Conditions for stable equilibrium of dielectric spheres in an electrostatic field

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(Submitted 25 March 1982)
Zh. Eksp. Teor. Fiz. **83**, 2170–2175 (December 1982)

It is known [L. D. Landau and E. M. Lifshitz, Electrodynamics of Continuous Media, Pergamon, 1959] a probe charge cannot be in equilibrium in an electrostatic field (the Earnshaw theorem), since the potential φ as a harmonic function can have a maximum or a minimum only at the boundaries of the region. Therefore, for example, all the free charges are gathered just on the surface of a charged conductor. Polarization charges are produced on the surface of a dielectric placed in an external field. Without a preliminary investigation, on the basis of the Earnshaw theorem alone, it is impossible to state whether such a system of surface charges will be stable, since these charges are at the boundary of the harmonicity region of the potential φ. If this system of charges turns out to be stable, the body on which it is located will also be kept by the electrostatic forces in a fixed position. In this paper we study theoretically and experimentally the conditions for levitation of dielectric spheres in electrostatic and gravitational fields. The results of the theory and of the experiment are quantitatively compared.

PACS numbers: 41.10.Dq

1. THEORY

We consider the equilibrium of a dielectric sphere in an electrostatic field and a gravitational field. We place a sphere of radius R_0 (dielectric constant ε_i in a liquid dielectric (dielectric constant ϵ_e) near a round opening of radius a drilled in a horizontal ($z=0$) grounded metallic plate. In external homogeneous field E_∞ is specified in the upper and lower half-spaces at infinity ($z\to\pm\infty$).

Under these conditions, in a uniform gravitational field $\mathbf{g}=-g\mathbf{k}$ ($\mathbf{k}$ is the unit vector of the polar axis z) the dielectric sphere is acted upon by a gravitational force $\mathbf{F}_1$ and by an electric force $\mathbf{F}_2$ (Ref. 1):

$$\mathbf{F}_1={}^4/_3\pi R_0^3(\rho_e-\rho_i)\mathbf{g},$$
$$\mathbf{F}_2=\frac{\varepsilon_e}{4\pi}\oint\left[\mathbf{E}(\mathbf{nE})-\frac{E^2\mathbf{n}}{2}\right]df. \tag{1.1}$$

Here ρ_i and ρ_e are the densities of the sphere and of the outer medium, and $\mathbf{n}$ is the unit vector of the outward normal to the surface element df of the sphere.

At certain values of the parameters, these two forces can cancel each other and the sphere can be at equilibrium. Let us find these equilibrium conditions. To this end we determine the field $\mathbf{E}=-\nabla\varphi$.

We choose a spherical coordinate system (r, ν, φ) with a center at the midpoint of the opening. The angle ζ is measured from the polar axis z, and the plane $z=0$ coincides with the grounded plate. The equation of the surface of the sphere, displaced a small distance from the origin, is specified by the equation $r=R(\zeta,\varphi)$. In particular, if the center of the sphere and the origin coincide, then $R\equiv R_0$. The potentials φ_e in the medium and φ_i in the sphere satisfy the following system of equations:

$$\Delta\varphi_i=0,\qquad \Delta\varphi_e=0;$$
$$z\to\pm\infty:\varphi_e=-E_\infty|z|;$$
$$z=0,\quad r>a:\varphi_e=0;\quad R<r<a:\partial\varphi_e/\partial z=0; \tag{1.2}$$
$$r=R:\varphi_e=\varphi_i,\quad \varepsilon_e\partial\varphi_e/\partial n=\varepsilon_i\partial\varphi_i/\partial n.$$

The problem (1.2) we solved by the method of joinable asymptotic expansions.[2] To this end we consider the case of small spheres $R_0\ll a$. Then at distances that are large compared with the radius R_0 of the sphere but are comparable with the size a of the opening, the influence of the sphere is insignificant and its presence can be neglected. The distribution of the potential φ_0 in this approximation satisfies the mixed (Dirichlet and Neumann) boundary conditions on the surface $z=0$, and the remaining equations are the same as in (1.2).

$$\Delta\varphi_0=0;\quad z\to\pm\infty:\varphi_0=-E_\infty|z|;$$
$$z=0,\quad r>a:\varphi_0=0;\quad r<a:\partial\varphi_0/\partial z=0. \tag{1.3}$$

The solution of the problem (1.3) is regarded as the boundary condition "at infinity" for the problem of the distribution of the potentials φ_i and φ_e near the surface of the sphere $r=R$ ($r\ll a$, $r\sim R_0$):

$$\Delta\varphi_i=0;\quad \Delta\varphi_e=0;$$
$$r\to\infty:\varphi_e=\varphi_0; \tag{1.4}$$
$$r=R(\vartheta,\varphi):\varphi_e=\varphi_i,\quad \varepsilon_e\partial\varphi_e/\partial n=\varepsilon_i\partial\varphi_i/\partial n.$$

An exact solution of the system of equations (1.3) can be expressed in terms of the Weber-Schafheitlin integrals[3,4]:

$$\varphi_0=\frac{2E_\infty}{\pi}r|\cos\vartheta|\times\arcsin\frac{2a}{(r^2+a^2+2ar\sin\vartheta)^{1/2}+(r^2+a^2-2ar\sin\vartheta)^{1/2}}$$
$$-\frac{2E_\infty}{\pi}\left\{\frac{a^2-r^2+((a^2-r^2)^2+4a^2r^2\cos^2\vartheta)^{1/2}}{2}\right\}^{1/2} \tag{1.5}$$
$$-E_\infty r|\cos\vartheta|.$$

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The function φ_0 has a branch point $(r = a, \zeta = \pi/2)$. The choice of the required branch of the solution must be subjected to requirements that follow from the boundary conditions for φ_0 at $z = 0$. We expand (1.5) in powers of r:

$$\frac{\pi\varphi_0}{2E_\infty a} = -1 - \frac{r^2}{a^2}P_2 + \frac{r^4}{3a^4}P_4 + \ldots \equiv \sum_{l=0}^{\infty} Q_l r^l P_l(\cos\vartheta). \quad (1.6)$$

Here and elsewhere P_l (cos ζ) are Legendre polynomials. Using (1.6), we obtain the solution of the problem (1.4) in series form $(R \equiv R_0)$:

$$\frac{\pi\varphi_e}{2E_\infty a} = \sum_{l=0}^{\infty}\left[1 + \frac{l(\varepsilon_e - \varepsilon_i)}{(l+1)\varepsilon_e + l\varepsilon_i}\left(\frac{R_0}{r}\right)^{2l+1}\right] Q_l r^l P_l. \quad (1.7)$$

Equation (1.7) enables us to estimate the accuracy of the employed calculation method. Since it has been assumed that at distances of the order of a the influence of the sphere is negligible, the second term in the square brackets of (1.7) which takes the influence of the sphere into account, should be small compared with unity at $r \sim a$.

The force $\mathbf{F}_2$ calculated from formulas (1.1) and (1.7) is zero because of the symmetry of the problem. Thus, the center of symmetry of the problem is the sphere's "equilibrium" point in the electrostatic-force field (without the gravitational field). To ascertain whether this equilibrium position is stable, we must find the forces acting on the sphere when it is slightly displaced from the equilibrium position.

At a virtual displacement $\delta_1 \ll R_0$ of the center of the sphere along the z axis, the functions φ_i and φ_e change with changing δ_1/R_0. To estimate them we must solve the problem (1.4), in which the equation of the surface is

$$r = R(\vartheta) = R_0 + \delta_1\cos\vartheta - {}^1/_2\delta_1^2\sin^2\theta + \ldots$$

and

$$\frac{\partial}{\partial n} = \left(1 - \frac{\delta_1^2\sin^2\vartheta}{2R_0^2}\right)\frac{\partial}{\partial r} + \frac{\delta_1\sin\vartheta}{rR_0}\frac{\partial}{\partial\theta}.$$

Solving (1.4) by expanding the solution in powers of the small parameter δ_1/R_0, we can find with the aid of the obtained solution the resultant force $(F_2)_z$ exerted on the sphere by the electric field:

$$(F_2)_z = \frac{4\varepsilon_e E_\infty{}^2 a^2}{\pi^2}(\varepsilon_i - \varepsilon_e)\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\delta_1{}^n l(l+1)\ldots(l+k)$$
$$\times\frac{(l+1)(l+2)\ldots(l+n+1-k)}{(n-k)!k!}\frac{Q_{l+k}Q_{l-k+n+1}R_0^{2l+1}}{(l+1)\varepsilon_e + l\varepsilon_i}. \quad (1.8)$$

It can be seen from (1.8) that a stable state of equilibrium of the sphere corresponds to the inequality $\varepsilon_i < \varepsilon_e$.

We estimate from the first terms of the expansion (1.8)

$$(F_2)_z \sim Q_2\delta_1 + 8\delta_1^3 Q_4 + \ldots$$

the maximum distance δ_* up to which the force $(F_2)_z$ increases with increasing δ_1: $\delta_* \approx a/8$. Since the solution method is based on the assumption that $\delta_1 \ll R_0$, and also in view of the inevitable difference between the theoretical and experimental models of the setup, δ_* must be taken with an empirical adjustment coefficient k: $\delta_* = ka/\sqrt{8}$.

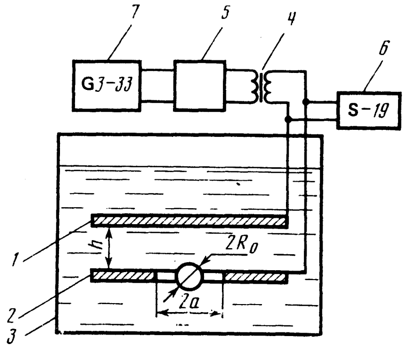

FIG. 1. Diagram of experimental setup.

Equating the Archimedean force $(F_1)_z$ from (1.11) to the electric force $(F_2)_z$ from (1.8) at $\delta_1 \leqslant \delta_*$, we can find the stable-equilibrium condition for a sphere in the electric and gravitational fields. We write down this condition, introducing in place of $\varepsilon_e E_\infty$ the experimentally measured potential difference U, divided by the distance h between the plates in the capacitor, $\varepsilon_e E_\infty = U/h$:

$$\left(\frac{U}{h}\right)^2 \geqslant \frac{g\pi^2}{k3\sqrt{2}}|\rho_e - \rho_i|\frac{\varepsilon_e(2\varepsilon_e + \varepsilon_i)}{|\varepsilon_e - \varepsilon_i|}a. \quad (1.9)$$

For a virtual displacement of the sphere by $\delta_2 \ll R_0$ along the x axis, the force $(F_2)_x$ turns out to be

$$(F_2)_x = \frac{4\varepsilon_e E_\infty{}^2 a^2}{\pi^2}(\varepsilon_i - \varepsilon_e)\sum_{l=0}^{\infty}\sum_{m=0}^{l}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\left(\frac{1}{2} + \frac{\delta_{m0}}{2}\right)\delta_2{}^n$$
$$\times\frac{l(l+1)\ldots(l+k)(l+1)(l+2)\ldots(l-k+n)}{(n-k)!k!}$$
$$\times\frac{(l+1-k+m+n)!}{(l-k-m+n)!}\frac{Q_{l+k}^{(m)}Q_{l+n+1-k}^{(m)}R_0{}^{2l+1}}{l\varepsilon_i + (l+1)\varepsilon_e}. \quad (1.10)$$

Here

$$Q_l^{(m)} = Q_l(2 - \delta_{m0})(-1)^m\frac{(l-m)!}{(l+m)!}P_l^{(m)}\left(\frac{\pi}{2}\right),$$

δ_{m0} is the Kronecker delta, and $P_l{}^m$ are generalized Legendre polynomials. It can be seen from (1.10) that a stable

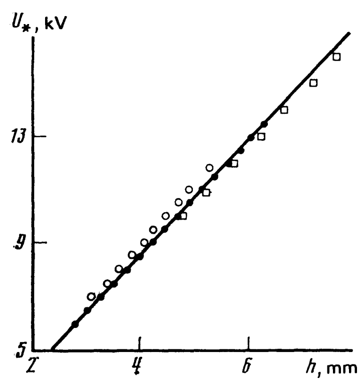

FIG. 2. Dependence of the critical field E_* on the distance h between the electrodes for bubbles of different diameters $2R_0$(● —R_0 = 1.1 mm, □—R_0 = 0.7 mm, ○—R_0 = 0.34 mm). Hole diameter 5.2 mm.

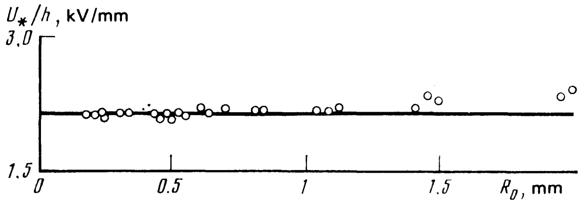

FIG. 3. Dependence of U_*/h on the bubble radius R_0. Hole diameter 5.2 mm.

position of equilibrium relative to the displacement in the plane $z = 0$ corresponds to the same inequality $\varepsilon_i < \varepsilon_e$.

Thus, the critical field $(U_*/h)^2$ determined by the equality in (1.9) does not depend on the dimensions of the sphere and increases linearly with the radius of the opening.

2. EXPERIMENTAL RESULTS

We investigated in the experiments the equilibrium conditions for air bubbles in toluene. Figure 1 shows the experimental setup. The horizontal electrodes 1 and 2 were copper plates 1 mm thick and placed in a glass vessel 3 filled with a dielectric liquid. A hole of the required radius a was drilled in the lower electrode. A potential difference 2–20 kV from a highvoltage transformer 4, which served as the load for power amplifier 5, was applied to the plates and was measured with a C-19 kilovoltmeter (6). A signal from a G 3-33 generator (7) was applied to the input of the amplifier.

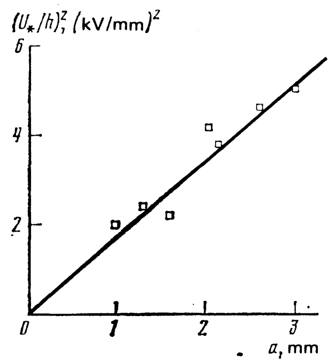

FIG. 4. Dependence of $(U_*/h)^2$ on the hole radius a.

To decrease the intensity of the electroconvective motion and alternating electric field of frequency 150 Hz was produced between the plates. In view of the small electroconductivity of toluene ($\sigma \sim 10^{-12}\,\Omega^{-1}\,\text{cm}^{-1}$) and the small dimensions of the model (~ 10 cm), this field can be regarded as electrostatic.

An air bubble blown through a glass tube was fixed by the electric and gravitational forces near the opening at a certain voltage U_* between the electrodes. In this position, a cathetometer was used to measure the bubble diameter and the diameter of the opening. While the plates were gradually moving apart, we measured the maximum distance h between the electrodes, at which the bubble was still at equilibrium. The results of these measurements are shown in Fig. 2. It can be seen from the plot that the bubble equilibrium in a hole of a specified radius can be characterized by a uniform-field intensity U_*/h.

Experiments performed with bubbles having different diameters have shown that U_*/h does not depend on their radius (Fig. 3), as follows from (1.9). This result has the following simple explanation. The bubble located in the electric field is a dipole whose moment is proportional to the radius of the bubble, and the charge induced by the field on its surface is proportional to the square of the radius. Consequently, the electric force and the Archimedes force are proportional to R_0^3.

Figure 4 shows the measured dependences of the square of the critical field $(U_*/h)^2$ on the hole dimensions, performed with bubbles of different radii.

On all the plots, the solid line was drawn in accordance with Eq. (1.9) with $k = 3.27$. Analysis of the data obtained has shown that the discrepancy between the results of the theory and the experiment is within the measurement error.

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Translated by J. G. Adashko