Perturbation transformations on a fast shock wave in a longitudinal magnetic field

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The transformation coefficients of low-amplitude magnetohydrodynamic and entropy waves in a rapid shock wave moving in a longitudinal magnetic field are investigated. Cases of appreciable wave amplification are considered. It is shown that in a weak homogeneous magnetic field it is possible to enhance the field's perturbations considerably.

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Perturbations can change their spatial and temporal scales considerably and also be significantly amplified in interactions with shock waves. Transformations of a perturbation by a shock wave were first considered by Blokhintsev\(^1\) and also by Kontorovich in ordinary hydrodynamics and magnetohydrodynamics.\(^2,3\) A numerical study of the transformation coefficients in magnetohydrodynamics was carried out by J. F. McKenzie and K. O. Westphal.\(^4,5\)

Compact expressions are derived in the present paper for the transformation coefficients of magnetohydrodynamic waves in the case of oblique incidence on a fast shock wave in a longitudinal magnetic field. The dependence of the transformation coefficients on the angle of incidence is investigated analytically. The possibility of significant amplification of the perturbations of the magnetic field in supernova fragments via excitation of vortex motions in a shock wave is discussed.

1. STATEMENT OF THE PROBLEM

We shall consider a shock wave in a system of coordinates in which the unperturbed discontinuity is at rest and coincides with the \(x\) plane. We shall assume the plasma on both sides of the discontinuity to be an ideal conducting liquid, which we shall describe by the equations of ideal magnetohydrodynamics.\(^6\) In the case of small perturbations that are of interest to us, we shall limit ourselves to a linear approximation; we set \(\partial / \partial z=0\), and take the dependence of the perturbations on the time \(t\) and the coordinate \(y\) in the form \(\exp(-\theta t-\phi y)\). On each side of the discontinuity, the perturbations can be represented in the form of the superposition of families of waves: entropy, two slow magnetosonic, two fast magnetosonic, and two Alfvén waves, which, in correspondence with the direction of the group velocity,\(^7\) divide into waves incident on the discontinuity and into waves going out from it. According to the evolutionarity condition\(^8\) there are only incident waves in front of the fast shock wave \((x<0)\) and only a single fast magnetosonic wave behind the shock wave. This latter is an incident wave, the others are outgoing waves.

Perturbation of the velocity and the magnetic field in the \(xy\) plane, and also perturbations of the entropy and pressure, are due to the entropy and magnetosonic waves. We express these perturbations in terms of the amplitude of the waves \(\delta u,\delta B\), and the polarization vector \(\mathbf{y}_i\):

\[
\begin{pmatrix}
\delta p_v \\
\delta B_{xv} \\
\delta B_{yv} \\
\delta B_{zv} \\
\delta u_v \\
\delta T_{s}v_v
\end{pmatrix} = A_i \mathbf{y}_i
\]

The index \(i\) here is equal to unity in the region in front of the shock wave, and is absent in back of the shock. \(\delta x\) denotes the type of wave; \(\delta p_v\), \(\delta B\), and \(T_s\) are the unperturbed density, flow velocity, and temperature, respectively; \(\delta u_v\), \(\delta T_{s}v_v\), \(\delta B_{xv}\), \(\delta B_{yv}\), and \(\delta B_{zv}\) are the perturbations, created by the \(i\)-th wave, of the pressure, velocity, magnetic field and entropy.

The polarization vector \(\mathbf{y}_i\) is determined from the linearized equations of magnetohydrodynamics:

\[
A_i \mathbf{y}_i = k_{ui} \delta u_v
\]

where \(k_{ui}\) is the derivative of the component of the wave vector, and the matrices \(A_i, B_i\) have the form

\[
A_i = \begin{pmatrix}
M_{i1} & 0 & 0 & -Q_{i1}/M_{i1} \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

\[
B_i = \begin{pmatrix}
M_{i1} & 1 & 0 & 0 \\
0 & M_{i1} & 0 & 0 \\
0 & 0 & 0 & 0 \\
v_i & 0 & 0 & 0
\end{pmatrix}
\]

Here \(M_{i1} = \sigma_i/\mu_i\) is the Mach number, \(\mu_{vi} = v_i/\mu_{vi}\) is the magnetic Mach number, \(\mu_{vi} = B/(\rho \sigma p_v)\) is the Alfvén velocity,

\[
\sigma_i = \frac{\partial \rho_v}{\partial \rho_v}, \quad \mu_i = \frac{\partial \mu_v}{\partial \mu_v}, \quad \nu_i = \frac{\partial \nu_v}{\partial \nu_v}
\]

Equation (2) determines the polarization vectors \(\mathbf{y}_i\) within a multiplicative factor. We represent its solutions for the entropy waves in the form

\[
y_{sv} = (0, 0, 0, 0, 1),
\]

and for the magnetosonic waves,

\[
y_{sv} = \left(1, \frac{1 - \nu}{v}, -\frac{M_v\nu^3 - (1 - \nu)^4}{\nu v^2}, 0, 0\right)
\]

where \(v_i = \omega_i/k_{ui}\) is the normalized frequency of the waves in the system of coordinates of the plasma at rest.

Perturbations of the projections of the velocity and the magnetic field intensity on the \(x\) axis are due exclusively to

the Alfvén waves, which we shall describe with the help of the amplitudes $\Delta B_{x_{inc}}$ and $\Delta B_{z_{inc}}$ in correspondence with the dispersion law $\omega = k_{x_{inc}} n_1 + 1/\sqrt{M_{x_{inc}}}$.

Waves departing from the discontinuity appear obviously as the result of the interaction of the incident waves with the discontinuity. This interaction takes place with conservation of the projections of the mass and energy fluxes and momentum-flux tensor normal to the discontinuity $I$. The normal component of the magnetic field intensity and the tangential component of the electric field intensity are also continuous. This means that the surface of the discontinuity itself undergoes oscillations near the $zy$ plane under the action of the perturbations. These oscillations have the form

$$ R(y, t) = y \exp[-i(\omega t - q y)], $$

where $R(y, t)$ is the deviation of the surface element of the discontinuity from the $zy$ plane along the $x$ axis, and $q$ is the amplitude of the oscillations. From the conservation conditions at the discontinuity, we arrive at the following set of equations which connect the amplitudes of the incident and departing (entropy and the magnetosonic) waves:

$$ \sum \delta B_{x_{inc}} \cdot \hat{n} = \sum \delta B_{x_{out}} \cdot \hat{n} = \sum \delta B_{z_{out}} \cdot \hat{n}, $$

for $x > 0$.

2. TRANSFORMATION COEFFICIENTS. GENERAL PROPERTIES

The equations for the amplitudes of the Alfvén waves are separable; the perturbations of the magnetic fields in the transmitted and incident Alfvén waves are connected in the following way:

$$ \delta B_{x_{inc}} = (M_1 + 1)^{-1/2} \delta B_{x_{out}} (\gamma_1 + 1) \gamma_1 \gamma_1^{-1} + \delta B_{x_{out}} (-y_1 + 1) \gamma_1 \gamma_1^{-1}, $$

$$ \delta B_{z_{inc}} = (M_1 + 1)^{-1/2} \delta B_{z_{out}} (\gamma_1 + 1) \gamma_1 \gamma_1^{-1} + \delta B_{z_{out}} (-y_1 + 1) \gamma_1 \gamma_1^{-1}, $$

where $\gamma$ is the compressibility in the shock wave.

From Eq. (8) we obtain for the amplitudes $\Delta A_i$ of the waves departing from the discontinuity

$$ \Delta A_i = \sum \Pi_{inc} \Delta A_i + O_i \Delta A_{inc}. $$

In the solution of (8) by Cramer’s rule, the transformation coefficients $\Pi_{inc}$ and $O_i$ turn out to be functions of the wave vectors $k_1, k_2, \ldots, k_{x_{inc}}$ of all three magnetosonic waves departing from the discontinuity (the angular dependence.) This complicates the analysis of the transformation coefficients, since the longitudinal wave vectors of the magnetosonic waves are the roots of the dispersion equation

$$ \Delta \omega = k_{x_{inc}} n_1 + 1/\sqrt{M_{x_{inc}}}, $$

which is of the fourth degree, and has the following form in the variables $Q, \varphi$:

$$ Q = \begin{pmatrix} (M_1 + 1)^{-1} & (M_1 + 1)^{-1} \\ (M_1 + 1)^{-1} & (M_1 + 1)^{-1} \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, $$

$$ Q - (M_1 + 1)^{-1} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0. $$

The function $Q(\varphi)$ is shown in Fig. 1, where we have used the notation

$$ \gamma_1 = (\gamma_1 + 1)^{-1}, \quad \varphi = (\gamma_1 - 1)^{-1}, $$

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FIG. 1. Dispersion curve (12).

\[ \left[ P \left( \frac{\partial V}{\partial p} \right) \right] _{\frac{1}{2}} + 1 \left( 1 + \nu [M^2_{\text{inc}} - (1 - \gamma \omega)] \right) \]

\[ - 2 \nu = (1 - M^2) = 0. \]  

(18)

Here \( J \) is the mass flow across the discontinuity, \( \partial V / \partial p \) is the derivative of the specific volume \( V = \frac{1}{p} \) with respect to the pressure \( p \) along the shock adiabat. Satisfaction of (18) at \( \text{Im} \omega = 0 \) corresponds to spontaneous radiation of waves by the discontinuity; \( \text{Im} \omega > 0 \) it corresponds to absolute instability. \( \text{Im} \omega = 0 \) is located near the boundary of this region, in particular, in the case \( \nu = v_{k1} < v_{k2} < v_{k3} \). In the latter case, the reflection coefficients \( O_{1} \) are as before bounded by virtue of the multiplier \( \Pi_{1} = B_{1} \), which vanishes in the case \( \nu = v_{k1} < v_{k2} < v_{k3} \).

The transformation coefficients \( \Pi_{m} \) can be large at \( \nu = v_{k1} < v_{k2} < v_{k3} \) while at sufficiently large distance from \( \nu = v_{k1} \) (17) simplifies to

\[ \Pi_{m} = \frac{\text{Im} \omega}{\text{Im} \omega - \nu} \text{ const.} \]  

Moreover, the reflection and refraction coefficients become proportional to one another:

\[ O_{1}/O_{2} = \Pi_{1}/\Pi_{m}. \]  

(20)

In an ideal gas with constant heat capacity \( \Pi_{m} \) is large in two cases. First, in a strong shock wave (as \( M_{1} \rightarrow \infty \) ) and in a bounded magnetic field \( B \) when the larger root of (18), \( \nu = v_{k1} \) and the boundary of the real \( v_{m} \) tend to the same limit \( v = (1 - M_{1}^2)^{-\frac{1}{2}} \) from different sides (a similar situation occurs in ordinary hydrodynamics). Second, in a sufficiently strong magnetic field in the case of bounded Mach numbers \( M_{1} \rightarrow M_{\text{lin}} \). Here \( M_{\text{lin}} = \frac{B_{1}^2}{4 \pi n I} \) is the smallest of the Mach numbers at which the shock wave is absolutely stable (that is, spontaneous radiation of waves does not occur).

3. REFLECTION OF A FAST MAGNETOSONIC WAVE

For the medium we take the equation of state of an ideal gas with constant heat capacity and give the expression for the transformation coefficients \( O_{1} \).

The departing waves in the region \( x > 0 \) can be traveling and surface waves, depending on the angles of incidence of the waves at the discontinuity. The traveling waves (\( \text{Im} \omega = 0 \), \( \text{Im} \omega = 0 \)) correspond to portions of the real axis (Fig. 1). In the general case there exist two regions of real values of \( v_{m} \). The region \( v_{m} > v_{\text{inc}} \) arises in the case of
FIG. 3. Dependence of the transformation coefficient $O_v$ on $v_{in}$.

arbitrary values of the magnetic field ($M > 1$). The roots $(12)$, corresponding to the departing waves, lie in the intervals $v_{in} < v_1 < v_2$ (fast), $v_2 < v_1 < v_3$ and $v_4 < v_1 < v_3$ (slow). The region $v_1 < v_{in} < v_2$ is possible only in sufficiently strong magnetic fields; the roots corresponding to the departing waves lie in the intervals $v_4 < v_1 < v_2$. The boundaries of the regions I-IV in the complex $\omega$ plane correspond to the surface waves (Im $\omega = 0$). The physical picture behind the shock wave is a picture of spatial beats of the perturbations of the magnetic field develops (a component of the perturbation of the velocity $\nu$ beats in antiphase). In this sense we can speak of a significant amplification of the perturbations of the magnetic field of the shock in a weak uniform magnetic field.

4. PASSAGE OF THE WAVES ACROSS THE DISCONTINUITY

The transformation coefficients $B_v$ [17] are in the general case more complicated functions of the wave vectors than $O_v$. Here we give the relation between the perturbation $(\delta T'/T_v)$, of the temperature in an incident entropy wave and the perturbation $\delta B_{m,v}$ of a magnetic field by the magnetosonic waves excited by it:

$$\frac{\delta B_{m,v}}{B_v} = \frac{x}{x_0} \left[ \frac{(x_0-1)}{(x_0-1)+AM'} \left( \frac{\delta T'}{T_v} \right) \right]$$

Here $x_0 = (\gamma + 1)/(\gamma - 1)$ is the maximum compressibility in the shock wave, $\gamma$ is the adiabatic coefficient. For traveling waves, the modulus of $O_v$ increases monotonically with the magnetic field but remains bounded in the region of stability. Qualitatively, the properties of $O_v$ are shown in Fig. 3 for the region $v_1 < v_{in} < v_2$. At normal incidence ($v_{in} = v_2$) the coefficient $O_v$ does not depend on the magnetic field. At arbitrary $v_{in}$ the modulus of $O_v$ does not exceed

$$\left( v_{in} - v_2 \right) \left( v_{in} - v_1 \right) \left( v_{in} - v_3 \right) \left( v_{in} - v_4 \right)$$

The fields in the drawing have values $B_1 < B_2 < B_3 < B_4$, where $B_3$ corresponds to the boundary of the instability.

The magnetic waves are described by the transformation coefficients $O_v$, which can be expressed in terms of the $O_v$ in the following way:

$$O_v = O_v(v_{in})/O_v(v_{in})$$

We note that at $v_{in} = v_2 (j = 1, 2, 3)$ only a single outgoing wave is formed, $l$, for which $v_{in} - v_2$ (Fig. 1). Here $O_v$.
We now consider the possibility of this mechanism of excitation and amplification of the perturbations of the magnetic field under astrophysical conditions. According to Ref. 17, the galactic magnetic field and the mean concentration of particles in interstellar space amount to $B = (2-3)10^{-6}$ Oe and $n_i = 1$ cm$^{-3}$. From the condition $M_x < 1$ we obtain $v_i = (\pi B)^{1/2} \approx 1.1 \times 10^5$ cm/s, which is generally achievable in supernova fragments. The direct application to them of the results set forth above is connected with two conditions: 1) The emission of energy from the shock wave should be negligible, a condition satisfied in the adiabatic phase of expansion; 2) the width of the shock front $\delta$ should be significantly less that the radius of the relic $R_s$.

We shall use a model of a supernova with ejection of mass $\approx 4M_\odot$ and initial velocity $5 \times 10^3$ cm/s with typical density and temperature of interstellar space $n_i = 1$ cm$^{-3}$ and $T_i = 10^2$ K. The adiabatic phase in this case begins at a velocity $v_1 = 5 \times 10^3$ cm/s and ends at $v_2 = 1.7 \times 10^5$ cm/s (when the temperature behind the shock front drops to $10^6$ K). Condition 2) begins to be fully satisfied at expansion velocities $v < 7 \times 10^5$ cm/s. At this stage, the radius of the relic $R_s \approx 12$ pc, the temperature behind the front $T_s = (3/2)\mu mT_i/k_2 \approx 1.3 \times 10^{-4} v_i^2 \approx 6.4 \times 10^9$ K; consequently, the free path length is $\lambda = 2 \times 10^{-7} T_s^{1/2} \approx 2 \times 10^{17}$ cm $\times 7 \times 10^{-2}$ pc, $\delta = A _1 \Delta_x$. The relic slows its motion from a speed $v_1 = 7 \times 10^5$ cm/s to $v_2 = 2.7 \times 10^5$ cm/s in a time of $2.4 \times 10^4$ yrs, which is about 80% of the adiabatic stage.

The magnetic Mach number at a speed of expansion $v_1 = 5 \times 10^5$ cm/s in a magnetic field $B = 3 \times 10^{-6}$ Oe amounts to $M_x = v_i/(\pi B)^{1/2}/\lambda = 36$ which assures a significant gain in amplification (excitation) of the perturbations of the magnetic field. At a temperature of interstellar space $T_i = 10^2$ K, we get from (27)

$$\frac{\Delta B}{B_0} \Delta (T_i/T_1)_{\text{max}} = 1.2 M_x = 0.45 \times 10^5,$$

i.e., relative perturbations of the temperature of the order of $10^{-3}$ can lead to a significant $(\Delta B/B \sim 1)$ perturbation of the magnetic field.

An observational test of the given mechanism of formation of inhomogeneities of the magnetic field can be provided by a dual structure of radio details. Actually, when a shock front passes through a hot (or cold) region, vertical perturbations of the velocity are formed behind it, compressed by a factor of 4 along the $x$ axis. This perturbation in a weak magnetic field is the superposition of two wave packets of the slow magnetosonic waves, moving with different group velocities $v_g = v_1 \pm v_2$. Consequently, the packets move apart within a certain time (by 1 pc within $5 \times 10^4$ yr) and form two details with amplified perturbation of the magnetic field $\Delta B$.

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1 D. I. Blokhintsev, Akustika neodnorodnich dviruchshchitya troy (Acoustics of a Moving Inhomogeneous Medium) Moscow, Nauka, 1981, Sec. 32.