Superradiance in a two-dimensional model

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A semiclassical theory valid for arbitrary Fresnel numbers is developed for collective spontaneous emission of two-level quantum emitters in multidimensional (mainly two-dimensional) space. It is shown that uncorrelated spontaneous decay can be taken into account within the framework of the semiclassical approximation. The kinetics of formation of a correlated state in an ensemble of excited emitters is considered. It is shown that to observe superradiance in a concentrated system calls for a symmetrical arrangement of the emitters (the Dicke model). A general expression is obtained for the emission frequency shift in the Dicke model. A feature of the dynamics of extended emitter systems is that the angular distribution of the emission intensity does not reach a maximum simultaneously in different directions, in contrast to the results predicted by the existing superradiance models. The question of the anisotropy of collective emission in the case of two-dimensional periodic structures is investigated.

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Spontaneous decay of a group of excited quantum emitters can differ radically from their independent decay and turns into an essentially collective process.[1,2] Of interest are four cases in which collective properties of spontaneously emitting atoms manifest themselves: 1) the case of a concentrated system, when N atoms decay in a spatial region of size $R \ll \lambda$, where λ is the wavelength of the radiation field; this problem was first considered by Dicke,[1] who obtained for the radiation intensity a value proportional to N^2 and introduced the concept of super-radiant state; 2) decay of a finite number of atoms at arbitrary distances between them was investigated in detail only for two atoms (a list of the basic papers is given in Ref. 3); 3) decay of one excited radiator in the presence of analogous unexcited emitters: in Ref. 4 was considered a simple model of linear oscillators with damping, and in Ref. 5 (see the bibliography therein) was investigated decay in the presence of one unexcited radiator; 4) collective emission of an extended system of radiators with $R \gg \lambda$, having arbitrary dimensions and shape—this problem in so general a formulation is very complicated, so that satisfactory results were obtained only in the one-dimensional approximation.[2] In a number of papers (see, e.g., Refs. 6–8) various methods were proposed for the description of the angular anisotropy of the radiation and for taking into account the role of the different modes in the development of collective emission, but a common shortcoming of the employed methods is the assumption of spatial homogeneity of the decay process and of constancy of the phases of the emitters. These conditions are realized only during the initial stage of the emission, when the intensity of the radiation is low, and are substantially violated when the intensity is increased, yet it is precisely the case of high radiation intensities which is of interest.

This variety of physical problems gives rise to a corresponding variety of methods of their solution, and it is sometimes difficult to establish common ideas that connect these solutions. Furthermore, it is still too early to speak of a complete understanding of the physical mechanism of the superradiance effect. All this stimulates a search for new theoretical-description methods that would be more universal and free of unjustified assumptions.

One can point out two difficulties faced by a theory of collective spontaneous emission: one is due to the vector character of the radiation field and the other to the spatial multidimensionality of the real problem. As for the first difficulty, we note that in the problems listed above were used, explicitly or implicitly, special initial conditions for the orientations and phases of the radiating currents (several possible types of such conditions are listed in Ref. 9), or else the polarization properties of the radiation were neglected.[6–8] With respect to the second difficulty we note that at large Fresnel numbers $F = S/\lambda L > 1$, where S and L are respectively the cross section and the length of the sample, the one-dimensional approximation works well, therefore a three-dimensional description is necessary only at $F < 1$, in particular at transverse dimensions comparable with the radiation -field wavelength.

In the present paper, on the basis of a semiclassical description of the "atoms + field" system, we consider a multidimensional model of collective spontaneous emission. Principal attention is paid here to the two-dimensional model since, first, within the framework of such a model it is physically justified to neglect the polarization of the radiators and, second, this model contains a number of distinguishing features that are characteristic of three-dimensional systems and is at the same time very simple and illustrative.

The basis of the semiclassical description is chosen to be a method cited in Ref. 10, which makes it possible to describe the inhomogeneities of a medium of arbitrary length; this is important when considering active media with small transverse dimensions. At the same time, the possibility is demonstrated of taking into account, within the framework of the semiclassical approach, the quantum fluctuations of the atomic variables and of the field, which are important during the initial stage of the development of the collective radiation. Allowance for the state of each radiator, and not for the integral characteristics of the system, makes it possible to consider a large number of problems of all the types listed

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above. In the present paper, on the basis of the proposed model, we consider three fundamental cases: the Dicke model ($R\ll\lambda$), solid samples characterized by small Fresnel numbers $F\ll 1$, and systems of radiators with ordered arrangement—two-dimensional periodic systems whose periods are comparable with the wavelength of the radiation field.

1. PHYSICAL MODEL

In a two-dimensional system all the functions of the state of the medium (the field amplitude, the inverted population, the polarization of the radiators) depend only on two spatial coordinates. In a three-dimensional realization of such a system, each radiator constitutes a straight infinite filament along which are located identical two-level atoms whose states are the same at all instants of time. If the polarization direction is assumed constant and directed along the filament, the radiation of such a two-dimensional radiator is isotropic in a two-dimensional subspace orthogonal to the direction of the filament, and in this space the radiation field can be regarded as scalar. The two-dimensional case, as well as the three-dimensional one, differs from the one-dimensional one in that the force of the self-action of the radiator is infinite, and in that the retarded solution of Maxwell's equation near the radiator increases without limit. To overcome this difficulty we shall use Dirac's method,[11] wherein the self-action field of a point radiator (and a filament is a point in two-dimensional space) is chosen to be not the retarded field, but the limiting value of half the difference between the retarded and advanced solutions of Maxwell's equation. This method has two advantages: first, the self-action field of a point source is always finite, and second, this field describes correctly the radiation-friction force. We note that in the one-dimensional case[10] Dirac's procedure leads to the same result as allowance for only the retarded field, inasmuch as the advanced field of the radiator at the location of the latter is finite in magnitude and differs only in sign from the retarded solution.

We consider now a two-level quantum system and, in accord with Ref. 12, divide the total intensity of the spontaneous emission of such a system into two parts:

$$I=\gamma\langle R^+R^-\rangle=\gamma(\langle R_1\rangle^2+\langle R_2\rangle^2)+\gamma(\langle\Delta R_1^2\rangle+\langle\Delta R_2^2\rangle+\langle R_3\rangle),\quad \gamma=4d^2\omega^4/3c^3, \tag{1}$$

where d is the matrix element of the dipole-moment operator, ω is the transition frequency, c is the speed of light, $R^\pm=R_1\pm iR_2$, $R_i=\sigma_i/2$, and σ_i are Pauli matrices. The first term in the right-hand side of (1) is due to the nonzero average value of the dipole moment, and can therefore be called the regular or classical part of the spontaneous emission. The second term is due to quantum fluctuations of the dipole moment (the quantity $\langle R_3\rangle$ in this term can be associated with quantum fluctuations of the electromagnetic field[13]), and can therefore be called the fluctuating part. It is natural to assume that the classical parts of the radiation field due to different atoms interfere with one another while the fluctuation parts do not interfere. Using this assumption, we consider several examples.

Thus, for a one-dimensional radiator—a plane—the fluctuation part of the total spontaneous-emission density is given by

$$I_{\rm fl}=\frac{\gamma n}{4}\left(1-\cos\theta-\frac{\sin^2\theta}{2}\right), \tag{2}$$

where n is the surface density of the atoms, θ is the Bloch angle defined by the relation $\cos\theta=-2\langle R_3\rangle$. The classical part of the radiation density is equal to[10]

$$I_{\rm cl}=(2\pi/c)(n\omega d)^2\sin^2\theta, \tag{3}$$

and then we obtain for the total intensity the expression

$$I=I_{\rm cl}+I_{\rm fl}=\frac{4\pi}{c}(n\omega d)^2\left[\frac{\sin^2\theta}{2}+\beta_1\left(1-\cos\theta-\frac{\sin^2\theta}{2}\right)\right], \tag{4}$$

where the parameter $\beta_1=\pi/3n\lambda^2$ can be regarded as the "classicism" parameter of a plane radiator. Indeed, at $\beta_1\gg 1$ all the radiation is classical. We note that relations (3) and (4) were obtained under the assumption that the emission of the atoms that fill the plane adds up coherently only in one direction and with a definite polarization, i.e., only one working mode is segregared. At $\beta_1\sim 1$, i.e., $n\lambda^2\sim 1$, additional diffraction directions appear—additional working modes, and Eq. (4) no longer holds.

We can consider similarly spontaneous decay of a two-dimensional radiator—a filament (see below). In this case one obtains an equation analogous to (4):

$$I=\pi cs^2k^3d^2[{}^1/_2\sin^2\theta+\beta_2(1-\cos\theta-{}^1/_2\sin^2\theta)], \tag{5}$$

where $k=\omega/c$, $\beta_2=4/3\lambda s$, and s is the number of atoms per unit length of the filament. Using the connection between the Bloch angle θ and the radiator energy, we can obtain from (4) and (5) the law governing the collective decay of the radiator. Thus, at $\beta_{1,2}\ll 1$ both the one-dimensional and the two-dimensional radiators decay in accord with a characteristic super-radiant law, with the radiation intensity taking the form

$$I(t)=I_{max}\mathrm{sech}^2[(t-t_0)/2\tau_c], \tag{6}$$

where we have for a one-dimensional radiator

$$I_{max}=2\pi(n\omega d)^2/c,\quad \tau_c=\hbar c/16\pi n\omega d^2,\quad t_0=\tau_c\ln(\beta_{1,2}^{-1}),$$

and for a two-dimensional one

$$I_{max}=\pi cs^2k^3d^2/2,\quad \tau_c=\hbar c/2\pi ks\omega d^2.$$

We note that the radiation intensity is referred to a unit length of the two-dimensional radiation and to unit area of the surface of the one-dimensional one. For a three-dimensional radiator $\beta=1$ and the decay is exponential. Spontaneous decay of a plane was considered in Ref. 14 in a semiclassical approximation without allowance for atomic and field fluctuations, so that no criterion was given for the collective character of the decay. The problem of decay of a two-dimensional radiator (filament) has apparently not been considered before.

We examine now a system of two-dimensional radiators (filaments parallel to one another), retaining for the Bloch angles θ and φ the notation adopted in Ref. 10. In the presence of an external resonant field $E(t) = \mathscr{E} e^{-i\omega t}$ the behavior of the two-level system is described by the equation[10]

$$\frac{1}{\cos\theta}\frac{d}{dt}(\sin\theta e^{i\varphi}) = \frac{2id\mathscr{E}}{\hbar}, \quad (7)$$

in which no account is taken of the fluctuation terms. To find the field it is necessary to take into account the fact that in the considered model the charge density is zero, therefore Maxwell's equation for the field E takes the form

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)E(x,y,t)=\frac{4\pi}{c^2}\frac{\partial j(x,y,t)}{\partial t}, \quad (8)$$

where x and y are the spatial coordinates and j is the current density. Since the current density in a resonant medium can be represented in the form

$$j(x,y,t)=J(x,y,t)e^{-i\omega t}, \qquad \partial J/\partial t \ll \omega J,$$

we obtain from (8) an expression for a slowly varying amplitude of the field $\mathscr{E}$ as a function of the current-density amplitude in the medium:

$$\mathscr{E}(\mathbf{r},t)=-\frac{\pi k}{c}\iint H_0^{(1)}(k|\mathbf{r}-\mathbf{r}'|)J(\mathbf{r}',t)\,d^2\mathbf{r}', \quad (9)$$

where $H_0^{(1)}$ is a Hankel function, and the time delay has been left out (see Ref. 10). In the case of a discrete two-dimensional system of radiators we have in place of (9)

$$\mathscr{E}(\mathbf{r}_n,t)=-\frac{\pi k}{c}\sum_m H_0^{(1)}(k|\mathbf{r}_n-\mathbf{r}_m|)I_0(\mathbf{r}_m,t), \quad (10)$$

where I_0 is the current amplitude defined in analogy with Ref. 10:

$$I_0=se^{i\omega t}\frac{\partial d(t)}{\partial t}=\frac{\omega sd}{2i}\sin\theta e^{i\varphi}. \quad (11)$$

We note, however, that expression (10) is incorrect because $H_0^{(1)}(0) = -i\infty$. This singularity is due to the infinite self-action of a point source of radiation. To obtain the correct value of the self-action force by Dirac's method[11] it is necessary to determine the advanced solution of Eq. (8)

$$\mathscr{E}_{adv}(r,t)=(\pi k/c)H_0^{(2)}(kr)I_0(t) \quad (12)$$

and calculate next the correct value for the self-action field $\mathscr{E}_s$:

$$\mathscr{E}_s=\frac{1}{2}\lim_{r\to 0}(\mathscr{E}-\mathscr{E}_{adv})=-\frac{\pi k}{c}J_0(0)I_0=-\frac{\pi k}{c}I_0, \quad (13)$$

where J_0 is a Bessel function, $J_0(0) = 1$. Substituting further the relation (11) in (10) and next (10) in (7), as well as taking (13) into account, we obtain a system of equations that describe the evolution of an ensemble of radiators:

$$\frac{1}{\cos\theta_n}\frac{d}{d\tau}(\sin\theta_n\exp(i\varphi_n))$$

$$=-{\sum_m}' H_0^{(1)}(k|\mathbf{r}_m-\mathbf{r}_n|)\sin\theta_m\exp(i\varphi_m), \quad (14)$$

where we have introduced the dimensionless time $\tau = t(\pi d^2k^2s/\hbar)$, and the prime on the summation sign denotes that $H_0^{(1)}$ is taken to be unity at $m = n$. In the system (14), account is taken only of the classical parts of the radiation field. To take into account the fluctuation parts, it is necessary to add to the left-hand side of (14) the term

$$(2\beta_2/\sin\theta_n)(1-\cos\theta_n-{}^1/_2\sin^2\theta_n)$$

and remember that the quantum fluctuations of the dipole moment cause phase fluctuations

$$|\Delta\varphi_n|\approx\mathrm{arctg}\,(2/(\lambda s)^{1/2}\sin\theta_n)$$

about a mean value φ_n that obeys Eq. (14). To conclude this section we note that on going over to the continual limit, Eq. (14) can be represented in differential form:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k^2\right)\left[\frac{1}{\cos\theta}\frac{\partial}{\partial\tau}(\sin\theta e^{i\varphi})\right]=-4ik^2\sin\theta e^{i\varphi}, \quad (15)$$

which may turn out to be more convenient when approximate calculation methods are developed.

2. SPONTANEOUS DECAY OF A SYSTEM OF RADIATORS WITH DIMENSION $R \ll \lambda$

The system (14) can be investigated analytically in the case $R \ll \lambda$. To this end it is convenient to introduce new variables W_n and P_n with the aid of the following relations:

$$P_n\equiv\sin\theta_n\exp(i\varphi_n), \qquad W_n\equiv-\cos\theta_n, \quad (16)$$

and represent the system (14) in a universal form that does not depend on the dimensionality of space:

$$dP_n/d\tau=W_n\sum_k G_{nk}P_k, \qquad dW_n/d\tau=-\mathrm{Re}\left(P_n^*\sum_k G_{nk}P_k\right), \quad (17)$$

where G_{nk} is an element of the interaction matrix defined by the relations

$$G_{\substack{nk\\n\neq k}}=\begin{cases}\exp(ik|x_n-x_k|), & \tau=t\dfrac{2\pi nkd^2}{\hbar}, \quad D=1\\ H_0^{(1)}(k|\mathbf{r}_n-\mathbf{r}_k|), & \tau=t\dfrac{\pi d^2k^2s}{\hbar}, \quad D=2,\\ \dfrac{\exp(ik|\mathbf{r}_n-\mathbf{r}_k|)}{ik|\mathbf{r}_n-\mathbf{r}_k|}, & \tau=t\dfrac{4d^2k^3}{3\hbar}, \quad D=3\end{cases} \quad (18)$$
$$G_{nn}=1,$$

where D is the dimensionality of space; scalar interaction is considered at $D = 3$. At $R \ll \lambda$ we have the asymptotic expressions

$$G_{\substack{nk\\n\neq k}}=\begin{cases}1, & D=1\\ 1+(2i/\pi)\ln(k|\mathbf{r}_n-\mathbf{r}_k|), & D=2,\\ 1-i/k|\mathbf{r}_n-\mathbf{r}_k|, & D=3\end{cases} \quad (19)$$
$$G_{nn}=1.$$

The principal quantity that characterizes the collective state of an ensemble of radiators is the collective Bloch vector $\mathbf{S}$ whose components in energy-spin space are defined in the following manner:

$$\mathbf{S}=\left\{\mathrm{Re}\sum_i P_i,\ \mathrm{Im}\sum_i P_i,\ \sum_i W_i\right\}. \quad (20)$$

For the absolute value of the vector $\mathbf{S}$ we get from (17) the equation

$$\frac{d|\mathbf{S}|^2}{d\tau}=2\sum_{i,j,k}\operatorname{Im}G_{ij}[W_j\operatorname{Im}(P_i^*P_k)+W_k\operatorname{Im}(P_i^*P_j)].$$

We see therefore that in the multidimensional case, in contrast to the one-dimensional case, the absolute value of the vector $\mathbf{S}$ is generally speaking not conserved, since $\operatorname{Im}G_{ij}\neq 0$ (moreover, $|\operatorname{Im}\ G_{ij}|\to\infty$), so that the possibility of superradiance in multidimensional systems is not so obvious.

We consider first a very simple system consisting of two radiators and introduce, besides the collective vector $\mathbf{S}=\mathbf{S}_1+\mathbf{S}_2$, the vector $\boldsymbol{\Delta}=\mathbf{S}_2-\mathbf{S}_1$. For the vector $\boldsymbol{\Delta}$ we get from (17) the equation

$$\frac{d\boldsymbol{\Delta}}{d\tau}=[\boldsymbol{\Delta},[\mathbf{S}\mathbf{e}_z]]+\frac{\operatorname{Im}G}{2}S_z[\mathbf{e}_z\boldsymbol{\Delta}]+\operatorname{Im}G[\boldsymbol{\Delta}\mathbf{S}]-\frac{\operatorname{Im}G}{2}\Delta_z[\mathbf{e}_z\mathbf{S}],$$
$$G\equiv G_{12}. \qquad (21)$$

From the definition of the vectors $\mathbf{S}$ and $\boldsymbol{\Delta}$ it follows that the question of conservation of the modulus of the vector $\mathbf{S}$ reduces to the question of the conservation of the modulus of the vector $\boldsymbol{\Delta}$. The change of the length of the vector $\boldsymbol{\Delta}$ is due only to the presence of the last term in the right-hand side of (21). The first term in the right-hand side of (21) describes the change of the orientation of the vector $\boldsymbol{\Delta}$ when the joint system of vectors $\mathbf{S}_1$ and $\mathbf{S}_2$ is rotated as a unit in the vertical plane around the vector $\mathbf{S}\times\mathbf{e}_z$. This rotation corresponds to relaxation of the system to the ground state. In the one-dimensional case this plane is immobile, and in the multidimensional case, however, it rotates around a vertical axis with angular velocity $\Omega=(1/2)\operatorname{Im}GS_z$, and the two vectors $\mathbf{S}_1$ and $\mathbf{S}_2$ rotate with it. The change of the orientation of the vector $\boldsymbol{\Delta}$ in this rotation is described by the second term in the right-hand side of (21). We note also that this rotation describes a frequency shift amounting to $\Delta\omega=-1/2S_z\operatorname{Im}G$. Finally, the remaining two terms in the right-hand side of (21) describe the rapid rotation of the vector $\boldsymbol{\Delta}$ in a plane perpendicular to the vector $\mathbf{S}$. An analysis of this motion shows that the end point of the vector $\boldsymbol{\Delta}$ traces an ellipse with semi-axes $\boldsymbol{\Delta}_\varphi$ and $\Delta_\theta=\Delta_\varphi(1-1/2\sin\theta)^{-1/2}$ in a plane perpendicular to the vector $\mathbf{S}$. Thus, in a system of two radiators the lengths of the vector $\mathbf{S}$ oscillates rapidly about a certain mean value with oscillation amplitude $\propto|\Delta|^2$, and the radiators remain in phase (the decay is coherent) if $|\boldsymbol{\Delta}|\ll 1$ at the initial instant. In terms of stability theory it can be stated that the state with $\boldsymbol{\Delta}=1$ is an equilibrium state of the "center" type.

We consider next a system of three radiators. In such a system, in contrast to the preceding one, there can be several unequal off-diagonal elements of the matrix G, and this gives rise to new peculiarities in the behavior of the system. We choose the simplest type of such a system, when there are two different off-diagonal elements of this matrix: $G_1\equiv G_{12}=G_{23}$ and $G_2\equiv G_{12}$. Under this condition the system is symmetrical with respect to interchange of the first and third radiators. In view of the corresponding symmetry of Eqs. (17) it can be assumed in the simplest case that the states of these two radiators are always the same, i.e., $\mathbf{S}_1(t)=\mathbf{S}_2(t)$. Equation (17) leads then to the system

$$dP_1/d\tau=W_1[(1+G_2)P_1+G_1P_2],\quad dP_2/d\tau=W_2[P_2+2G_1P_1],$$
$$dW_1/d\tau=-2|P_1|^2-\operatorname{Re}(G_1P_1^*P_2), \qquad (22)$$
$$dW_2/d\tau=-|P_2|^2-2\operatorname{Re}(G_1P_1P_2^*).$$

From this we get an equation that describes the evolution of the vector $\boldsymbol{\Delta}$:

$$d\boldsymbol{\Delta}/d\tau=[\boldsymbol{\Delta},[\mathbf{S}\mathbf{e}_z]]+2W_1\operatorname{Im}G_1[\mathbf{e}_z\boldsymbol{\Delta}]+3\operatorname{Im}G_1[\boldsymbol{\Delta}\mathbf{S}_1]$$
$$+\Delta_z\operatorname{Im}G_1[\mathbf{S}_1\mathbf{e}_z]+W_1\operatorname{Im}(G_2-G_1)[\mathbf{S}_1\mathbf{e}_z]. \qquad (23)$$

The first four terms in the right-hand side are similar here to the corresponding terms in (21), while the last term is due to the asymmetry in the arrangement of the radiators; in contrast to the first four, this term does not depend on $\boldsymbol{\Delta}$. Since the first two terms describe the rotation of the entire system of vectors as a unit, we shall consider, just as in the case of two radiators, the remaining terms separately. Assuming that the vector $\mathbf{S}_1$ does not change over a time $\sim(\operatorname{Im}G_1)^{-1}$ (this assumption is generally speaking incorrect, but it makes it possible to reveal in simpler fashion the qualitative result), we obtain from (23) equations for the components of the vector $\boldsymbol{\Delta}$ in a plane perpendicular to the vector S_1:

$$\frac{d\Delta_\varphi}{d\tau'}=\Delta_\theta\left(1-\frac{\sin^2\theta}{3}\right)+\frac{\varepsilon}{2}\sin 2\theta-(\Delta_\varphi^2+\Delta_\theta^2)\frac{\sin 2\theta}{12},$$
$$\frac{d\Delta_\theta}{d\tau'}=-\Delta_\varphi,\qquad \tau'\equiv\tau\cdot 3\operatorname{Im}G_1,\qquad \varepsilon=\frac{\operatorname{Im}(G_2-G_1)}{3\operatorname{Im}G_1}. \qquad (24)$$

The nonlinear term in the first equation is small and will here-after be left out. From (24) we obtain the equation for the trajectory of the vector $\boldsymbol{\Delta}$ in a plane perpendicular to the vector $\mathbf{S}_1$:

$$\Delta_\varphi^2+\left[\Delta_\theta\left[1-\frac{\sin^2\theta}{3}\right]^{1/2}+\frac{\varepsilon}{2}\frac{\sin 2\theta}{[1-{}^1/_3\sin^2\theta]^{1/2}}\right]^2=\text{const.} \qquad (25)$$

Thus, the vector $\boldsymbol{\Delta}$ rotates with a frequency $\sim 3\operatorname{Im}G_1$ along an ellipse whose center is shifted relative to the point $\Delta_\varphi=0$, $\Delta_\theta=0$ by an amount $\sim\varepsilon$, which indicates the degree of dephasing of the radiators. It is of interest to estimate the value of ε for different forms of the interaction function G_{nk}. Thus, in the two-dimensional model, according to (19), we have

$$\varepsilon=\frac{\ln(k|\mathbf{r}_1-\mathbf{r}_3|)-\ln(k|\mathbf{r}_1-\mathbf{r}_2|)}{3\ln(k|\mathbf{r}_1-\mathbf{r}_2|)}=\frac{\ln[|\mathbf{r}_1-\mathbf{r}_3|/|\mathbf{r}_1-\mathbf{r}_2|]}{\ln(k|\mathbf{r}_1-\mathbf{r}_2|)}.$$

In a self-similar decrease of the geometric dimensions of the system of radiators we have $\varepsilon\to 0$ and the degree of dephasing turns out to be insignificant. We note that this result remains in force also for an arbitrary arrangement of an arbitrary number of radiators in a two-dimensional concentrated system. In the three-dimensional case, if the function G_{nk} is chosen in the form (18) (this choice corresponds to allowance for only the radiative part of the electromagnetic interaction)

$$\varepsilon=\frac{1}{3}\left(\frac{|\mathbf{r}_1-\mathbf{r}_3|^{-1}-|\mathbf{r}_1-\mathbf{r}_2|^{-1}}{|\mathbf{r}_1-\mathbf{r}_2|^{-1}}\right)=\frac{1}{3}\left(\frac{|\mathbf{r}_1-\mathbf{r}_3|}{|\mathbf{r}_1-\mathbf{r}_2|}-1\right)$$

and at sufficient asymmetry of the system of radiators, ε can be much larger than unity. It must be borne in mind, however that at $R\ll\lambda$ an important role is assumed in the three-dimensional case by the dipole-dipole interaction of the radiators. Taking this interaction into account, the function G for parallel dipoles takes the form[15,19]

$$G(x)=\frac{3}{2}\left\{\left[\left(\frac{1}{x^3}-\frac{1}{x}\right)\sin x-\frac{\cos x}{x^2}\right]-i\left[\left(\frac{1}{x^3}-\frac{1}{x}\right)\cos x+\frac{\sin x}{x^2}\right]\right\}, \quad x\equiv kr. \tag{26}$$

At other types of mutual orientation of the dipoles, superradiance is impossible; it is not clear at present, however, how a concentrated system of excited radiators goes over into a state with parallel orientation of the currents. Using (26), we obtain

$$\varepsilon=\frac{1}{3}\left(\left|\frac{\mathbf{r}_1-\mathbf{r}_2}{\mathbf{r}_1-\mathbf{r}_3}\right|^3-1\right).$$

Consequently, when account is taken of the dipole-dipole interaction, the parameter ε is even more sensitive to the degree of asymmetry of the system of radiators.

In the case of a concentrated system containing N radiators, the mathematical expression for the symmetry condition consists in the fact that the quantity $\Sigma_i G_{in}$ does not depend on the index n. Only under this condition does the system of equations (17) have a solution with identical states of all the radiators, and the character of the stability of this solution becomes clear from the foregoing analysis of the simplest cases. At $P_i=P$ and $W_i=W$, the system becomes substantially simpler:

$$dP/d\tau=W[N+i(N-1)\operatorname{Im}G]P,$$

$$\frac{dW}{d\tau}=-N|P|^2, \quad \operatorname{Im}G\equiv\frac{1}{N-1}\operatorname{Im}\sum_{i\neq k}G_{ik}. \tag{27}$$

The solution of this system is of the form

$$W=-\operatorname{th}[N(\tau-\tau_0)], \quad |P|^2=\operatorname{sech}^2[N(\tau-\tau_0)],$$

and the radition intensity is determined by Eq. (6). At the same time, the radiation frequency is shifted relative to the fundamental frequency ω by an amount

$$\Delta\omega=-(\tau/t)(N-1)\operatorname{Im}GW. \tag{28}$$

In the two-dimensional case we have

$$\Delta\omega=-(2d^2k^2s/\hbar)(N-1)W\ln(kR), \tag{29}$$

where R is the dimension of the region occupied by the radiators; in the three-dimensional case,

$$\Delta\omega=(2d^2/\hbar R^3)(N-1)W. \tag{30}$$

We note that if the function G is chosen in the form (18) for $D=3$, the expression for the frequency shifts coincides with the expression obtained in Ref. 12, in second-order perturbation theory, for a quantum description of the radiation field neglecting the dipole-dipole interaction.

3. CASE OF CONTINUOUS MEDIUM

We compare first of all the considered method with the one proposed earlier[6–8,13] for the analysis of extended media. To this end, we write down the equation for the integrated characteristic of the system (the total energy $\Sigma_i W_i$) an equation that follows from (17):

$$\frac{d}{d\tau}\left(\sum_i W_i\right)=-\sum_{j,k}\operatorname{Re}G_{jk}(P_j^*P_k+P_jP_k^*). \tag{31}$$

If account is taken of only the radiating part in the interaction function, and the vector character of the radiation field is neglected, then Eq. (31), with the spontaneous processes taken into account, coincides with the corresponding equations of the preceding method, since

$$\operatorname{Re}G_{ij}=\sin(kr_{ij})/kr_{ij}.$$

A matrix of this type was considered in Ref. 16, and in greater detail in Ref. 7 (in a two-dimensional system, $\operatorname{Re}G_{ij}=J_0(kr_{ij})$). The use of the matrix $\{\operatorname{Re}G_{ij}\}$ is permissible under the condition that the direction of the polarization of all the radiators is the same and is perpendicular to the vectors $\mathbf{r}_{ij}=\mathbf{r}_i-\mathbf{r}_j$, but these requirements can be satisfied only in the two-dimensional model. In addition, the transition from (17) to (31) entails the loss of information on the system, e.g., information connected with the frequency shift (28). At the same time, to solve equations such as (31) it is necessary to know the time dependence of the quantities $\langle P_i^*P_j\rangle$. Therefore, when investigating the relaxation of a multidimensional system and the anisotropy of the emission, use is made of various assumptions that facilitate the solution of the problem, namely: it is assumed that at $R>c\tau_c$ the relaxation process is homogeneous over the entire sample[2] and that the polarization phases φ_i of the different radiators are constant in time over the extent of the entire process.[13] It was shown in Ref. 10 that both these conditions are violated even at $R\sim\lambda/2\ll c\tau_c$. We shall show now that in the multidimensional case this leads to a number of singularities in the anisotropy of the radiation.

Figure 1 shows the results of a numerical solution of the problem of relaxation of a two-dimensional continuous sample ($r_{ij}\ll\lambda$) of length $L=3\lambda$ and width $b=0.6\lambda$, excited by a coherent resonant pumping pulse directed along the sample. The dimensions of such a system are close to the dimensions of the sample investigated in Ref. 17. To simplify the numerical calculation, the quantum fluctuations of the dipoles and of the radiation field were left out, and for the initial angle θ_0 we chose the same value 0.1 for all the radiators. Since in the absence of inhomogeneous broadening the initial angle θ_0 is connected with the density of the radiators by the approximate relation

$$\theta_0\sim2/(\rho\lambda^2L)^{1/2} \quad (I_{\text{fl}}\sim I_{\text{cl}}),$$

it follows that $\theta_0=0.1$ the dimensionless density of the radiators is $\rho\lambda^3\approx2\times10^2$. It must be indicated that the initial angle θ_0 is different for radiators located in different places in the active sample, since the classical part of the radiation is not the same at different points of the sample (e.g., owing to the anisotropy of the radiation). For the considered sample the Fresnel number is $F=b^2\lambda L=0.12\ll1$, so that the one-dimensional approximation is not valid here; nevertheless, one can see distinctly the residual excitation previously observed[10] in the sample in the one-dimensional approximation. In a two-dimensional sample this excitation dissipates

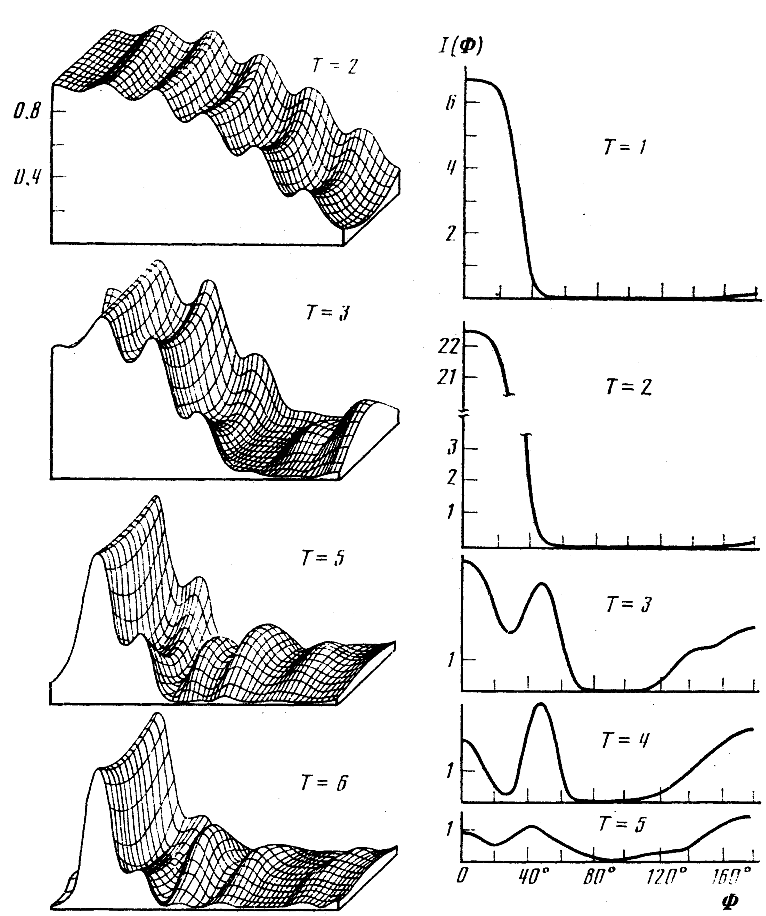

FIG. 1. Relaxation of an active sample with dimension $3\times 0.6\,\lambda^2$, excited by a resonant pump pulse directed along the sample. On the left is shown the temporal dynamics of the inverted population, and on the right the angular distribution of the intensity of the radiation at various instants of time. The angle Φ is measured from the longitudinal axis of the sample, along which the exciting pulse is directed; $T = t(\pi d^2 k^2 \rho L b/10\hbar)$.

more rapidly because of the possibility of emission in lateral directions. We call attention to the presence of a "shallow" structure of the inversion density. The formation of this structure takes place during the initial stage of the collective relaxation and can be investigated with the aid of the linearized equation (17), by putting $W_i = 1$. The spatial period of the "shallow" structure is equal to $\lambda/2$. The nonlinear model predicts unusual dynamics of the anisotropy of the emission. Thus, whereas for homogeneous inversion and at constant phases of the radiator currents $\varphi(\mathbf{r}, t) = kx$ the dependence of the intensity of the radiation on the angle in a two-dimensional sample is of the form

$$I(\Phi, t) \propto \frac{\sin^2[{}^1/_2 kL(1-\cos\Phi)]}{k^2L^2(1-\cos\Phi)^2} \cdot \frac{\sin^2({}^1/_2 kb\sin\Phi)}{k^2b^2\sin^2\Phi}\,\mathrm{sech}^2\left(\frac{t-t_0}{2\tau_c}\right), \qquad (32)$$

where Φ is the angle between the longitudinal axis of the sample and the radiation direction, the nonlinear model predicts that the directions of the lateral maxima are not described by Eq. (32), the maximum intensity in the lateral beam is reached considerably later than the maximum of intensity in the principal direction. The ratio of the maximum values of the intensities in the lateral and in the principal directions likewise does not agree with Eq. (32); according to (32) we have

$$I_1/I_0 \leqslant \sin^2(3\pi/2)/(3\pi/2)^2 \approx 0.05,$$

whereas it follows from Fig. 1 that $I_1/I_0 \approx 0.17$.

Figure 2 illustrates the collective emission from a needlelike sample with dimensions $L = 50\lambda$, $b = 1.75\lambda$, $\theta_0 = 0.1$, corresponding to $\rho\lambda^3 \approx 20$. The inversion longitudinal inhomogeneity that occurs here is reminiscent of the inversion inhomogeneity which takes place in the one-dimensional approximation[10] and accompanies the oscillations of the radiation intensity. The character of the radiation anisotropy is similar to that obtained in the preceding case: one observes the most powerful pulse in the longitudinal direction, and lateral pulses in which the intensity maximum is reached later than the maximum of the main beam.

Figure 3 shows the relaxation of an excited sample with parameters $L = 25\lambda$, $b = 1.75\lambda$, $\theta_0 = 0.005$, corresponding to $\rho\lambda^3 \sim 10^4$. The most powerful radiation pulse is observed here in a direction opposite to the dirction of the pump pulse. In the forward direction, however, the radiation increases more slowly and turns out to be less powerful. Such an unusual behavior of the system is due to the strong interaction of two opposing coherent-radiation waves. This interaction recalls the interaction of two coupled pendulums, and in the case of a homogeneous state of the medium it depends on the density and dimensions of the active sample. This phenomenon was analyzed in greater detail in a separate paper.[18]

In all the examples considered, the initial values of the phases were chosen in the form $\varphi(\mathbf{r}, t=0) = kx$, corresponding to coherent resonant pumping. No less interesting is the case when the initial phases are specified by a random function. In this case, as shown by analysis, we have a relatively

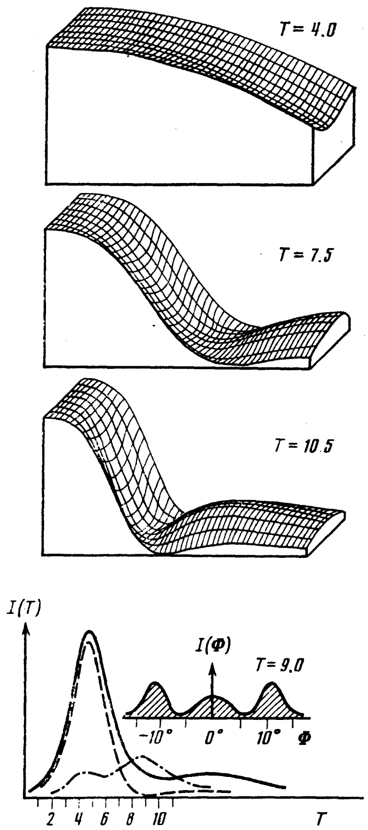

FIG. 2. Relaxation of sample measuring $50 \times 1.75\ \lambda^2$. Below—temporal change of the intensity of the radiation I in various directions and the angular distribution of the intensity at a fixed instant of time; solid curve—$\Phi = 0°$, dashed—$\Phi = 6°$, dash-dot—$\Phi = 12°$.

(in comparison with the relaxation times) phasing of radiators with an initially weak radiation wave propagating from the geometrical center of the sample to its boundary. As a result, the amplitude of this wave increases considerably in the directions of the elongations of the sample, and the radiators phased by this wave have phases $\varphi(\mathbf{r}) = \mathbf{k}\cdot\mathbf{r}$, where $\mathbf{k}$ is the wave vector of the radiation field. Consequently, the system assumes a state with coherent conditions. The emission is here symmetrical with respect to the geometric center of the sample, and the dynamics of the radiation anisotropy is characterized by the pecularities noted above (assuming that the active sample is elongated in shape).

4. CASE OF TWO-DIMENSIONAL PERIODIC STRUCTURE

As shown in Ref. 10, the rate of emission from a one-dimensional periodic lattice depends on the relation between the period of the lattice and the wavelength of the radiation field, so that it is of interest to consider a two-dimensional lattice with nonquadratic unit cell. The most important case here is the one in which the Bragg diffraction condition with a Bragg angle $\theta_B = \pi/2$ is satisfied in one of the directions. In accordance with Ref. 10 one should expect a strong anisotropy of the radiation with an intensity maximum in the Bragg-diffraction direction. Figure 4 shows the temporal relaxation of the inverted population in a lattice of this type. At random initial phases of the radiators, the radiation does indeed turn out to be anisotropic with a maximum in the Bragg direction, while the inversion in the relaxation re-

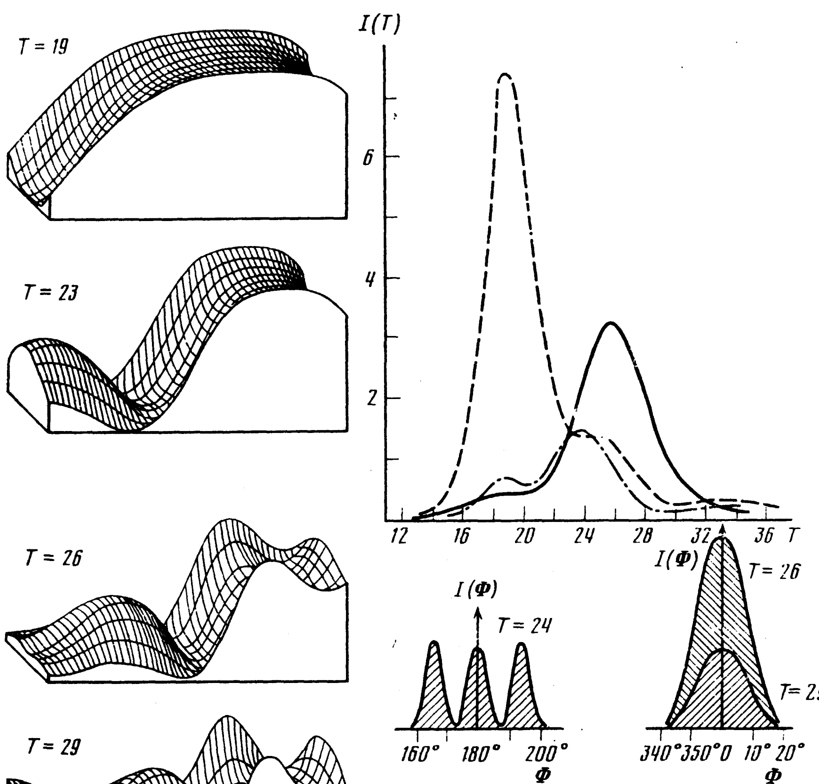

FIG. 3. Relaxation of sample measuring $25 \times 1.75\ \lambda^2$. On the right is shown the temporal variation of the intensity of the radiation in different directions and the angular structure of the radiation at various instants of time; solid curve—$\Phi = 0°$, dashed—$\Phi = 180°$, dash-dot—$\Phi = 180° \pm 14°$.

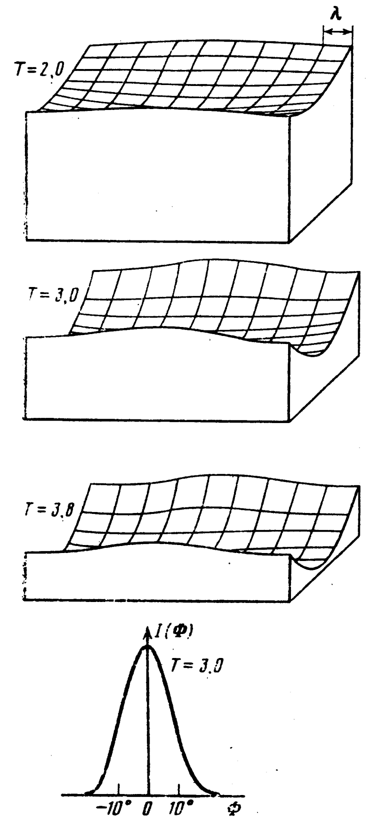

FIG. 4. Collective relaxation of a two-dimensional lattice consisting of 100 radiators. The periods of the lattice in mutually perpendicular directions are equal to λ and $\lambda/4$. Shown below is the angular dependence of the radiation intensity. The angle Φ is reckoned from the Bragg-diffraction direction; $T = t(\pi d^2 k^2 S/10\hbar)$. Analogous results were obtained for a different choice of lattice periods under the condition that in one direction the Bragg diffraction relation with angle $\theta_B = \pi/2$ is satisfied.

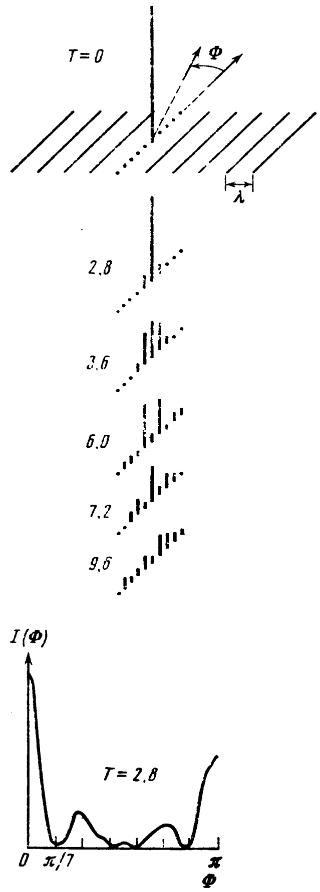

FIG. 5. Spontaneous decay of one excited radiator in the same lattice (λ, $\lambda/4$) as in Fig. 4. The dynamics of the excitation of the radiators in the Bragg plane containing the excited radiator is shown. Below is represented the angular distribution of the intensity of the radiation. The remaining radiators located in other planes are much less excited, but their fields are in phase, and this is in fact the cause of the "screening" of the radiation.

mains weakly inhomogeneous. Figure 5 shows the spontaneous decay of an excited radiator in a lattice consisting of analogous unexcited radiators. In full accordance with the assumptions made in Ref. 10, the radiation that leaves such a structure is anisotropic: the Bragg planes "screen" the spontaneous emission in the Bragg diffraction direction, and the transport of the radiation and excitation is effected predominantly along that Bragg plane on which the excited radiator is located. One can note here the phenomenon of the return of the excitation to the radiator, when the inversion of the radiator executes damped oscillations. The absorption will influence strongly the angular distribution of the radiation and quench it in those directions where the diffraction condition is not satisfied. As a result, Kossel lines will be observed near the Bragg angles (see also Ref. 19).

In connection with the problem of one excited radiator, interest attaches to the case when all the radiators located in a single Bragg plane are initially excited. The radiation is then clamped by the remaining Bragg planes into a narrow channel because of the aforementioned screening, and will be strongly anisotropic.

5. CONCLUSION

Let us make a few remarks concerning the proposed model.

1. From the point of view of the semiclassical description, collective decay proceeds as follows: during the initial stage, each radiator emits independently because of its phenomenologically considered individual quantum fluctuations of the dipole moment and of the field. Next, against the background of the fluctuations, a polarization P appears and is the source of the classical field, the initial phase φ of this polarization being random. Under the influence of the classical resonant fields of the other radiator, the phase of each radiator "floats." As a result of this process there can occur in a concentrated system a state in which the phases of all the radiators are equal; the total polarization is then proportional to N and the radiation intensity is proportional to N^2.

2. To establish such a state in a three-dimensional concentrated system ($R \ll \lambda$) it is necessary to have complete symmetry in the arrangement of the radiators. Since the satisfaction of the last condition in the case of a gaseous medium is problematic, the reasons why the effect cannot be observed in a concentrated system becomes understandable. The symmetry condition is not required in a two-dimensional system, and all the more in a one-dimensional system.

3. An experimental investigation of the dynamics of the

angular anisotropy of collective emission is the most accessible method of checking the proposed model.

4. What is paradoxical is the existence of resonant fields in the spatial region of size smaller than the wavelength of the radiation field. This at first glance contradictory property of the field is explained by the fact that in a resonant medium the connection between the frequency ω and the wave vector k is not the usual relation $\omega \neq ck$, as in vacuum; at the same time, because of the nonlinearity of the model there is no singled valued connection whatever between ω and k.

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