessary to study the structure stability of the trap relative to the perturbation δU or, in other words, the structure stability of a separatrix path.[1,5]

The attractive feature of integrable models for two coupled nonlinear oscillators is the possibility they provide for the complete analysis of both the nonresonant case (ω_1^2 and ω_2^2 are incommensurable) and in the case of exact frequency resonance (ω_1^2 and ω_2^2 are commensurable). Moreover, the class of multi-parametric integrable potentials we have found may be used to study a problem of undoubted importance for phase-transition models, the behavior of a system as each of the eigenfrequencies passes through zero independently and in turn, and also when two eigenfrequencies simultaneously pass through zero. In the first case, the change of the system behavior may be associated with two successive one-parameter bifurcations, and in the second, with an

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essentially two-parameter bifurcation.

The value of multi-parametric integrable models for applications is clear from the fact that in a number of cases nonintegrable physical models may be "approximated" to some degree through an appropriate selection of the structure parameters. For example, a potential of the form (1.1) with $\delta U \equiv 0$ and arbitrary values of the parameters is nonintegrable. However, if $\delta U \equiv 0$ and $\beta = 1/2$, then, as will be shown below, we have an integrable case. If it is assumed that $\delta U \neq 0$ is determined by higher-order nonlinearities (by comparison with the basic nonlinearity determined by the cubic form of the variables X_1 and X_2), the integrable case may be found for any value of the parameter β. In particular, if $\delta U \equiv 0$ and $\omega_1^2 = \omega_2^2 = 1$, $A = 1/3$, and $\beta A = -1$, the nonintegrable potential (1.1) is the potential of the most intensively studied Henon–Hales system[2,3] directly related to the above motion problems for a star in the gravitational field of the galaxy or the nonlinear oscillations of particles in an accelerator under intrinsic resonance conditions. An integrable model analogous to the potential of the Henon–Hales system (to within a fourth-order form) is constructed in the present article. Moreover, our discussion of the propagation problem for standing waves described by the system of nonlinear coupled equations

$$u_{tt}-u_{xx}=\partial U/\partial u, \quad v_{tt}-v_{xx}=\partial U/\partial v,$$

where $U = U(u,v)$, also leads to problems we shall study below. Note that methods of approximating nonintegrable models by means of integrable potentials may differ depending on the nature of the problem. In the case of problems involving interphase boundaries in magnetic or ferroelectric media, the basic "proximity" or "approximation" criterion is one according to which the corresponding separatrix paths that determine the feasible types of interphase boundaries coincide.

Certain special examples of integrable potentials related to the motion of a point in a two-dimensional potential well have recently been presented.[6] Note that the more general integrable models presented in the current article arise naturally if we assume a zero Gaussian curvature for the two-dimensional manifold (configuration space) of previous studies.[1,7]

To say that the classical problem is integrable means that a pair of commuting operators can be produced for the corresponding quantum problem, and that the Schrödinger equation admits a separation of the variables.

Finally, we will present examples of physical problems that demonstrate the need for an analysis of quantum analogs of integrable models. A study of the rotational spectrum of linear molecules linked at one end to the interphase boundary leads to the problem of a quantum rotator which is immersed in a potential well and whose configuration space is contracted, for example, to a hemisphere.[8] By means of singular integrable potentials, a complete classification of the eigenvalue spectrum may be given in terms of the eigenvalues of a pair of commuting operators corresponding to classical first integrals of the form (2.1) and (2.2). Here it may be proved that the transition from the classical first integrals to operators is unique, and that the Schrödinger equation admits a separation of variables in a suitable coordinate system.

The transition-matrix systems with two order parameters leads (in the thermodynamic limit[9,10]) to an eigenvalue problem for a Schrödinger-type linear operator. The eigenvalue spectrum of this type of problem defines in essence both the mean values of the obvervables and the behavior of the correlation functions. Integrable models that admit an exact classification of states relative to two eigenvalues and a separation of variables can be used in the study of the properties of quasi-one-dimensional systems with two order parameters.

2. Suppose that the motion of a classical particle on a plane in a potential $U(x_1,x_2)$ is defined by a Hamiltonian of the form

$$H={}^1/_2p_1{}^2+{}^1/_2p_2{}^2+U(x_1,x_2). \tag{2.1}$$

We assume that the system is completely integrable and that it possesses an additional first integral of the form

$$Q={}^1/_2ap_1{}^2+bp_1p_2+{}^1/_2cp_2{}^2+V(x_1,x_2), \tag{2.2}$$

in which a, b, and c are functions of the coordinates x_1 and x_2. The conditions under which the Poisson bracke $\{H,Q\}$ (which is a cubic polynomial in the momenta p_1 and p_2) vanishes determines the feasible class of metric coefficients of the additional first integral (2.2). Namely,

$$\begin{aligned} a&=2\lambda x_2{}^2+2a_1x_2+a_0,\\ b&=-2\lambda x_1x_2-a_1x_1-c_1x_2+b_0,\\ c&=2\lambda x_1{}^2+2c_1x_1+c_0. \end{aligned} \tag{2.3}$$

Here λ, a_1, c_1, a_0, b_0, and c_0 are arbitrary constants. The two conjugate potentials U and V are related as

$$\frac{\partial V}{\partial x_1}=a\frac{\partial U}{\partial x_1}+b\frac{\partial U}{\partial x_2}, \quad \frac{\partial V}{\partial x_2}=b\frac{\partial U}{\partial x_1}+c\frac{\partial U}{\partial x_2}. \tag{2.4}$$

The integrability conditions for the system (2.4) $\partial^2 V/\partial x_1\partial x_2 = \partial^2 V/\partial x_2\partial x_1$ reduce to the equation

$$b\frac{\partial^2 U}{\partial x_1{}^2}+(c-a)\frac{\partial^2 U}{\partial x_1\partial x_2}-b\frac{\partial^2 U}{\partial x_2{}^2}+3\frac{\partial b}{\partial x_1}\frac{\partial U}{\partial x_1}-3\frac{\partial b}{\partial x_2}\frac{\partial U}{\partial x_2}=0, \tag{2.5}$$

whose unique solutions determine the class of integrable potentials (whether regular or singular). Note that when $\Delta = ac - b^2 \neq 0$, by solving the system (2.4) for $\partial U/\partial x_1$ and $\partial U/\partial x_2$ and using the integrability condition $\partial^2 U/\partial x_1\partial x_2 = \partial^2 U/\partial x_2\partial x_1$, we arrive at the equation

$$\frac{\partial}{\partial x_1}\left[\Delta^{-1}\left(-b\frac{\partial V}{\partial x_1}+a\frac{\partial V}{\partial x_2}\right)\right]=\frac{\partial}{\partial x_2}\left[\Delta^{-1}\left(c\frac{\partial V}{\partial x_1}-b\frac{\partial V}{\partial x_2}\right)\right], \tag{2.6}$$

whose solutions completely determine the class of conjugate potentials. It can be proved that if $U(x_1,x_2)$ is a solution of (2.5), the function $V = \Delta U$ satisfies (2.6). From this assertion, we obtain the recursion relations

$$\begin{aligned} \frac{\partial U_{n+1}}{\partial x_1}&=c\frac{\partial U_n}{\partial x_1}-b\frac{\partial U_n}{\partial x_2}+\frac{\partial c}{\partial x_1}U_n,\\ \frac{\partial U_{n+1}}{\partial x_2}&=-b\frac{\partial U_n}{\partial x_1}+a\frac{\partial U_n}{\partial x_2}+\frac{\partial a}{\partial x_2}U_n, \end{aligned} \tag{2.7}$$

by means of which it is possible to construct the succeeding integrable potential U_{n+1} using the known integrable potential U_n. Once we have applied (2.7), we can construct a sequence of integrable potentials which is a generalization of a previously obtained sequence[6] for the case in which $\lambda = a_0 = b_0 = c_0$ and $U_0 = 1$ in (2.3). Here $V_{n+1} = U_n\Delta$ will be the potential conjugate to U_{n+1}.

However, an integral representation of the solutions of equations (2.5) and (2.6) previously used by us[1] is more general and, in particular, makes it possible to define a relation between the class of integrable potentials and the metric of the coordinate systems that admit a separation of variables in the Hamilton–Jacobi equations. Note, too, a number of results[7] which represent a mathematically more complete study (by comparison with Ref. 1) of coordinate systems on the sphere and admit a separation of variables for systems such as (2.1) and (2.2).

The solutions of equations (2.5) may be represented in the form

$$U(x_1, x_2) = \int \frac{\rho(z)\,dz}{(a-z)(c-z)-b^2}, \tag{2.8}$$

where $\rho(z)$ is an arbitrary function and the functions a, b, and c are defined by (2.3). Suppose that $q_1(x_1,x_2)$ and $q_2(x_1,x_2)$ are the roots of the equation

$$z^2-(a+c)z+ac-b^2=0; \tag{2.9}$$

so that (2.8) may be written in the form

$$U(x_1, x_2) = \int \frac{\rho(z)\,dz}{(z-q_1)(z-q_2)}, \tag{2.10}$$

which is completely analogous to (A.15), (Appendix to Ref. 1). If the functions $q_1(x_1,x_2)$ and $q_2(x_1,x_2)$ are independent, then, if we make them new generalized coordinates (the case in which these functions are not independent will be discussed below), we find that they specify a curvilinear orthogonal reference grid on the (x_1,x_2) plane. Since

$$ac-b^2=q_1q_2,$$

we may write for (2.6) a solution that determines the integrable potential conjugate to (2.10), in the form

$$V(x_1, x_2) = q_1q_2 \int \frac{\rho(z)\,dz}{z(z-q_1)(z-q_2)}. \tag{2.11}$$

Based on the results of the appendix to Ref. 1 (or an analysis of (2.10) and (2.11) for the case $\rho(z) = z_{n+1}/2\pi i$, where n is a positive or negative integer), we may verify that the class of single-valued multi-parametric potentials admits of the representation

$$U(x_1, x_2) = \sum_{n=-\infty}^{\infty} C_n U_n(q_1, q_2). \tag{2.12}$$

Here

$$\begin{gathered} U_n=q_1^n+q_1^{n-1}q_2+\ldots+q_1q_2^{n-1}+q_2^n, \quad n\geqslant 1, \\ U_0=1, \quad U_{-1}=0, \quad U_n=-(q_1q_2)^{n+1}U_{-n-2}, \quad n\leqslant -2, \end{gathered} \tag{2.13}$$

while the C_n are arbitrary constants. We associate with the case of positive integers n regular integrable potentials, and with integers $n \leqslant -2$, singular integrable potentials. The expression for the potential V conjugate to (2.12) is given by the relation

$$V(x_1, x_2) = \sum_{n=-\infty}^{\infty} C_n V_n, \quad V_n=q_1q_2U_{n-1}. \tag{2.14}$$

We present explicit expressions for the first two regular integrable potentials

$$\begin{gathered} U_1(x_1, x_2)=q_1q_2=a+c, \\ U_2(x_1, x_2)=q_1^2+q_1q_2+q_2^2=(a+c)^2-ac+b^2. \end{gathered} \tag{2.15}$$

Note that each of the regular integrable potentials constitutes a polynomial in the variables q_1+q_2 and q_1q_2. Also worth noting is the simple physical meaning of the potential U_1. If $\lambda > 0$, it is a two-dimensional harmonic oscillator with eigenfrequency λ and equilibrium state at the points $x_1 = -c_1/2\lambda$ and $x_2 = -a_1/2\lambda$. If $\lambda = 0$, it is the potential of a particle in a constant electric field.

We associate with different values of the arbitrary constant coefficients in (2.3) different grids of orthogonal curvilinear coordinate systems q_1,q_2. Thus, if in (2.3) only the coefficients a_1 and c_1 are nonzero, we obtain an orthogonal grid that corresponds (apart from rotations) to the coordinates of a parabolic cylinder,[11] while if λ, a_1 and c_1 are nonzero, we obtain an orthogonal grid that corresponds to the coordinates of an elliptical cylinder. If only the coefficient λ is nonzero, the functions q_1 and q_2 are no longer independent ($q_1 = q_2$). Then, if we let q_1 be one of the independent variables and add it to the orthogonal coordinate system, we obtain a grid corresponding to polar coordinates.

3. Using the above class of integrable potentials, we construct a simple integrable model that corresponds to the motion of a point in a two-dimensional potential well. As the structure parameter is varied, the corresponding potential is continuously transformed, changing from a potential well with a single global minimum to a well with two minima and a single local maximum. For the one-dimensional case, this type of model is a source of well-known analogies with the phenomenological theory of phase transitions and self-organization phenomena.[10,12]

Generalizations of our model (some of which are presented below) show, in particular, how to systematically identify cases when the important problem of two coupled nonlinear oscillators is completely integrable.

Let us consider the integrable potential

$$U(x_1, x_2)=C_1U_1(x_1, x_2)+C_2U_2(x_1, x_2). \tag{3.1}$$

Here U_1 and U_2 are defined in (2.15) and C_1 and C_2 are arbitrary constants. As an example, let us consider the case in which $a_0 = b_0 = c_0 = 0$ in (2.3). The shift transformation

$$\bar{x}_1=x_1+c_1/2\lambda, \quad \bar{x}_2=x_2+a_1/2\lambda \tag{3.2}$$

and rotation transformation

$$sX_1=c_1\bar{x}_1+a_1\bar{x}_2, \quad sX_2=-a_1\bar{x}_1+c_1\bar{x}_2, \tag{3.3}$$

where $s^2 = a_1^2 + c_1^2$, lead to

$$\begin{gathered} q_1+q_2=a+c=2\lambda(X_1^2+X_2^2)-s^2/2\lambda, \\ q_1q_2=ac-b^2=-s^2X_2^2 \end{gathered} \tag{3.4}$$

and the potential (3.1) may be written in the form

$$U(X_1, X_2)=(\alpha-2\gamma)X_1^2+(\alpha-\gamma)X_2^2+\beta(X_1^2+X_2^2)^2. \quad (3.5)$$

Here we have introduced the notation

$$\alpha=2\lambda C_1, \quad \beta=4\lambda^2C_2, \quad \gamma=s^2C_2. \quad (3.6)$$

When $\alpha-2\gamma>0$ and $\alpha-\gamma>0$, the potential (3.5) corresponds to the integrable problem of two coupled oscillators with distinct eigenfrequencies and nonlinear isotropic potential interaction.

The first integrals (2.1) and (2.2) in this model take the form

$$H={}^1/_2p_1^2+{}^1/_2p_2^2+U(X_1, X_2), \quad (3.7)$$

$$Q=\lambda M^2-(s^2/4\lambda)p_2^2+V(X_1, X_2). \quad (3.8)$$

Here $M=X_2p_1-X_1p_2$ is the angular momentum of the particle, and the potential conjugate to (3.5) has the form

$$V(X_1, X_2)=-(s^2/2\lambda)X_2^2[\alpha-\gamma+\beta(X_1^2+X_2^2)]. \quad (3.9)$$

As $s^2\to 0$, the expressions (3.5) reduce to a potential invariant to rotations. The additional integral (3.8) degenerates to the law of conservation of the angular momentum.

As the structure parameters α, β, and γ vary, both the type of equilibrium state and their quantity vary. Let us consider the case $\beta>0$. In this case, the model has a null equilibrium state ($X_1=X_2=0$) for all values of the structure parameters, and when $\alpha>2\gamma$, this state is unique and corresponds to the global minimum of the potential energy $U(X_1,X_2)$ defined by (3.5). If $\alpha=2\gamma$, two equilibrium states split off from this null state:

$$X_1^2\equiv\varkappa_1^2=(2\gamma-\alpha)/2\beta, \quad X_2=0, \quad (3.10)$$

and correspond to two distinct (by virtue of symmetry) global minima of the potential energy. Two separatrix loops corresponding to solutions with a single degree of freedom ($X_2\equiv 0$) are created in this case from the null equilibrium state.

Further, as the second critical value $\alpha=\gamma$ passes out of the null equilibrium state, two additional saddle-center type equilibrium states are split off:

$$X_1=0, \quad X_2^2\equiv\varkappa_2^2=(\gamma-\alpha)/2\beta. \quad (3.11)$$

The equilibrium states (3.11) are related by separatrices whose projections on the configuration space X_1, X_2 form the ellipse

$$\frac{X_1^2}{\Gamma^2+\varkappa_2^2}+\frac{X_2^2}{\varkappa_2^2}=1, \quad \Gamma^2=\frac{s^2}{4\lambda^2}. \quad (3.12)$$

In the case when a potential of the form (3.5) corresponds to a model of an ordered medium with two order parameters, such separatrices are the image of interphase boundaries associated with a simultaneous variation of both order parameters (unlike the degenerate cases, which are related to the variation of only one of the two order parameters as the interphase boundaries are crossed).

When $\alpha<\gamma$, the null equilibrium state corresponds to a local maximum of the potential energy; two additional degenerate separatrix loops corresponding to one-dimensional motion with $X_1\equiv 0$ are created in this state, and since the problem is integrable, a continuous spectrum of closed separatrix loops is created.

4. A new class of integrable potentials arises if we set $\lambda=0$ in the expressions for the metric coefficients (2.3). In this case, the shift transformation

$$\bar{x}_1=x_1+c_0/2c_1, \quad \bar{x}_2=x_2+a_0/2a_1 \quad (4.1)$$

and rotation transformation

$$sX_1=c_1\bar{x}_1+a_1\bar{x}_2, \quad sX_2=a_1\bar{x}_1-c_1\bar{x}_2 \quad (4.2)$$

lead to relations with structure different from that of (3.4), that is,

$$q_1+q_2=a+c=2sX_1,$$
$$q_1q_2=ac-b^2=-s^2X_2^2+(2/s)B_0[2a_1c_1X_1+(a_1^2-c_1^2)X_2]-B_0. \quad (4.3)$$

Here we have set

$$B_0=b_0+a_1c_0/2c_1+a_0c_1/2a_1. \quad (4.4)$$

The expressions for the first two integrable potentials have the form (to within minor constants)

$$U_1=2sX_1,$$
$$U_2=s^2(4X_1^2+X_2^2)-(2/s)B_0[2a_1c_1X_1+(a_1^2-c_1^2)X_2],$$
$$U_3=4s^3X_1(2X_1^2+X_2^2)-4B_0X_1[4a_1c_1X_1+2(a_1^2-c_1^2)X_2-sB_0]. \quad (4.5)$$

These expressions show that, for example, the problem for a two-dimensional nonlinear oscillator with potential

$$U={}^1/_2\omega_1^2X_1^2+{}^1/_2\omega_2^2X_2^2+\Lambda(X_1^3+{}^1/_2X_1X_2^2) \quad (4.6)$$

is integrable for arbitrary values of the eigenfrequencies ω_1 and ω_2 and of the parameter Λ.

Note that in this class of integrable potentials ($\lambda=0$), the additional integral (2.2) assumes the form

$$Q=sMp_2+B_0\left[\frac{a_1c_1}{s^2}(p_1^2-p_2^2)+\frac{a_1^2-c_1^2}{s^2}p_1p_2\right]+V.$$

The class of integrable potentials of the form (4.6) includes potentials corresponding to the intrinsic resonance of the frequencies of a two-dimensional oscillator. The interesting case of simultaneously vanishing frequencies and the transition to the region of purely imaginary eigenvalues can then be studied. This transition into the region of purely imaginary eigenvalues may be accomplished in different ways, depending on whether or not it occurs when there is a resonance relation between the frequencies.

The particular case of the intrinsic resonance $\omega_1^2=4\omega_2^2$ for the present class of integrable potentials ($\lambda=0$) with the additional constraints

$$a_0=b_0=c_0=0, \quad c_1=0$$

was given in Ref. 6. However, a more general potential of the form

$$U=C_1U_1+C_2U_2+C_3U_3+C_4U_4 \quad (4.7)$$

with $B_0\neq 0$ must be investigated if we wish to analyze critical perturbations[12] without violating integrability. (Our aim here is a proper analysis of the bifurcations associated with the simultaneous vanishing of both frequencies.)

The class of integrable potentials of the form (4.7) includes, in particular, the potential

$$U={}^1/_2X_1^2+{}^1/_2X_2^2+{}^1/_3X_1^3-X_3X_2^2+\varepsilon(16X_1^4+12X_1^2X_2^2+X_2^4), \tag{4.8}$$

where $\varepsilon = 7/36$, which in a finite region of the plane X_1, X_2, approximates the potential of the Henon-Hales problem[2]:

$$U={}^1/_2X_1^2+{}^1/_2X_2^2+{}^1/_3X_1^3-X_1X_2^2. \tag{4.9}$$

Equation (4.8) indicates that when $X_1^2, X_2^2 \ll 1$ and for correspondingly small values of the energy of a system of two coupled nonlinear oscillators in the nonintegrable Henon-Hales problem (for nonintegrability, see Ref. 13), an integrable model "similar" to (4.7) may be constructed. Results of numerical computations[2] convincingly attest to the fact that the Henon-Hales system behaves like an integrable system at relatively low values of the total energy.

Thus, the discussed class of multi-parametric integrable potentials in the problem of particle motion in a plane, can be used to completely investigate integrable models of a host of physical problems, and also provides new methods of approximating a nonintegrable model by means of integrable models.

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