A classification of the collective modes of the order parameter in superfluid ^{3}He

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The collective modes of the A-phase of superfluid ^{3}He are classified in terms of the irreducible representations of the symmetry group H of the "vacuum." In distinction from the modes in ^{3}He-B which are characterized by one quantum number J and a single parity (with respect to complex conjugation), the modes in ^{3}He-A are characterized by two quantum numbers: Q and S_z, and two parities P^1 and P^2. The charges Q and S_z assume the values 0, ± 1, ± 2, and 0, ± 1, respectively. Owing to the parities P^1 and P^2 those modes which differ in the sign of either S_z or Q turn out to be degenerate. Consequently, if the wave vector **q** is parallel to the orbital anisotropy axis the spectrum of modes will consist of two four-fold degenerate branches, four two-fold degenerate branches, and two nondegenerate branches. The splitting of the mode spectrum under the lowering of the symmetry H of the vacuum due to spin-orbit interaction and magnetic fields is investigated. An additional degeneracy of the spectrum of modes is exhibited in the weak coupling limit, on account of an enlargement of the group H owing to hidden symmetry. The consequences are analyzed of the asymmetry between quasiparticles and holes near the Fermi surface, which results in a splitting of the spectrum of modes with different signs of Q. This classification allows one to indicate necessary conditions for mutual mode pinning. All results have been obtained without the use of any equations describing the collective modes.

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I. INTRODUCTION

In superfluid ^{3}He the order parameter $A_{i\alpha}$ is a complex 3×3 matrix. The oscillations $\delta A_{i\alpha}$ of the order parameter around its equilibrium $A_{i\alpha}{}^0$ (the "vacuum" of the system) form 18 collective modes in each of the superfluid phases of ^{3}He (see the review papers[1,2]). The majority of these modes were observed in experiments using NMR or ultrasound absorption (see the latest experiments on ultrasound absorption in the A and B phases of ^{3}He in Refs. 3 and 4). In the B phase the collective modes are classified[5] in terms of the magnitude J of the angular momentum, taking the values 0, 1, 2, its projection J_z, and the parity with respect to complex conjugation. As far as we know, in the A phase there does not exist a satisfactory classification of the modes in terms of quantum numbers, although these modes are grouped into different "clapping" and "flapping" modes.[1]

We carry out a group-theoretical classification of the collective modes of the order parameter, expanding them in terms of irreducible representations of the symmetry group H of the vacuum, since it is this group which determines the symmetry of the dynamical equations which yield the spectrum of the collective modes. Those variables $\delta A_{i\alpha}$ which are transformed among themselves under an irreducible representation of the group H have the same oscillation spectrum, and the multiplicity (degree of degeneracy) of this spectrum coincides with the dimension of this representation.[6]

In Section II we define the quantum numbers which characterize the various irreducible representations. In Section III we consider the splitting of the mode spectrum under a lowering of the symmetry by the spin-orbit (dipole) interaction and the magnetic field. In Section IV we consider the additional degeneracy of the mode spectrum which arises in the so-called weak coupling approximation, when the symmetry H of the state becomes higher. We note that we do not need to investigate the dynamical equations, since all the required information about their symmetries is contained in the group H.

II. COLLECTIVE MODES AND IRREDUCIBLE REPRESENTATIONS OF THE SYMMETRY GROUP OF THE VACUUM

The symmetry of the dynamical equations which describe the collective oscillations of the order parameter around one of its equilibrium values $A_{i\alpha}{}^0$ is determined by the symmetry H of the vacuum state $A_{i\alpha}{}^0$. The group H is a subgroup of the general group G describing the "symmetry of the laws of physics," i.e., the symmetry group of liquid ^{3}He in its normal state. The latter group has the form

$$G=SO_3^{orb}\times SO_3^{sp}\times U(1)\times Z_2. \quad (1)$$

Here $SO_3^{\rm orb}$ is the group of three-dimensional rotations of coordinate space, under which the matrix $A_{i\alpha}$ transforms as a vector in the first (orbital) index. Under the action of the group $SO_3^{\rm sp}$ of rotation of the three-dimensional spin space the matrix $A_{i\alpha}$ transforms as a vector in the second (spin) index. The group $U(1)$ of gauge transformations takes $A_{i\alpha}$ into $A_{i\alpha}e^{i\varphi}$. Finally, the group Z_2 consists of the two elements 1 and $T(TT=1)$ where T is the operation of time reversal, which in addition to $t\to -t$ subjects the matrix to a complex conjugation $C:A_{i\alpha}\to A_{i\alpha}{}^*$.

We select the vacuum state of the A phase in the following form:

$$A_{i\alpha}{}^0=\hat{z}_\alpha(\hat{x}_i+i\hat{y}_i), \quad (2)$$

where $\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}}$ are the unit vectors of a Cartesian coordinate

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system. The subgroup H of the group G leaving this state invariant is

$$H=U^1(1)\times U^2(1)\times Z_2^1\times Z_2^2. \tag{3}$$

Here $U^1(1)$ denotes the rotations of the spin space around the z axis; this group is generated by the spin angular momentum operator S_z, and $U^2(1)$ is a combination of a rotation of the orbital space around the z axis by an arbitrary angle φ, combined with a simultaneous multiplication of the order parameter by the phase factor $\exp(i\varphi)$. The generation Q of this transformation has the following expression in terms of the angular momentum operator $\mathbf{L}$ and the gauge transformation operator I: $Q=I-L_z$. The discrete subgroups Z_2^1 and Z_2^2 consist of two elements each, to wit: 1 and P^1 ($P^1P^1=1$) and 1 and P^2 ($P^2P^2=1$). The operation P^1 is a combination of a spin rotation by π around the x (or y) axis with a rotation by π around the z axis of orbital space. We write this in the form: $P^1=S_y(\pi)L_z(\pi)$. Each of the rotations $S_y(\pi)$ and $L_z(\pi)$ changes the sign of $A_{i\alpha}{}^0$, so that together they leave it invariant. The element $P^2=L_x(\pi)T$ is a combination of time reversal T with an orbital rotation by π around the x axis.

The operators $\mathbf{L}$, $\mathbf{S}$ and I act on the variable $\delta A_{i\alpha}=U_{i\alpha}+iv_{i\alpha}$ according to the formulas

$$Iu_{i\alpha}=iv_{i\alpha},\ Iv_{i\alpha}=-iu_{i\alpha},\ L_i\delta A_{k\alpha}=-ie_{ikl}\delta A_{l\alpha},\ S_\alpha\delta A_{i\beta}=-ie_{\alpha\beta\gamma}\delta A_{i\gamma}.$$

The variables $u_{i\alpha}$ and $v_{i\alpha}$ are the basis of an 18-dimensional representation of the group H which has to be decomposed into irreducible representations of this group. The group H is nonabelian, since Q does not commute with P^2 and S_z does not commute with P^1. Therefore the group has representations of dimension higher than one, i.e., the collective-oscillation spectrum will be degenerate.

The table on the next page lists a classification of the modes in terms of the quantum numbers of the irreducible representations of the group H. Each nondegenerate representation is characterized by a single set of quantum numbers Q, S_z, P^1, P^2, or by several, if it is degenerate (the second column of the table). The first column indicates the basis of the representation. The subscripts 1, 2, 3 denote x, y, z, respectively. Since $\delta A_{i\alpha}$ is a vector both in spin and in orbital space, it transforms according to the representations with $S=1$ and $L=1$. Therefore S_z and L_z can take on the values 0 and ±1. Since the operator I has the eigenvalues ±1 the operator $Q=I-L_z$ will have the eigenvalues 0, ±1, ±2.

The table permits one to elucidate the character of the collective-mode spectrum $\omega(\mathbf{q})$ for $\mathbf{q}\|\hat{\mathbf{z}}$. The case of $\mathbf{q}$ not parallel to the orbital axis requires separate consideration, since it violates the symmetry with respect to rotations around the z axis and the charge Q is no longer conserved. Thus, the 18-dimensional representation of $\delta A_{i\alpha}$ decomposes into two one-dimensional representations, four two-dimensional representations, and two four-dimensional representations. Accordingly, for $\mathbf{q}\|\hat{\mathbf{z}}$ there are a total of eight branches of the spectrum $\omega(\mathbf{q})$ of which four are two-fold

TABLE I.

Modes	Variables	Quantum Numbers: In the absence of dipole interaction and magnetic field				Taking into account dipole interaction		In magnetic field			In weak coupling	
		Q	P^2	S_z	P^1	$\tilde{Q}$	$\tilde{P}^2$	Q	P^2	P^1	$\tilde{S}$	Q
Sound	$u_{23}-v_{13}$	0	−1	0	+1	0	−1	0	−1	+1		
Spin waves	$u_{11}+v_{21}$, $u_{12}+v_{22}$	0	+1	±1	–	±1	–	0	+1 ($u_{11}+v_{21}$)	+1	1	0
								0	+1 ($u_{12}+v_{22}$)	−1		
Orbital modes	u_{33}, v_{33}	±1	–	0	−1	±1	–	±1	–	−1		
Spin-orbit modes	u_{31}, v_{31}; u_{32}, v_{32}	±1	–	±1	–	0 ($u_{31}+v_{32}$)	−1	±1	– (u_{32}, v_{32})	+1	1	±1
						0 ($v_{31}-u_{32}$)	+1					
						±2 ($u_{32}+v_{31}$, $u_{31}-v_{32}$)	–	±1	– (u_{31}, v_{31})	−1		
Pseudosound	$u_{13}+v_{23}$	0	+1	0	+1	0	+1	0	+1	+1		
Pseudospin modes	$u_{22}-v_{12}$, $v_{11}-u_{21}$	0	−1	±1	–	±1	–	0	−1 ($v_{11}-u_{21}$)	+1	1	0
								0	−1 ($u_{22}-v_{12}$)	−1		
"Clapping" modes	$u_{23}+v_{13}$, $u_{13}-v_{23}$	±2	–	0	+1	±2	–	±2	–	+1	1	±2
	$u_{11}-v_{21}$, $u_{21}+v_{11}$, $u_{12}-v_{22}$, $u_{22}+v_{12}$	±2	–	±1	–	±1 ($u_{11}-v_{21}+u_{22}+v_{12}$, $u_{21}+v_{11}-u_{12}+v_{22}$)	–	±2	– ($u_{11}-v_{21}$, $u_{21}+v_{11}$)	+1		
						±3 ($u_{11}-v_{21}-u_{22}-v_{12}$, $u_{21}+v_{11}+u_{12}-v_{22}$)	–	±2	– ($u_{12}-v_{22}$, $u_{22}+v_{12}$)	−1		

degenerate, and two are four-fold degenerate. Among these branches there are Goldstone bosons with $\omega(0)=0$. These correspond to the variables $\delta A_{i\alpha}$ which are obtained from $A_{i\alpha}{}^0$ through the action of elements of the group G. We enumerate these modes, starting with the Goldstone modes.

Sound. A nondegenerate Goldstone mode with charges $Q=0$, $S_z=0$, which at low temperatures has the spectrum $\omega_1=cq$ where c is the speed of sound.

Spinwaves. Two-fold degenerate Goldstone modes with charges $Q=0$, $S_z=\pm 1$. The degneracy is related to the fact that the symmetry P^1 takes the state with $S_z=+1$

$$\psi_{+1}=v_{21}+u_{11}-i(u_{12}+v_{22}),$$

into the state with $S_z=-1$,

$$\psi_{-1}=v_{21}+u_{11}+i(u_{12}+v_{22}),$$

and vice versa: $P^1\psi_{+1}=\psi_{-1}$, $P^1\psi_{-1}=\psi_{+1}$. Therefore the waves have identical spectra $\omega_{2,3}=c^{\rm sp}q$, where $c^{\rm sp}$ is the speed of spin waves (Ref. 7).

Orbial modes. These are Goldstone modes with charges $S_z=0$, $Q=\pm 1$ which for high temperatures have a difussive character, and for low temperatures acquire a wave character (see, e.g., Ref. 7). In general there is no degeneracy, since the operator P^2 includes the replacement $t\to -t$. Therefore P^2 takes the states with $Q=+1$ into states with $Q=-1$, but with frequency of opposite sign; as a result of this $\omega_{Q=+1}\neq\omega_{Q=-1}$. There is, however, degeneracy if we neglect the very small asymmetry between quasiparticles and holes near the Fermi surface. In this case, above the transition the system is symmetric under complex conjugation $C: A_{i\alpha}\to A_{i\alpha}{}^*$, and below the transition it is symmetric with respect to $P^2=L_x(\pi)C$. Now P^2 does not contain the substitution $t\to -t$, and the orbital modes with $Q=+1$ and $Q=-1$ have the same spectrum, which for $T\to 0$ has the form $\omega_{4,5}=c^{\rm orb}|q_z|$. If we take into account the asymmetry between quasiparticles and holes the spectrum splits and for $q\to 0$ has the form

$$\omega_{Q=-1}=L_0/\chi_{orb},\quad \omega_{Q=+1}=q^2\chi_{orb}(c^{orb})^2/L_0. \tag{4}$$

Here $x_{\rm orb}$ is the orbital susceptibility, $L_0\sim\hbar\rho(\Delta/\varepsilon_f)^2$ is the magnitude of the spontaneous orbital angular momentum in the A phase, arising from the particle-hole asymmetry (see the review[6]).

Spin-orbit modes. The modes with the charges $Q=\pm 1$, $S_z=\pm 1$, which are four-fold degenerate if the asymmetry between quasiparticles and holes is neglected. The modes have identical gaps in the spectrum, gaps which vanish in the so-called weak-coupling limit on account of the hidden symmetry (see Section IV). In this limit their spectrum coincides with the spectrum of the orbital waves $\omega_{6\text{-}9}=c^{\rm orb}|q_z|$.

The remaining modes always have a gap in the spectrum. The nondegenerate *pseudosound* mode and the doubly degenerate *pseudospin* mode have quantum numbers differing from those of the sound and spin modes only in the P^2 parity. Moreover, there are the "clapping" modes—two-fold degenerate modes with $Q=\pm 2$, $S_z=\pm 1$.

We call attention to the fact that all representations are different, i.e., are labelled by different quantum numbers, and therefore for $\mathbf{q}\|\hat{\mathbf{z}}$ the collective modes are not pinned to one another. Mode-pinning and splitting of the spectrum occur when the symmetry H is broken, a topic we consider next.

The magnitude of the splitting of modes with differing signs of Q on account of the particle-hole asymmetry is determined by the parameter $(\Delta/\varepsilon_F)^2$.

III. THE SPLITTING OF THE COLLECTIVE-MODE SPECTRUM ON ACCOUNT OF THE SPIN-ORBIT INTERACTION AND THE MAGNETIC FIELD

The *spin-orbit interaction* (also known as the dipole interaction) lowers the symmetry G and consequently also H. In the group G the separate rotations of the orbital and spin spaces disappear, and only the joint rotation generated by the total angular momentum operator $\mathbf{J}=\mathbf{L}+\mathbf{S}$ survives:

$$G=SO_3\times U(1)\times Z_2. \tag{5}$$

The order parameters $A_{i\alpha}{}^0$ at equilibrium is given by the previous formula (2), since the minimum of the dipole energy is realized when the spin axis $\mathbf{d}$ and the orbital axis $\mathbf{l}$ coincide (in Eq. (2) $\hat{\mathbf{d}}=\hat{\mathbf{l}}=\hat{\mathbf{z}}$).[8] The subgroup H of G leaving this state invariant became smaller compared to Eq. (3):

$$H=U(1)\times Z_2. \tag{6}$$

Here $U(1)$ is a one-parameter group defined by the generator $\tilde{Q}=I-J_z$, and Z_2 is the group consisting of the two elements 1 and $\tilde{P}^2$ ($\tilde{P}^2\tilde{P}^2=1$), where $\tilde{P}^2=I(\pi)J_x(\pi)T$, $I(\pi)$ denotes multiplication by $\exp(i\pi)$, $J_x(\pi)$ is a joint rotation by π around the x axis of the orbital and spin space.

The splitting of the collective-mode spectrum on account of the lowering of symmetry is shown in the third column of the table. If one takes into account the weak asymmetry between quasiparticles and holes then the splitting becomes complete, since states with opposite signs of the charge $\tilde{Q}$ have different spectra. We have neglected this effect in the table compared to the dipole splitting, and considered these states as belonging to one doublet.

It can be seen from the table that there are different modes transforming according to the same representations, which signifies that there is pinning between these modes. Thus, sound is coupled (pinned) to a spin-orbit wave having the same quantum numbers $\tilde{Q}=0$, $\tilde{P}^2=-1$. It is possible that pinning is the reason for the recently observed[9] enhanced damping of ultrasound at low pressures, when the gap in the spectrum of the spin-orbit waves, a gap due to strong-coupling effects, becomes small. Similarly, there occurs a pinning between spin and orbital waves with identical charges $Q=+1$ or $Q=-1$. This halves the number of Goldstone modes.

The magnitude of the dipole splitting is determined by the parameter Ω_L/Δ, where Ω_L is the Leggett frequency.[7]

The magnetic field. The fourth column of the table lists the mode splitting scheme in a magnetic field without taking into account the dipole interactions and neglecting the asymmetry between quasiparticles and holes. The magnetic field $\mathbf{H}$ is directed along the y axis so that the vacuum state (2) should correspond to equilibrium in the magnetic field (the spin axis $\hat{\mathbf{d}}\|\hat{\mathbf{z}}$ at equilibrium must be perpendicular to $\mathbf{H}$, Ref. 8). In the absence of dipole interaction and quasiparticle-hole asymmetry the symmetry group G in a magnetic field is

$$G=SO_3^{orb}\times U^{sp}(1)\times U(1)\times Z_2, \qquad (7)$$

where $U^{sp}(1)$ denotes the rotations around the y axis in spin space and Z_2 consists of the identity and the complex conjugation operator C. The subgroup H of this group describing the symmetry of the vacuum state (2) differs from (3) by the absence of the rotations $U^1(1)$ of spin space around the z axis:

$$H=U^2(1)\times Z_2^1\times Z_2^2. \qquad (8)$$

Therefore the representations are described by the three quantum numbers: Q, P^2, and P^1.

We call attention to the fact that near the transition point into the A_1 phase the order parameter differs from Eq. (2), namely:

$$A_{i\alpha}^0=(\hat z_\alpha+i\lambda\hat x_\alpha)(\hat x_i+i\hat y_i),$$

where the coefficient $\lambda\to 0$ far from the A_1 phase and $\lambda=1$ in the A_1 phase. This order parameter does not exhibit the P^2 symmetry, and as a result of this only two quantum numbers Q and P^1 are conserved. In this case sound and one of the spin waves, namely the one without a gap, have identical quantum numbers: $Q=0$, $P^1=+1$. This indicates the possibility that they are pinned in a magnetic field near the transition into the A_1 phase. This pinning was first described in Ref. 10; it leads to an increase of the speed of second sound near the transition point into the A_1 phase.[11] The magnitude of the splitting of the mode spectrum in a magnetic field is determined by the parameter Ω_H/Δ, where Ω_H is the Larmor frequency.

IV. COLLECTIVE MODES IN THE WEAK-COUPLING APPROXIMATION

An experimental investigation of the jump in heat capacity at the transition point to the superfluid state[12] shows that for low pressures one may neglect the strong coupling effects. In these conditions the A phase at equilibrium acquires additional degrees of freedom[13] leading to four additional Goldstone modes.[14,15] The spectra of all 18 collective modes have been obtained in the weak coupling approximation by Brusov and Popov[16] by means of the functional integration method. We are interested in classifying these modes in terms of their quantum numbers by using symmetry considerations, without deriving dynamical equations.

The weak-coupling approximation consists in assuming that the pairings in the two types of quasiparticle (with spin up and with spin down) are independent. In these conditions the order parameter of the A phase may be written at equilibrium in the following general form[13]:

$$A_{i\alpha}^0=\tfrac{1}{2}(e_\alpha e_i^{(1)}+e_\alpha^* e_i^{(2)}), \qquad (9)$$

where $\mathbf{e}$, $\mathbf{e}^{(1)}$, and $\mathbf{e}^{(2)}$ are three complex vectors, each of which satisfies the following (scalar product) equalities

$$\mathbf{e}\cdot\mathbf{e}=0,\quad \mathbf{e}\cdot\mathbf{e}^*=2.$$

The order parameter (9) describes two subsystems. The Cooper pairs of the first subsystem have spin $\hbar\mathbf{s}$, where $\mathbf{s}=\mathbf{e}^*\times\mathbf{e}/2i$ is a unit vector, and orbital momentum $\hbar\mathbf{l}^{(1)}$, where $\mathbf{l}^{(1)}=\mathbf{e}^{*(1)}\times\mathbf{e}^{(1)}/2i$. The pairs of the other subsystem have the opposite spin $-\hbar\mathbf{s}$ and orbital momentum $\hbar\mathbf{l}^{(2)}$. If the interaction between the subsystems (i.e., strong coupling) is taken into account, the orbital angular momenta at equilibrium become parallel, $\mathbf{l}^{(1)}=\mathbf{l}^{(2)}$, so that the order parameter (9) at equilibrium takes on its ordinary form. Indeed, setting

$$\mathbf{l}^{(1)}=\mathbf{l}^{(2)}=\hat{\mathbf{z}},\quad \mathbf{e}^{(1)}=\mathbf{e}^{(2)}=\hat{\mathbf{x}}+i\hat{\mathbf{y}},\quad \mathbf{e}=\hat{\mathbf{z}}+i(\hat{\mathbf{y}}\cos\mu-\hat{\mathbf{x}}\sin\mu),$$

we obtain the expression (2) for arbitrary angle μ.

In this section we neglect the interaction between subsystems, which corresponds to the weak-coupling approximation. We will, however, consider the oscillations of the order parameter about the vacuum (2), to be able to include strong coupling effects. On account of the indeterminacy of the angles μ, i.e., of the direction of the spin axis $\mathbf{s}=\hat{\mathbf{x}}\cos\mu+\hat{\mathbf{y}}\sin\mu$, this vacuum belongs to a singular submanifold of the eight-dimensional region over which the order parameter varies (see Refs. 15, 17). This manifests itself, in particular, in the fact that the vacuum (2) has, as will be seen nine Goldstone modes, whereas the vacuum with $\mathbf{l}^{(1)}\neq\mathbf{l}^{(2)}$ has only eight such modes,[15] corresponding to the eight degrees of freedom of the order parameter (9). Owing to this singularity, the dynamical-equations symmetry group describing the transformations of the collective modes does not generally coincide with the symmetry group of the vacuum, and we shall look for this group by means of other symmetry considerations.

We fix the $\mathbf{s}$ axis, prescribing an angle μ, e.g., $\mu=0$. We thus distinguish between two types of Cooper pairs, with spins $\hbar\hat{\mathbf{x}}$ and $-\hbar\hat{\mathbf{x}}$. We first consider oscillations of the order parameter which do not mix the pairs of the two kinds. In this case the pairs of each kind have their own order parameter, a complex vector ψ such that

$$A_{i\alpha}=\tfrac{1}{2}(e_\alpha(\mu=0)\psi_i^{(1)}+e_\alpha^*(\mu=0)\psi_i^{(2)}). \qquad (10)$$

The variations $\delta\psi^{(1)}$ and $\delta\psi^{(2)}$ near equilibrium yield 12 collective modes, and on account of the independence of the subsystems, the Lagrangian which describes the dynamics of the $\delta\psi^{(1)}$ and $\delta\psi$ decomposes into a sum of two identical Lagrangians:

$$\mathscr{L}=\tilde{\mathscr{L}}(\delta\psi_i^{(1)})+\tilde{\mathscr{L}}(\delta\psi_i^{(2)}). \qquad (11)$$

In order to find the dependence of $\mathscr{L}$ on all 18 modes $\delta A_{i\alpha}$ in terms of the $\delta\psi^{(1,2)}$:

$$\delta A_{i3}=\tfrac{1}{2}(\delta\psi_i^{(1)}+\delta\psi_i^{(2)}),\quad \delta A_{i1}=0,$$

$$\delta A_{i2}=\tfrac{1}{2}i(\delta\psi_i^{(1)}-\delta\psi_i^{(2)}).$$

Since the Lagrangian $\mathscr{L}$ (11) is always quadratic in $\delta A_{i\alpha}$, it can be rewritten as a sum of independent Lagrangians for δA_{i3} and δA_{i2}:

$$\mathscr{L}=2\tilde{\mathscr{L}}(\delta A_{i3})+2\tilde{\mathscr{L}}(-i\delta A_{i2}).$$

The dependence of $\mathscr{L}$ on the remaining 6 modes δA_{i1} can be uniquely reconstructed if the symmetry of $\mathscr{L}$ with respect to rotations around the spin z axis is taken into account:

$$\mathscr{L}=2\tilde{\mathscr{L}}(\delta A_{i3})+2\tilde{\mathscr{L}}(-i\delta A_{i2})+2\tilde{\mathscr{L}}(-i\delta A_{i1}). \qquad (12)$$

It is clear that to the symmetry group (3), which is valid also in the presence of interaction between the subsystems, additional symmetry elements are added, and these are dictated by the form of the Lagrangian $\mathscr{L}$ (12). These are quasirotations around the x and y spin axes, constructed in the following manner. Under a pseudorotation by an angle α_x around the x axis the collective variables transform according to:

$$\delta A_{i3}\to\delta A_{i3}\cos\alpha_x+i\delta A_{i2}\sin\alpha_x,$$
$$\delta A_{i2}\to -i\delta A_{i3}\sin\alpha_x+\delta A_{i2}\cos\alpha_x. \quad (13)$$

The operators of these transformations are: $\tilde{S}_x = IS_x$, and $\tilde{S}_y = IS_y$, where I is the generator of the gauge transformations. Since $I^2 = 1$ the operators IS_x and IS_y together with the operator S_x form the three-dimensional group of pseudorotations $\tilde{\mathbf{S}} = (IS_x, IS_y, S_z)$, so that the effective symmetry H in the weak coupling approximation has the form

$$H=SO_3^{pseudo}\times U^2(1)\times Z_2^2. \quad (14)$$

Since P^2 does not commute with the components of $\tilde{\mathbf{S}}$ the collective modes have only the two quantum numbers Q and $\tilde{S}$. Due to the pairing with spin 1 the quantum number $\tilde{S}$ can only take on the value $\tilde{S} = 1$.

The classification of modes in terms of the quantum numbers is given in the fifth column of the table. According to the table there are two three-dimensional and two six-dimensional representations. Sound and spin waves belong to one representation with $\tilde{S} = 1$ and $Q = 0$. In the gas approximation they have the same spectrum $\omega = v_F q/\sqrt{3}$.

The orbital and spin-orbit waves also belong to one six-dimensional representation with $\tilde{S} = 1$ and $Q = \pm 1$, and therefore the spectrum of the spin-orbit waves has no gap in the weak coupling approximation, i.e., the number of Goldstone modes is effectively increased by 4, becoming equal to 9. In the gas approximation the spectrum of all six modes is identical: $\omega = v_F|q_z|$.[15,16] The Fermi-liquid corrections split the spectrum for $q\neq 0$, since dynamically, i.e., for $\omega\neq 0$ they couple subsystems with different spins. The number of additional Goldstone modes does not change.

Pseudosound and pseudospin waves with $\tilde{S} = 1$ and $Q = 0$ also form one representation. Owing to Landau damping their spectrum is not real.[16] The six-fold degenerate modes $\tilde{S} = 1$ and $|Q| = 2$ have in the gas approximation the spectrum $\omega = \Delta$.[16]

Strong coupling effects split the mode spectrum (the splitting scheme is indicated in the second column of the table). The spin-orbital waves acquire a gap proportional to the strong-coupling parameter.[15]

CONCLUSION

The classification of the collective modes in terms of quantum numbers of the irreducible representations of the symmetry group of the vacuum allows one to determine the multiplicity of the degeneracy of the mode spectrum and its splitting under the influence of various symmetry-breaking interactions, without solving the dynamical equations. It also allows one to indicate necessary conditions for coupling of modes to one another.

The classification can be extended to the A_1 phase. Here, as in the A phase, the collective modes are characterized by two charges: $Q_1 = I - L_z$ and $Q_2 = I - S_y$, each of which can take on the values 0, ± 1, ± 2.

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