The hydrodynamics of superfluid turbulence

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The equations of motion of superfluid helium, taking account of a chaotic cluster of vortex filaments (superfluid turbulence), are obtained by means of a method close to the Bekarevich-Khalatnikov theory. These equations are used for an investigation of the propagation of linear and nonlinear second sound in a supercritical helium countercurrent. A relation is obtained between the transit time of a nonlinear wave and the parameters of superfluid turbulence. This relation can be considered as a new method of probing the vortex cluster.

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1. THE VINEN-SCHWARZ MODEL

As already mentioned, Vinen was the first to attempt a quantitative theory of superfluid turbulence. In doing this he was guided by the following qualitative picture, developed by Feynman. Assume that the vortex cluster forms a homogeneous state characterized by the total length $L(t)$ of the filaments per unit volume. It is clear that one may talk about homogeneity only if the average distance $\delta = L(t)^{1/3}$ between the filaments is much smaller than the size of the system. Due to the Magnus effect the friction against the normal component causes the elements of the filaments to reduce their curvature, and this leads to an increase of the quantity $L(t)$. For large densities intersection effects come into play. As a result of intersection the filaments get fragmented, forming rings of smaller sizes, which in turn fragment, etc., until clumps are formed of a size for which the hydrodynamic description is no longer applicable, and kinetic theory effects become appreciable. Something of the nature of a diffusion in the space of sizes of vortex rings takes place. This situation is very reminiscent of classical turbulence, where an energy flow takes place from large scales to small scales, with subsequent dissipation. On the basis of these considerations Vinen has derived a rate equation for the quantity $L(t)$

$$\frac{dL}{dt} = -\alpha V_\infty L^3 - \beta L^2.$$  \hspace{1cm} (1)

Here $V_\infty = \gamma - \upsilon$ is the relative velocity and $\alpha$ and $\beta$ are the empirical parameters of the theory. The first term describes the increase of the length on account of the Magnus effect, and the second term describes the fragmentation effect.

It follows from Eq. (1) that the characteristic relation time of a vortex cluster is of the order $\tau = \beta/\alpha^2 V_\infty^3$. This time can be large, of the order of the damping time of hydrodynamic modes, and then it becomes necessary to take into account the dynamics of the cluster itself. It is necessary here to make an important remark. In general, the quantity $L$ represents an integral characteristic (a moment) of some distribution function (the distribution function) which carries the detailed information about the vortex cluster. There arises the problem of the damping of the other moments of this
L. Strictly speaking, there are no theoretical reasons to as-
sume that the relaxation of the other degrees of freedom of the
cluster occurs faster, justifying the fact that we have limited
our attention to the quantity L. However, numerous experi-
ments on the evolution of the vortex structure show that this
evolution agrees well with the Vinen equation (1). We there-
fore make the natural assumption that the relaxation of the
other degrees of freedom occurs at a faster rate than the
relaxation of L.

The statistical nature of the quantity L brings up an-
other problem. Although we restrict our attention to the
function L, it is nevertheless important to deal with quanti-
ties which are higher moments of the distribution function.
Such quantities are: the mean velocity \( \langle v \rangle \) of the cluster as
a whole and the work \( R' \) done by the friction forces against the
frozen filament system. Continuing to assume consistently
that their relaxation times are small, we make use of empiri-
cal relations obtained under static conditions for their deter-
mination. Thus, for \( \langle v \rangle \) we make use of the following depen-
dence determined in Ref. 5:

\[
\langle v \rangle = b \nu_n \nabla \nu_n.
\]

Here \( b = b(T) \) is a known function of the temperature.

To calculate the work of the friction forces against the
"frozen" filament system we make use of the well-known
Gorter-Mellink formula\(^1\) for the mutual friction between the
normal and superfluid components:

\[
P_G^M = \alpha(T) \rho_0 \nu_n \nabla \nu_n.\]

The work done by this force per unit time is, accordingly

\[
R'^{GM} = \alpha(T) \rho_0 \nu_n \nabla \nu_n.\]

This work consists of the total work \( R' \) plus the energy that
the vortex cluster drains in order to increase its length and
then transports along the turbulence spectrum to smaller
scales. As can be seen from Eq (1), this energy equals

\[
R' = \epsilon_0 \nu_n L \nu_n.\]

where, recognizing that \( L = \langle \nu / \beta \rangle \nu_n \), we obtain for \( \alpha_1 \), from Eq (5)

\[
\alpha_1 \equiv \alpha_1(T) \rho_0 x_n \beta = \epsilon_0 \nu_n.\]

In the sequel we shall describe the state of a vortex clus-
ter by means of the quantity \( L \) (\( \rho_0 \)). In distinction from the
case discussed by Vinen, we attribute to this quantity a coor-
dinate dependence, having in mind the investigation of non-
homogeneous and nonstationary cases. Equation (1) will be
the basis for closing the system of equations of motion.

2. THE HYDRODYNAMICS OF TURBULENT SUPERFLUIDITY

In deriving the equations of motion we make use of a
method analogous to the one used in the derivation of the
Bekarevich-Khalatnikov\(^3\) equations. In distinction from these
equations we cannot relate the number of vortex fila-
ments to the curl of the average velocity, since the vortices
are normally oriented. Therefore we adopt Eq (1) in lieu of
the missing equation, rewriting it in the form

\[
\frac{\partial L}{\partial t} + \nabla \cdot L = 0,\]

In addition, we complement the equations of two-velocity
fluid dynamics with as-yet unknown terms which may ap-
ppear on account of the vortices:

\[
\frac{\partial \rho_0}{\partial t} + \nabla \cdot \rho_0 v = 0,\]

\[
\frac{\partial S}{\partial t} + \nabla \cdot \left( \rho_0 v^2 + \Sigma \right) = \frac{R}{\rho_0},\]

\[
\frac{\partial \rho_0}{\partial x_n} \frac{\partial L}{\partial x_n} - \frac{\partial \rho_0}{\partial x_n} + \frac{\partial \rho_0}{\partial x_n} = 0,\]

\[
\frac{\partial E}{\partial t} + \nabla \cdot \left( \rho_0 v \nabla \nu_n \right) = 0,\]

Here \( \rho_0, \nu_n, L, \) \( \Sigma \), and \( \rho_0 \) denote the dissipative function,

The equations (6)-(11) together with (12)-(15) are 11 re-
lations for the 10 quantities \( \rho, L, S, E, x_n \), and \( \rho_0 \). This over-
determinacy suffices for a unique determination of the addi-
tional terms introduced.\(^3\) For this we proceed in the same
manner as in Ref. 3. We differentiate Eq (14) with respect to
time and in place of the derivatives \( \partial / \partial t \) we introduce the
adjoint values \( \partial / \partial t \) and \( \partial / \partial t \). After straightforward but tedious transformations we arrive at the
energy balance equation

\[
\frac{\partial E}{\partial t} + \nabla \cdot \left[ \rho_0 v \nabla \nu_n + L \left( \nu_n - \nu \right) \nabla \nu_n \right] = \int D S \]

where \( E_0 \) is the energy per unit length of the vortex fila-
ment, defined as \( E_0 = (\rho_0 \nu_n / m) \left[ (1/4 \pi \ln(b/a)) \right] \). The quantity \( \beta \)
is a characteristic intervortex distance, and \( a_0 \) is an assumed
core radius. Owing to the logarithm, the dependence of \( E_0 \)
on \( L \) is weak, and for realistic conditions (1/4 \pi \ln(b/a)) is
close to unity.

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core radius. Owing to the logarithm, the dependence of \( E_0 \)
on \( L \) is weak, and for realistic conditions (1/4 \pi \ln(b/a)) is
close to unity.
In order to take into account the entropy flux we add to both sides of Eq. (16) the quantity
\[
\text{div} \left[ S^i T (v_i - v_i) + S^i (v_i - v_i) \right] = \text{div} (S^i T 
\]
Here \( S^i \) is the additional entropy due to the vortices, a quantity which cannot be determined within the framework of the proposed formalism. Its determination is an independent problem. Some considerations on the magnitude of \( S^i \) can be found in Appendix I.

After these transformations we obtain
\[
\frac{\partial}{\partial t} + \text{div} (Q + v_i T (v_i - v_i)) = \text{div} (S^i T (v_i - v_i) + S^i (v_i - v_i))
\]

Comparing (17) with the conservation equations for energy and entropy, we obtain
\[
\text{div} (S^i T (v_i - v_i)) = \text{div} (S^i T (v_i - v_i)) = \text{div} (S^i T (v_i - v_i))
\]

Comparing (17) with the conservation equations for energy and entropy, we obtain
\[
q = (n k T - \mu) \rho T = n k T \rho T + \mu \rho T
\]

In addition, supplementary quantities make their appearance in (18). The first of these \( L_e \) describes directly the energy flux of the vortices. The other, as can be seen, is related to the renormalization of the pressure. The meaning of the equality (20) is as follows. The total entropy flux consists of the entropy transport by the normal fluid component plus the entropy carried by the vortices.

The dissipative function \( R \) consists of the work \( R \) of the friction forces against the frozen system of vortices plus the energy transported by the cluster and released in the region of small scales. According to Eq. (1) we write
\[
R = a_1 L_e + b_1 L^2.
\]

Comparing this with Eq. (19) and making use of the fact that from symmetry considerations the vector \( f \) is collinear

with \( V_i \), we obtain
\[
\frac{\partial}{\partial t} + \text{div} (Q + v_i T (v_i - v_i)) = \text{div} (S^i T (v_i - v_i) + S^i (v_i - v_i))
\]

The quantity \( f \) contains \( L_e \) and \( V_e \), the so-called reactive terms. The other two terms describe the dissipation. The dissipative parts of the equations for \( v_i \) and \( L \) will be written in matrix form
\[
\begin{pmatrix}
\frac{\partial}{\partial t} + \text{div} (Q + v_i T (v_i - v_i)) = \text{div} (S^i T (v_i - v_i) + S^i (v_i - v_i))
\end{pmatrix}
\]

Since \( \text{div} (Q + v_i T (v_i - v_i)) = \text{div} (S^i T (v_i - v_i) + S^i (v_i - v_i)) \), the relation (22) describes the Onsager reciprocity principle for the kinetic coefficients. The antisymmetry of the coefficients follows from the different behavior of \( v_i \) and \( L \) under time reversal.

Before writing the definitive form of the equations we express the chemical potential \( \mu \) entering into them as a function of pressure and temperature. For this purpose we use the expression (15) for \( E \) and the following expression for the pressure:
\[
p = -E_o + T S f + \frac{1}{2} \rho (v_i - v_i) \rho T
\]

Going over to the renormalized pressure \( p \), one can obtain from Eq. (23)
\[
\mu = \mu (p, T) + (\rho_i / 2 \rho) \rho (v_i - v_i)
\]

In the sequel we shall deal only with the renormalized pressure \( p \) and omit the index \( r \).

The final equations of motion are:
\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \text{div} (\rho v_i) &= 0, \\
\frac{\partial v_i}{\partial t} + \frac{\partial}{\partial x_j} (\rho v_i v_j) &= 0,
\end{align*}
\]

\[
\begin{align*}
\frac{\partial S}{\partial t} + \text{div} \left[ S^i T (v_i - v_i) + S^i (v_i - v_i) \right] &= \frac{1}{T} \left[ a_1 L_e \rho (v_i - v_i) + b_1 L^2 \rho (v_i - v_i) \right], \\
\frac{\partial L_e}{\partial t} + \text{div} (L_e v_i) &= \frac{1}{T} \left[ a_1 L_e \rho (v_i - v_i) + b_1 L^2 \rho (v_i - v_i) \right].
\end{align*}
\]

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We have thus obtained a system of equations of motion applicable to the case when superfluid turbulence develops in the volume of the flowing helium. The equations can be used to solve problems such as the propagation of strong pulses (in a regime which is supercritical with respect to the current of propagation of second sound in a supercritical countercurrent). We impose the same condition on the perturbations of the velocities. For the perturbations (deviations from the average) $\nu_1$, $T$, $V_{1}$, we obtain the following system of equations:

$$
\begin{align*}
\frac{\partial \nu_{1}}{\partial t} + \nu_{1} \frac{\partial T}{\partial x} &= - \frac{\rho_0}{\rho_0^*} \left( \frac{\partial \nu_{1}}{\partial t} + \nu_{1} \frac{\partial T}{\partial x} \right) - \frac{3}{2} \frac{\rho_0}{\rho_0^*} \left( \frac{\partial \nu_{1}}{\partial t} + \nu_{1} \frac{\partial T}{\partial x} \right) + \left( \frac{\partial \nu_{1}}{\partial t} + \nu_{1} \frac{\partial T}{\partial x} \right), \\
\frac{\partial \nu_{2}}{\partial t} + \nu_{2} \frac{\partial T}{\partial y} &= - \frac{\rho_0}{\rho_0^*} \left( \frac{\partial \nu_{2}}{\partial t} + \nu_{2} \frac{\partial T}{\partial y} \right) - \frac{3}{2} \frac{\rho_0}{\rho_0^*} \left( \frac{\partial \nu_{2}}{\partial t} + \nu_{2} \frac{\partial T}{\partial y} \right) + \left( \frac{\partial \nu_{2}}{\partial t} + \nu_{2} \frac{\partial T}{\partial y} \right), \\
\frac{\partial \nu_{3}}{\partial t} + \nu_{3} \frac{\partial T}{\partial z} &= - \frac{\rho_0}{\rho_0^*} \left( \frac{\partial \nu_{3}}{\partial t} + \nu_{3} \frac{\partial T}{\partial z} \right) - \frac{3}{2} \frac{\rho_0}{\rho_0^*} \left( \frac{\partial \nu_{3}}{\partial t} + \nu_{3} \frac{\partial T}{\partial z} \right) + \left( \frac{\partial \nu_{3}}{\partial t} + \nu_{3} \frac{\partial T}{\partial z} \right),
\end{align*}
$$

where $r = (Q_{L} - [3/2]q_{0}/p_{0})L^{-1/2}V_{s}^{-1}$ is the relaxation time of a vortex cluster.

As always, we look for the solution of Eqs. (30)-(31) in the form $\exp \left( \alpha (t - k_{1}x - k_{2}y) \right)$ with the $y$ axis perpendicular to the countercurrent. The nontriviality condition, i.e., the vanishing of the determinant of the system (30)-(32), yields an equation of the fourth degree relating $\alpha$ and $k$ — the dispersion law. The fourth-degree determinant stems from the vector nature of Eq. (30), thus yielding $\alpha$ and $k$ components. One can solve the dispersion relation in the following way. We make use of the smallness (compared to the usual second sound) of the additional terms in Eqs. (30)-(32). Then the solution of the dispersion equation can obviously be written in the form

$$
\alpha(k) = c_{s}(k) + \xi(k), \quad \xi(k) = \frac{\rho_0}{\rho_0^*} \left( \frac{\partial \nu_{1}}{\partial t} + \nu_{1} \frac{\partial T}{\partial x} \right),
$$

Here $c_{s}$ is the velocity of second sound for a "frozen" cluster. Linearizing the dispersion equation with respect to $\xi(k)$ and solving it, we obtain the following result:

$$
\xi(k) = \frac{1}{2} \frac{\partial \nu_{1}}{\partial t} + \nu_{1} \frac{\partial T}{\partial x} - \frac{3}{2} \frac{\rho_0}{\rho_0^*} \left( \frac{\partial \nu_{1}}{\partial t} + \nu_{1} \frac{\partial T}{\partial x} \right),
$$

Here we have introduced the following notations:

$$
\begin{align*}
\Gamma &= \frac{1}{2} \left( \frac{\partial \nu_{1}}{\partial t} + \nu_{1} \frac{\partial T}{\partial x} \right), \quad A_1 = \frac{\partial \nu_{1}}{\partial t} \frac{\partial T}{\partial x}, \\
A_2 &= \frac{\partial \nu_{1}}{\partial t} + \nu_{1} \frac{\partial T}{\partial x} - \frac{3}{2} \frac{\rho_0}{\rho_0^*} \left( \frac{\partial \nu_{1}}{\partial t} + \nu_{1} \frac{\partial T}{\partial x} \right),
\end{align*}
$$

The following relation holds between the vectors $\nu$ and $k$:

$$
\begin{align*}
\frac{\nu_{1}}{\nu_{2}} &= \frac{\nu_{1}}{\nu_{2}} + \frac{1}{2} \frac{\partial \nu_{1}}{\partial \nu_{2}^*} + \frac{1}{2} \frac{\partial \nu_{2}}{\partial \nu_{1}^*}, \\
\frac{\nu_{3}}{\nu_{4}} &= \frac{\nu_{3}}{\nu_{4}} + \frac{1}{2} \frac{\partial \nu_{3}}{\partial \nu_{4}^*} + \frac{1}{2} \frac{\partial \nu_{4}}{\partial \nu_{3}^*}.
\end{align*}
$$

The relation (33) describes the spectrum of second sound oscillations. As can be seen, this spectrum is anisotropic.
The first three terms describe absorption and the last term describes both absorption and dispersion of the sound velocity. For small $k$, the addition to the velocity of sound depends only on the direction of the vector $k$, and estimates show that it has the order of magnitude $\Delta c_2 \approx 10^{-3} c_2$. For large $k$ this addition behaves approximately like $10^{-3} k/\alpha$. In addition the sound absorption depends both on the direction of $k$ and on the magnitude $k$. The source of the dispersion and of the $k$-dependent absorption is the following. The equation (33) for the quantity $L$ has typically a relaxation form, i.e., we encounter the classical Mandelshtam-Leontovich situation. A cluster which has been taken out of equilibrium by some external action relaxes to its equilibrium state during a characteristic time $\tau$. As is well known, this leads precisely to dispersion and to additional damping. It follows from the result (35) that the vector $v_1'$ is not collinear with the vector $k$. This signifies that when sound is excited in a certain direction at an angle to the flow there will also appear a sound in the $x$ direction. The reason is that the sound "latches on" to the cluster, modulating the value of $L$, and thus leading to oscillations in the $x$ direction.

The case $k_1 = 0$ (sound perpendicular to the flow) requires special consideration. In this case $v_1'$ also vanishes. The interaction of sound with the flow reduces then simply to a damping with decrement $\Gamma$. This is caused by the disappearance of the "latching on" of the sound to the cluster for perpendicular excitation. Indeed, the Vinen equation for $L$ contains only the magnitude of the relative velocity, which in first approximation does not change at $v_1'$ (The coupling via temperature is negligibly small in this case). In this sense Vinen's consideration of the cluster as a system of frozen vortex filaments is justified, one must however make one important restrictive remark. If the flow is stationary, then $L = \alpha/\mu \beta/[\nu_m]^2$, and the decrement equals $\alpha/\mu \beta/[\nu_m]^2$, i.e., in equal to the quantity used by Vinen. However, if $L$ changes with time, then $\Gamma$ is a complicated function, and the interpretation of Vinen's experiment should be based on Eq. (34) for $\Gamma$.

4. THE PROPAGATION OF NONLINEAR SECOND SOUND

The nonlinear acoustics of superfluids opens up great possibilities for He II. Thus, in Ref. 11 turbulence which appears in the wake of an intense heat pulse is studied by a nonlinear second-sound pulse. It is known that the transit time of a nonlinear wave depends on its amplitude (for the case of helium the calculations can be found in Ref. 12), and the latter, owing to damping on vortex clusters, depends on their parameters. It is therefore of practical interest to follow the evolution of a nonlinear second sound pulse in a turbulent medium and to relate its transit time $t_{tr}$ to the parameters of the vortex cluster.

As the analysis of the preceding section shows, the main result of the interaction between the countercurrent and the perpendicular sound wave is a damping with decrement. A small correction both into the main motion [without consideration of the vortex] and into the damping, which is already small. Therefore the terms related to the vortex cluster will be retained only in expressions which are linear in $v_1'$. As a result we are led to the following system of equations for $v_1'$ (see Ref. 13):

\[ \begin{align*}
\frac{\partial v_1'}{\partial t} + v \frac{\partial v_1'}{\partial y} &= -a \frac{\partial}{\partial y} \left( \frac{p}{\rho} + \frac{\partial v_1'}{\partial y} \right) + \frac{\partial}{\partial y} \left( \frac{\partial v_1'}{\partial y} \right), \\
\frac{\partial}{\partial y} \left( \rho v_1' \right) &= 0,
\end{align*} \]

\[ \frac{\partial v_1'}{\partial t} + v \frac{\partial v_1'}{\partial y} + \frac{\partial}{\partial y} \left( \frac{\partial v_1'}{\partial y} \right) = -a \frac{\partial}{\partial y} \left( \frac{p}{\rho} \right) + \frac{\partial}{\partial y} \left( \frac{\partial v_1'}{\partial y} \right) + \frac{\partial}{\partial y} \left( \frac{\partial v_1'}{\partial y} \right), ~ (36) \]

\[ \frac{\partial v_1'}{\partial t} + v \frac{\partial v_1'}{\partial y} + \frac{\partial}{\partial y} \left( \frac{\partial v_1'}{\partial y} \right) = -a \frac{\partial}{\partial y} \left( \frac{p}{\rho} \right) + \frac{\partial}{\partial y} \left( \frac{\partial v_1'}{\partial y} \right) + \frac{2v}{\rho} \frac{\partial v_1'}{\partial y} + \frac{\partial}{\partial y} \left( \frac{\partial v_1'}{\partial y} \right) = 0, \quad (37) \]

If the right-hand side were absent from Eq. (37) we would have a system of two quasilinear equations. It is known that such a system has solutions of the form of so-called simple waves, i.e., flows with a unique relation $T = T'(v_1')$. The addition of a small damping should change this relation, but in a higher order of smallness:

\[ T = T'(v_1') + \Psi(y, t). \quad (38) \]

Since $\Psi(y, t)$ contains only of higher order of smallness than the first order, it must satisfy the wave equation $\partial^2 \Psi / \partial t^2 = \epsilon_0 \partial^2 \Psi / \partial y^2$.

We call the solutions (38) quasilinear waves. Substituting further (38) into Eqs. (36) and (37), and making use of the known expression $T = T'(v_1')$ and of the fact that $\Psi = \epsilon_0 \partial \Psi / \partial y$, we obtain a system of nonhomogeneous algebraic equations for $v_1' / \partial t$ and $\partial \Psi / \partial y$. The requirement of compatibility of these equations allows one to express $\Psi$ in terms of $v_1'$ and to obtain the following equation for $v_1'$ itself (see Ref. 13):

\[ \frac{\partial v_1'}{\partial t} + a(T) v_1' + c_1 \frac{\partial v_1'}{\partial y} = -v_1' v_{1''}, \quad (39) \]

Here $a(T)$ is the nonlinearity coefficient (see Refs. 3, 12, 13).

Equation (39) can be integrated, and one can pose for it, e.g., the Cauchy problem $v_1'(y, 0) = \psi(y)$. We show how this is done. First we go over to a comoving (velocity $c_1$) coordinate system and introduce the new variable $\nu = v / c_1$. In the comoving frame we obtain for the equation

\[ \frac{\partial \nu}{\partial \xi} + a(T) \nu + c_1 \frac{\partial \nu}{\partial \eta} = 0, \quad (40) \]

We note that for a signal of finite extent (compact support)

\[ \frac{\partial \nu}{\partial \xi} \int \Psi(y, \nu = a(T) \nu + c_1 \frac{\partial \nu}{\partial \eta}) d\eta = 0, \quad (41) \]

i.e., the total area under the signal in the coordinates $\nu, \eta$ is conserved in time.

The Cauchy problem with the initial condition $\nu = \psi(y)$ is solved by means of the method of characteristics. The characteristics in the $\nu, \eta$ space satisfy the following equations with their respective parameters

\[ \frac{d\nu}{d\xi} = a(T) \nu + c_1 \frac{\partial \nu}{\partial \eta}, \quad \frac{d\eta}{d\xi} = 1, \quad \eta = \int \Psi(y, \nu = a(T) \nu + c_1 \frac{\partial \nu}{\partial \eta}) d\eta = 0, \quad \nu = \nu_{in}. \quad (42) \]

We determine the characteristic curve passing through the points $(\nu_{in}, \eta_{in})$.

\[ \nu = \frac{a(T) \nu + c_1 \frac{\partial \nu}{\partial \eta}}{\eta}, \quad \nu = \nu_{in}. \quad (43) \]
Let us relate the points \( y_0, y_0', y_0'' \) and the initial condition \( y_0 = 0, u_0 = (y_0) \). Then Eq. (43) gives a parametric solution of the problem. Eliminating the parameter \( y_0 \) we obtain the solution in implicit form

\[
u = \nu \left( \frac{\alpha(T)u}{1 + \alpha(T)ue^{-y}} \right).
\]

We note that for \( F \rightarrow 0 \), as expected, the solution goes over into the usual Riemann solution for a simple wave.

The result (44) is however valid only for smooth solutions. It is known that the nonlinear term \( a(T)u/\partial \nu/\partial y \) may lead to a steepening of the wave profile and ultimately to a discontinuity, when the solution (44) is no longer valid, if only because it gives an ambiguous answer (Fig. 1). In order to find the subsequent evolution of the signal one must make use of the area conservation law (41). Equation (41) implies that in a virtual "overspill" (Fig. 1) the profile has to be cut off by a shock front so that the two shaded regions should have the same areas:

\[
\frac{\alpha}{\partial \nu} \int (y - y_0) \, dy = 0.
\]

This yields

\[
\frac{\partial}{\partial \nu} \int (y - y_0) \, dy = \frac{\partial y_0}{\partial \nu}.
\]

Recognizing that according to the solution (44) \( \partial y_0/\partial t = \alpha(T)ue^{-y} \), and combining this with Eq. (46), we obtain

\[
\frac{\partial y_0}{\partial \nu} = \frac{\alpha(T)(u_0 + u_0')}{2} - e^{-y}.
\]

We have thus derived an equation for the motion of the discontinuity. The solution is smooth to the right and the left of the discontinuity, and one can use Eq. (44)

\[
y_0 = e^{y_0} - \frac{\alpha(T)u_0'}{1 + \alpha(T)ue^{-y_0}}, \quad y_0' = e^{y_0} - \frac{\alpha(T)u_0}{1 + \alpha(T)ue^{-y_0}}.
\]

The equations (47)-(49) are a complete set of equations for \( y_0, u_0, \) and \( u_0' \) (compare this derivation with Ref. 14).

As an example we calculate the evolution of a triangular profile (Fig. 2). In this case the Cauchy condition \( \partial y/\partial y \) is:

\[
\begin{align*}
0, & \quad y = 0, \\
\frac{u}{u_0}, & \quad 0 < y < L, \\
0, & \quad y > L.
\end{align*}
\]

We obtain the following equations for the coordinate of the discontinuity \( y_0 \) and the velocity at the crest \( u_0' \):

\[
\frac{\partial y_0}{\partial t} = \frac{\alpha(T)u_0}{2} e^{-y_0}, \quad y_0' = \frac{\alpha(T)u_0}{ \alpha(T)u_0 - 1} - \frac{\alpha(T)u_0}{2} (e^{-y_0} - 1).
\]

Solving this system we obtain, e.g., for \( u_0'(t) \) the following expression:

\[
u_0(t) = (\nu_0^0)^{-1} (\nu_0^0 - \alpha(T)\Gamma^{-1} (e^{-\nu_0^0} - 1))^{-1}.
\]

Making use of the result (52) one can determine the transit time of a pulse for a given distance \( B \) according to the following equation:

\[
\int \left[ \frac{\alpha(T)u_0(t)}{2} - e^{-y_0(t)} \right] \, dt = B.
\]

Equation (53) relates the transit time of a nonlinear signal with quantities characteristic for superfluid turbulence, and thus solves the problem we have posed. One can similarly calculate \( t_1 \) for more complicated cases.

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APPENDIX I

In Appendix I we estimate the entropy \( L^I \) transported by the vortices. That part which is related to the vortex component of the superfluid must be determined from the identity

\[
\delta p = p \, \delta T + S \, \delta V + \left( \mu \, \delta \nu \right)_V.
\]

The dependence of the pressure on the relative velocity has the form

\[
p = p_t(y_0, T) + \frac{p_t(y_0, T)}{2} \nu^2.
\]

Taking into account the fact that in view of the hydrodynamic equations the quantities \( T, \mu \) and \( \nu^2/2 \), and \( \nu \) cannot have singularities (as well as \( \partial p/\partial u = 0 \)), we obtain for the singular part of the pressure

\[
p_s = -\nu \frac{\partial p_s}{\partial \nu} \nu_v.
\]

In agreement with Eq. (I.1) the singular part of the entropy density is now equal to

\[
S_s = 1 \left( \frac{\partial p_s}{\partial \nu} \nu_v \right)^2.
\]

This involves the partial derivative for constant chemical potential

\[
\left( \frac{\partial p_s}{\partial \nu} \right)_V = -S \frac{\partial p_s}{\partial \nu} \left( \frac{\partial p_s}{\partial \nu} \right)_V.
\]

Integrating this expression near the vortex, one can obtain the entropy \( S^I \) carried away by them:

\[
\begin{align*}
\int \frac{\partial p_s}{\partial \nu} V dV & = 0, \\
\int \frac{\partial p_s}{\partial \nu} V dV + S \int \frac{\partial p_s}{\partial \nu} V dV & = S^I.
\end{align*}
\]
The lower cutoff is, however, not the radius of the vortex, but the mean free path of excitations, \( l \), i.e., the limit under which the hydrodynamic discussion is no longer justifiable. In the region \(- l\) the distribution function of the excitations differs from an equilibrium distribution, and this region contributes to \( S^0\) an addition which, on account of the smallness of the region, can hardly exceed the contribution \( (L/2)\). The same can apparently be said about the proper attributes to \( S^0\) an addition which, on account of the smallness of the entropy of the vortices, although it is extremely difficult to estimate it, without having an idea on the structure of the vortex cluster. Consequently, in the hydrodynamic equations one must consider \( S^0\) as a prescribed function (which should be obtained from microscopic considerations), transported with the velocity \( v_L\), and estimated by Eq. (L.2).

**APPENDIX II**

It is convenient to formulate the nondissipative hydrodynamic equations in the language of Poisson brackets [see the reviews 10, 11]. There arises a problem related to the fact that the Poisson brackets must satisfy the Jacobi identity. However, it is by far not always possible to construct adequate hydrodynamic equations by means of a system of Poisson brackets satisfying the Jacobi identities. For this reason Volovik and Kats, 12 who considered the hydrodynamics of liquid crystals, were forced to introduce the auxiliary quantity \( L\), which later had to be eliminated from the final equations. The reason for this is that the hydrodynamic degrees of freedom are slowly relaxing, surviving after one excludes from the complete set degrees of freedom the rapidly relaxing variables. Since the latter are not constants of the motion, such an exclusion from the expressions for the Poisson brackets (which for a complete set of observables satisfy the Jacobi identities) inevitably leads to Poisson brackets for the densities \( \rho, v, S\) and \( L\), which for a complete set of observables satisfy the Jacobi identities. For this reason the hydrodynamic equations in the language of Poisson brackets satisfying the Jacobi identities may accidentally still be valid 13 for example in the case of a classical fluid, or for superfluid He II, owing to limited number of hydrodynamic variables. In the sequel, formulating the hydrodynamic equations, we do not impose the Jacobi identities, but leave in force the requirement of antisymmetry of the Poisson brackets.

The hydrodynamic variables split into the densities of conserved (or slowly relaxing) quantities, and into variables describing the order parameter degrees of freedom. The expressions for the Poisson brackets involving densities generally do not depend on the concrete system under consideration and have a universal character determined by symmetry considerations. 14 These universal considerations guarantee that the corresponding conservation laws are satisfied for any system. The arbitrariness related to the violation of the Jacobi identities appears only in the expressions of the Poisson brackets for quantities related to the order parameter.

In the case considered here we have densities of mass \( \rho\) and momentum \( j\), and in addition there is the entropy density \( S\). Related to the order parameter \( S^0\) are the superfluid velocity \( v_S\), and the length \( L\) per unit volume of the vortex filaments. The enumerated quantities form a complete set of hydrodynamic variables for turbulent He II. The Poisson brackets for the densities \( \rho\) and \( j\) have the following universal form:

\[
\begin{align*}
\{j(\tau_1), \rho(\tau_2)\} & = v_S(\tau_1) \cdot \nabla (\tau_1 - \tau_2), \\
\{\rho(\tau_1), j(\tau_2)\} & = j(\tau_1) \cdot \nabla (\tau_1 - \tau_2) + j(\tau_2) \cdot \nabla (\tau_2 - \tau_1).
\end{align*}
\]

\[
\{\rho(\tau_1), j(\tau_2)\} = v_L(\tau_1) \cdot \nabla (\tau_1 - \tau_2).
\]

Also universal are the brackets

\[
\begin{align*}
\{j(\tau_1), S(\tau_2)\} & = \nabla (\tau_1 - \tau_2), \\
\{\rho(\tau_1), L(\tau_2)\} & = L(\tau_1) \cdot \nabla (\tau_1 - \tau_2).
\end{align*}
\]

The Poisson brackets between the momentum density \( j\) and the densities \( \rho\) and \( L\) are similar to (II.1): \( \{j(\tau_1), L(\tau_2)\} = L(\tau_1) \cdot \nabla (\tau_1 - \tau_2)\). The expressions for the Poisson brackets involving the superfluid velocity \( v_S\) can be derived on the basis of the fact that the quantities \( \rho\) and \( L\), and the entropy \( S\) are convected with the velocity \( v_L\). This fixes the following expressions:

\[
\begin{align*}
\{v_S(\tau_1), \rho(\tau_2)\} & = \frac{1}{\rho_S} \nabla \cdot \rho_S \frac{\partial \rho_S}{\partial \tau_1} \delta (\tau_1 - \tau_2), \\
\{v_S(\tau_1), L(\tau_2)\} & = -\frac{1}{\rho_S} \frac{\partial L}{\partial \tau_1} \delta (\tau_1 - \tau_2), \\
\{v_S(\tau_1), S(\tau_2)\} & = -\frac{1}{\rho_S} \frac{\partial S}{\partial \tau_1} \delta (\tau_1 - \tau_2) \cdot \nabla (\tau_1 - \tau_2).
\end{align*}
\]

The nondissipative hydrodynamic equations for turbulent He II can now be formulated by means of the Hamiltonian

\[
H = \int d\tau \left( \frac{\rho v^2}{2} + \rho v_S + E_S (\rho, L, L_s) \right).
\]

These equations have the following form:

\[
\begin{align*}
\frac{\partial \rho}{\partial \tau} - \{H, \rho\} & = \nabla S, \\
\frac{\partial L}{\partial \tau} - \{H, L\} & = \nabla H.
\end{align*}
\]

\[
\frac{\partial j}{\partial \tau} = \{H, j\} = \nabla (\rho v_S + j + L_s).
\]

Here the pressure is

\[
p = -E_S + \mu + T S (\rho + \rho v_S + j + L_s).
\]

Equations (II.11)-(II.15) lead to the following form of the energy conservation law:
\[
\frac{dE}{dt} = - V \left[ ST_{\nu_0} - \delta^2 \left( \nu_0 - \nu_1 \right) \right] + \nu_{\nu_0} \left( - \nu_0^2 \right)^2 \left[ (\nu_0 + \nu_1) \right] \]  
\text{(II.17)}

Schwarz has introduced the function \( \delta(\nu_0) \) representing the length of the filaments per unit volume, such that the selfinduced velocities (i.e., the velocities caused by the local curvature of the filament) are within the interval \( (\nu_0 + \nu_1) \). However, his attempt at deriving a kinetic equation for \( \delta \) directly from the dynamical equations of motion is incorrect. The Bogolyubov chain (BRGKY hierarchy) does not contain a small parameter in this case and its artificial truncation, as proposed by Schwarz, is illegitimate.

One does, of course, appeal to additional considerations or empirical facts for the construction of the dissipative function (cf. Ref. 3).

As is well known, in a countercurrent one encounters a "drag" of the second sound, i.e., \( c_2 = c_p + \Delta \omega \omega_{\nu_0} \). In order not to complicate the already long formulas we omit from the equations all terms leading to this effect. In view of the fact that both the Khalatnikov correction \( \Delta \omega_{\nu_0} \) as well as the corrections we have found, are small they will simply contribute additively to final effect.

In calculations one should substitute \( \mu = - \rho \nu_0 \).


