

Topological solitons in magnetohydrodynamics

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We use the Hopf mapping to construct a magnetic configuration consisting of closed field lines, each of which is linked with all the other ones. We obtain in this way a solution of the equations of magnetohydrodynamics of an ideal incompressible fluid with infinite conductivity, which describes a localized topological soliton.

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1. INTRODUCTION

Solutions of physical equations which have non-trivial topological properties have been studied for already more than five years. As examples we may give the "monopole"^{1,2} and the "instanton"³ in gauge field theories and the "pseudoparticle" in a two-dimensional isotropic ferromagnet.⁴ All these solutions are characterized by some topological index: the magnetic charge of the monopole and the number of pseudoparticles in the ferromagnet are equal to the degree of mapping of a two-dimensional sphere onto a two-dimensional sphere, the number of instantons is equal to the Pontryagin index of the mapping of the SU(2) group onto the three-dimensional sphere. In each case one can write this index as a volume integral of some "topological charge density". In this connection attention is drawn to the integral of motion

$$I = \int A \operatorname{rot} A d^3x \quad (1)$$

(A is the vector potential) which has been known for a long time in the magnetohydrodynamics of a perfectly conducting fluid and which is called the helicity of the magnetic field. Its topological nature is already indicated by the fact that no characteristics of the medium in which the magnetic field is present enter into (1). It has also been shown (see Refs. 6, 7) that if two field line tubes are linked the integral (1) is proportional to their linkage coefficient, i.e., the number of times which one tube is twisted around the other one.

It is thus clear that the helicity is a topological characteristic of the magnetic field. This topological nature of it is completely revealed if we note that (1) is the same as the Whitehead integral for the Hopf invariant which characterizes topologically different mappings of the three-dimensional sphere S^3 onto the two-dimensional S^2 .⁸⁻¹⁰ The topological meaning of the Hopf invariant is simple: it is equal to the linkage coefficient of the curves in S^3 which are the originals of different points of S^2 . Hence follows also a more constructive conclusion: knowing the mapping $S^3 \rightarrow S^2$ with a non-zero Hopf invariant, and the simplest such mapping was constructed by Hopf himself, we can find the vector field A corresponding to it and, then, the magnetic field $H = \operatorname{curl} A$ with non-zero helicity. The magnetic field lines of this field will be closed and each of them is linked with any other one. In the present paper we construct one such magnetic field configuration and we study its properties in magnetohydrodynamics.

2. STEREOGRAPHIC PROJECTION AND HOPF MAPPING

We establish first of all the connection between the physical space R^3 and the sphere S^3 . Equation (1) implies that the field H decreases sufficiently fast at infinity so that the helicity I is a gauge-invariant quantity: adding to A the gradient of any function does not change I as the additional term after integration by parts gives a surface term which does not contribute because H decreases rapidly, and a volume term which vanishes because $\operatorname{div} H = 0$. If the other physical conditions at infinity are also unique (say, we consider a homogeneous isotropic medium) we may assume that the Euclidean three-dimensional space R^3 is supplemented by a point at infinity. Such a "compactified" space becomes topologically equivalent to the three-dimensional sphere S^3 . If we embed S^3 in the four-dimensional Euclidean space with coordinates $(u_\mu; \mu = 1, 2, 3, 4)$ so that $S^3 = \{u_\mu; u_\mu^2 = 1\}$ we can establish the connection between R^3 and S^3 by the stereographic projection

$$x_i = u_i / (1 + u_4), \quad i = 1, 2, 3. \quad (2)$$

It is clear that the point at infinity corresponds to the "south pole" of the sphere with coordinates $(0, 0, 0, -1)$. The inverse transformation is realized by the formulae

$$u_i = 2x_i / (1 + x^2), \quad u_4 = (1 - x^2) / (1 + x^2), \quad i = 1, 2, 3, \quad (3)$$

where $x^2 = x_i^2$ is the square of the radius vector.

Let there now be in R^3 a vector field $A = (A_1, A_2, A_3)$. We find the formulae which express the connection between A and the corresponding vector field \tilde{A}_μ on S^3 , corresponds to it, where we impose on \tilde{A}_μ the condition that it be tangent to the sphere:

$$u_\mu \tilde{A}_\mu = 0. \quad (4)$$

This condition means that both A and \tilde{A}_μ also lie in the tangent spaces ("stratifications") to the appropriate configuration spaces R^3 and S^3 . We not use the condition for the invariance of the differential form¹¹

$$\omega_A = A_i dx_i = \tilde{A}_\mu du_\mu. \quad (5)$$

Since the variables u_μ are connected through the equation $u_\mu^2 = 1$ for the sphere in the two expressions on the right-hand side of Eq. (5), the number of independent differentials is equal to three. Taking as the independent variables on S^3 the first three Cartesian coordinates u_i we find from (5)

$$A_j \frac{\partial x_j}{\partial u_i} = \tilde{A}_i + \tilde{A}_4 \frac{\partial u_4}{\partial u_i}, \quad i = 1, 2, 3.$$

Substituting here (2) and using the equation $u_\mu^2 = 1$, we find three equations

$$\frac{A_i}{1+u_i} + \frac{u_i A_j}{(1+u_i)^2} \frac{u_i}{u_i} = \bar{A}_i - \frac{u_i}{u_i} \bar{A}_i, \quad i=1,2,3, \quad (6)$$

which together with (4) are sufficient to express \bar{A}_μ in terms of A and vice versa. As a result we get

$$\bar{A}_i = \frac{1}{2}(1+x^2)A_i - x_i(x_i A_j), \quad \bar{A}_i = -x_i A_j, \quad (7)$$

$$A_i = (1+u_i)\bar{A}_i - u_i \bar{A}_i, \quad (8)$$

where we must also make the coordinate substitutions (2) and (3) respectively.²⁾

We now describe how, using the known mapping $f: S^3 \rightarrow S^2$, one must construct the vector field \mathbf{A} which corresponds to it and which occurs in Eq. (1) for the Hopf invariant of this mapping. One must, as was shown by Whitehead (see Ref. 9 and also Ref. 13), start from the 2-form of the volume on the unit sphere S^2 . If that sphere is embedded in the three-dimensional Euclidean space with coordinates ξ_1, ξ_2, ξ_3 the 2-form of the volume has the form

$$\omega_2 = (4\pi)^{-1} (\xi_1 d\xi_2 \wedge d\xi_3 + \xi_2 d\xi_3 \wedge d\xi_1 + \xi_3 d\xi_1 \wedge d\xi_2) \quad (9)$$

(\wedge is the vector product sign); the coefficient is here chosen in such a way that the integral of (9) over the whole of the sphere S^2 equals unity. The mapping f induces a mapping f^* in the opposite direction from the space of forms on S^2 onto the space of forms on S^3 so that one can find the 2-form $f^*\omega_2$ on S^3 . One can show that any 2-form on S_3 can be written in the form of an external differential of some 1-form ω_1 where ω_1 is determined uniquely up to a differential of an arbitrary function. We thus can find a form ω_1 such that $f^*\omega_2 = d\omega_1$. By using a stereographic projection, we can associate with a vector field on S^3 , determined by the form ω_1 , a vector field \mathbf{A} in R^3 which we can use to evaluate the Hopf invariant through Eq. (1).

We now consider the Hopf mapping $f: S^3 \rightarrow S^2$ which has a Hopf invariant equal to unity (see Refs. 8, 13):

$$\xi_1 = 2(u_1 u_3 + u_2 u_4), \quad \xi_2 = 2(u_2 u_3 - u_1 u_4), \quad \xi_3 = u_1^2 + u_2^2 - u_3^2 - u_4^2. \quad (10)$$

Substituting this formula into (9) we find the form

$$f^*\omega_2 = \pi^{-1} (du_1 \wedge du_2 - du_3 \wedge du_4), \quad (11)$$

which is, clearly, the external differential of the following form [using the rule for the evaluation of an external differential $d(u_i du_j) = du_i \wedge du_j + u_i du_j \wedge du_i$]:

$$\omega_1 = (2\pi)^{-1} (-u_2 du_1 + u_1 du_2 + u_3 du_3 - u_4 du_4). \quad (12)$$

The vector field corresponding to this form $\omega_1 = \omega_A$ [see (5)]

$$\bar{\mathbf{A}} = (2\pi)^{-1} (-u_2, u_1, u_3, -u_4) \quad (13)$$

satisfies condition (4) so that we can use Eqs. (8) to find \mathbf{A} . Substituting $\bar{\mathbf{A}}$, thus found, into (1), indeed, gives $l=1$. However, it is important for us that \mathbf{A} , thus found, can be identified with the vector potential of a magnetic field with nonzero helicity.

3. MAGNETIC FIELD CONFIGURATION

We thus find, starting from the vector field (13) and using (8) and (3) a vector potential in the three-dimen-

sional physical space. We note that Eqs. (3) are clearly written in dimensionless form, i.e., all coordinates x_i refer to some characteristic dimension R . To change to dimensional units we must make the substitution $x_i \rightarrow x_i/R$, but in order to keep the formulae simple we stay in this section with dimensionless length units. The dimensional coefficient of proportionality which fixes the absolute value of the magnetic field strength has also so far been dropped. As a result of substituting (13) and (3) into (8) we get, apart from a proportionality factor

$$A_1 = \frac{x_1 x_3 - x_2}{2(1+x^2)^2}, \quad A_2 = \frac{x_2 x_3 + x_1}{2(1+x^2)^2}, \quad A_3 = \frac{2x_3^2 + 1 - x^2}{4(1+x^2)^2}. \quad (14)$$

Calculating the magnetic field corresponding to this potential we find

$$H_1 = \frac{2(x_1 x_3 - x_2)}{(1+x^2)^3}, \quad H_2 = \frac{2(x_2 x_3 + x_1)}{(1+x^2)^3}, \quad H_3 = \frac{2x_3^2 + 1 - x^2}{(1+x^2)^3} \quad (15)$$

or

$$\mathbf{H} = \text{rot } \mathbf{A} = 4\mathbf{A}/(1+x^2). \quad (16)$$

The square of the magnetic field strength equals

$$H^2 = 1/(1+x^2)^4, \quad (17)$$

so that the absolute magnitude of the magnetic field of the configuration which we have found is spherically symmetric.

We now find the field lines of the magnetic field (15). The equations of the lines of force have the form $d\mathbf{x}/dl = \mathbf{H}/H$, where dl is a line element, or

$$\frac{dx_1}{dl} = \frac{2(x_1 x_3 - x_2)}{1+x^2}, \quad \frac{dx_2}{dl} = \frac{2(x_2 x_3 + x_1)}{1+x^2}, \quad \frac{dx_3}{dl} = \frac{2x_3^2 + 1 - x^2}{1+x^2}. \quad (18)$$

One can easily solve this set of equations if we map it to begin with on the sphere S^3 . The vector field $\bar{\mathbf{H}}$ corresponding to \mathbf{H} is found by using Eqs. (7) and (2):

$$\bar{H}_1 = -\frac{1}{2}u_2(1+u_4), \quad \bar{H}_2 = \frac{1}{2}u_1(1+u_4), \quad (19)$$

$$\bar{H}_3 = \frac{1}{2}u_4(1+u_4), \quad \bar{H}_4 = -\frac{1}{2}u_3(1+u_4),$$

so that the equations for the lines of force on the sphere S^3 have the form

$$\frac{d\bar{u}_1}{d\varphi} = -u_2, \quad \frac{d\bar{u}_2}{d\varphi} = u_1, \quad \frac{d\bar{u}_3}{d\varphi} = u_4, \quad \frac{d\bar{u}_4}{d\varphi} = -u_3, \quad (20)$$

where $d\varphi$ is the corresponding line element on S^3 . The solution of the set (20) is clearly:

$$u_1 = a \cos(\varphi + \varphi_1), \quad u_2 = a \sin(\varphi + \varphi_1), \quad (21)$$

$$u_3 = b \sin(\varphi + \varphi_0), \quad u_4 = b \cos(\varphi + \varphi_0),$$

where the integration constants a and b are connected through the relation $a^2 + b^2 = 1$.

Again using (2) to change to the physical space we find that the solution of the set (18) has the form

$$x_1 = \frac{a \cos(\varphi + \varphi_1)}{1+b \cos(\varphi + \varphi_0)}, \quad x_2 = \frac{a \sin(\varphi + \varphi_1)}{1+b \cos(\varphi + \varphi_0)}, \quad x_3 = \frac{b \sin(\varphi + \varphi_0)}{1+b \cos(\varphi + \varphi_0)}, \quad (22)$$

where the l -dependence of φ is found from the differential equation

$$\frac{d\varphi}{dl} = \frac{2}{1+x^2} \quad (23)$$

which expresses the well known connection between the line elements in the two metrics: the Euclidean and the

stereographic (see Refs. 10, 11). It is clear already from Eqs. (22) that the lines of force are closed: when we change φ from 0 to 2π we completely traverse it and we return to the initial point. Substituting (22) into (23) we get

$$\frac{dl}{d\varphi} = \frac{1}{1+b \cos(\varphi+\varphi_0)}, \quad (24)$$

so that the length of a line of force is equal to

$$L = \int_0^{2\pi} \frac{d\varphi}{1+b \cos \varphi} = \frac{2\pi}{(1-b^2)^{1/2}} = \frac{2\pi}{|a|}. \quad (25)$$

The maximum and minimum values of the radius vector of the points belonging to a line of force are found from the formulae

$$x_{\max} = \left(\frac{1+|b|}{1-|b|}\right)^{1/2}, \quad x_{\min} = 1/x_{\max} = \left(\frac{1-|b|}{1+|b|}\right)^{1/2}. \quad (26)$$

The solution of Eq. (24) corresponding to the condition $l(0)=0$ has the form

$$l(\varphi) = \frac{2}{a} \left[\arctg \left(\frac{a}{1+b} \operatorname{tg} \frac{\varphi+\varphi_1}{2} \right) - \arctg \left(\frac{a}{1+b} \operatorname{tg} \frac{\varphi_1}{2} \right) \right], \quad (27)$$

whence we find

$$\operatorname{tg} \frac{\varphi+\varphi_1}{2} = \left(\frac{1+b}{a} \operatorname{tg} \frac{al}{2} + \operatorname{tg} \frac{\varphi_1}{2} \right) / \left(1 - \frac{1-b}{a} \operatorname{tg} \frac{al}{2} \operatorname{tg} \frac{\varphi_1}{2} \right). \quad (28)$$

Expressing the trigonometric functions in (22) in terms of $\tan[(\varphi + \varphi_1)/2]$ and substituting (28) we find the way the equations of the line of force depend on l . We shall not write down the general formulae in view of their complexity, but restrict ourselves to the case $\varphi_1 = \varphi_0$ as the curves differing only in the difference $\varphi_0 - \varphi_1$ can be superposed onto one another by a rotation over that angle around the x_3 -axis:

$$x_1 = \frac{(\cos \varphi_0 + b) \cos al - a \sin \varphi_0 \sin al}{a(1+b \cos \varphi_0)} - \frac{b}{a} \quad (29)$$

$$x_2 = \frac{(\cos \varphi_0 + b) \sin al + a \sin \varphi_0 \cos al}{1+b \cos \varphi_0}, \quad x_3 = \frac{b}{a} x_2.$$

Hence it is clear that the lines of force are plane curves. Evaluation of their curvature gives

$$k = |d^2x/dl^2| = |a|, \quad (30)$$

so that the lines of force turn out to be circles of radius $1/|a|$. This agrees with Eq. (25) for their length.

Although it follows from the way we have constructed the circles that they are linked, it is of interest to verify this also directly. We therefore consider two circles: C_1 and C_2 corresponding to values of the parameters $a = b = \sqrt{2}$, $\varphi_1 = 0$ and different values $\varphi_0 = 0$ and $\varphi_0 = \pi/2$ (one circle is rotated with respect to the other over $\pi/2$ around the x_3 -axis). Their parametric equations have the form

$$C_1 = \{x_1 = \sqrt{2} \cos(l/\sqrt{2}) - 1, x_2 = \sin(l/\sqrt{2}), x_3 = \sin(l/\sqrt{2})\}, \quad (31)$$

$$C_2 = \{x_1 = -\sin(l/\sqrt{2}), x_2 = \sqrt{2} \cos(l/\sqrt{2}) - 1, x_3 = \sin(l/\sqrt{2})\}.$$

The circle C_1 lies in the plane $x_2 = x_3$, and the circle C_2 in the plane $x_1 = -x_3$. These planes intersect along the line $x_1 = -x_2 = -x_3$. It is clear that if the circles C_1 and C_2 are linked, their points of intersection with this line must alternate with one another. One easily finds that C_1 intersects this line at the points $A_1 = (-1, 1, 1)$ and $B_1 = (\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3})$, and C_2 in the points $A_2 = (-\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and B_2

$= (1, -1, -1)$. The point A_2 lies between the points A_1 and B_1 and the point B_2 outside the section (A_1, B_1) so that these pairs of points alternate on the line $x_1 = -x_2 = -x_3$ and the circles C_1 and C_2 are linked.

When the parameter φ_0 changes from 0 to $\pi/2$ the circle is shifted in space from C_1 to C_2 covering a surface with boundaries C_1 and C_2 which can be obtained by joining two ends of a strip after twisting it over 360° . It is known (and one can easily verify this experimentally) that if one cuts such a strip along its boundaries following a closed line it falls apart into two such strips which are linked. Continuing this cutting exercise we shall obtain ever narrower strips which are linked with one another. It thus becomes clear that all circles forming the original strip are linked with one another.

When the parameter φ_0 changes from 0 to 2π the circle describes a closed surface (a torus obtained from a cylinder which is twisted 360° before it ends are joined; the generating cylinder then changes into linked closed curves) which is bounding a "plait" of closed lines of force. The lines of force thus lie on toroidal surfaces which are imbedded one into another, and are circles, each of which is linked with all the others.

We now consider a physical system in which the magnetic field configuration which we have described can be realized.

4. MAGNETOHYDRODYNAMIC SOLITON

We change to dimensional units so that the magnetic field (15) takes the form

$$\mathbf{H} = \frac{H_0 R^4}{(R^2 + x^2)^3} \{2R[\mathbf{k} \times \mathbf{x}] + 2(\mathbf{kx})\mathbf{x} + (R^2 - x^2)\mathbf{k}\}, \quad (32)$$

where \mathbf{k} is the unit vector along the x_3 -axis, R the size of the soliton, and H_0 the magnetic field strength at the origin. The square of the magnetic field strength is equal to

$$H^2 = H_0^2 R^4 / (R^2 + x^2)^4. \quad (33)$$

Using Eq. (1) to evaluate the helicity of the magnetic field we get

$$I = (\pi^2/16) H_0^2 R^4. \quad (34)$$

We note that through the mapping $\mathbf{x} \rightarrow -\mathbf{x}$ we get an "antisoliton," the magnetic field of which differs from (32) in the sign in front of the first term in the braces, while the helicity (34) also changes sign.

We shall consider a perfectly conducting liquid for which I is an integral of motion. We also restrict ourselves to the case of an incompressible ideal fluid. The equations of magnetohydrodynamics for stationary flow have the form (see, e.g., Ref. 14)

$$\operatorname{div} \mathbf{H} = 0, \quad \operatorname{div} \mathbf{v} = 0, \quad \operatorname{rot}[\mathbf{v} \times \mathbf{H}] = 0, \quad (35)$$

$$(\mathbf{v} \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla \left(p + \frac{H^2}{8\pi} \right) + \frac{1}{4\pi\rho} (\mathbf{H} \nabla) \mathbf{H}.$$

They are clearly satisfied (see Ref. 15) when the fluid moves along the magnetic field lines of force with a velocity

$$\mathbf{v} = \pm \mathbf{H} / (4\pi\rho)^{1/2} \quad (36)$$

while the pressure satisfies the equation

$$\rho + H^2/8\pi = p_\infty = \text{const.} \quad (37)$$

Thus, Eqs. (32), (36), and (37) give an exact solution of the equations of magnetohydrodynamics which describes a localized topological soliton.

We evaluate the soliton energy

$$E = \int \left(\frac{\rho v^2}{2} + \frac{H^2}{8\pi} \right) d^3x = \frac{\pi}{32} H_0^2 R^4. \quad (38)$$

For a physical interpretation of topological solitons we must bear in mind that they are metastable states, the energy of which is higher than the energy of a state at complete equilibrium. It is thus necessary for the stability of a soliton, at any rate, that there does not exist such a continuous deformation at which its energy diminishes while the topological invariant is conserved. Comparison of (38) and (34) shows that

$$E \propto I/R,$$

so that the soliton can diminish its energy for constant I by increasing its radius. However, in the case considered there is yet another integral of motion—the angular momentum [we take the + sign in Eq. (36)]

$$\mathbf{M} = \rho \int [\mathbf{x} \times \mathbf{v}] d^3x = \frac{1}{2} (\rho \pi^2)^{1/2} H_0 R^4 \mathbf{k},$$

which stabilizes the “spreading” of the soliton (cf. the remarks about “collapse” of solitons in Refs. 16, 17).

The radius R and the field H_0 are completely determined by the two conserved quantities I and M :

$$R = (M^2/4\pi\rho I)^{1/4}, \quad H_0 = 8(\rho/\pi)^{1/4} I/M. \quad (39)$$

For given I , the M -dependence of the energy has a unique decreasing spectrum, $E \propto M^{-1/2}$. One must, however, bear in mind that I and M are not completely independent quantities. As the pressure is always positive, it follows from (37) and (33) that

$$H_0^2 \leq 8\pi p_\infty, \quad (40)$$

and thus, according to (39), I and M must satisfy the thermodynamic inequality

$$\frac{I}{M} \leq \frac{\pi}{2} \left(\frac{p_\infty}{2\rho} \right)^{1/4}. \quad (41)$$

For a given external pressure p_∞ the radius and energy of the soliton this satisfy the inequalities

$$R \geq (2\pi^4 \rho p_\infty)^{-1/4} M^{1/2}, \quad E \leq (\pi^4 p_\infty^3 / 2^{10} \rho^3)^{1/4} M^{1/2}. \quad (42)$$

Combining these inequalities [or substituting (40) into (38)] gives

$$E \leq (\pi/2)^2 p_\infty R^2, \quad (43)$$

which is essentially the same as the well known inequality $E < 3pV$ which follows from the fact that the trace of the energy-momentum tensor is positive (see Refs. 18, 19).

The magnetic field of the soliton (32) is produced by currents which circulate along closed lines with a density

$$\mathbf{j} = \frac{c}{4\pi} \text{rot } \mathbf{H} = \frac{c}{2\pi} \frac{1}{R^2 + x^2} (2R\mathbf{H} + [\mathbf{H} \times \mathbf{x}]).$$

These currents are conserved since we neglect dissipa-

tive processes. When account is taken of the finite conductivity σ , magnetic field diffusion occurs. The considerations given here are applicable if the hydrodynamic velocities dominate the diffusion velocities, i.e., when $(\nu_m R/v)^{1/2} \ll R$ ($\nu_m = c^2/4\pi\sigma$ is the magnetic viscosity $\nu \sim H_0/\rho^{1/2}$), or

$$\text{Re}_m = \frac{\nu R}{\nu_m} \sim \frac{v^2 \sigma R}{c^2 \nu} \gg 1 \quad (44)$$

the magnetic Reynolds number must be much larger than unity. When this criterion is satisfied, the condition $\sigma R/v \gg 1$ that the displacement current is negligible (see Ref. 14), which is assumed to be true in magnetohydrodynamics, is satisfied automatically (the displacement current vanishes identically in a stationary case when there is no dissipation). We can estimate the lifetime of the soliton by dividing its energy E by

$$\frac{dE}{dt} = \frac{1}{\sigma} \int j^2 d^3x \sim \frac{c^2 H_0^2 R}{\sigma}.$$

As a result we get

$$t \sim \sigma R^2 / c^2. \quad (45)$$

When applying inequality (44) to this problem this means that the lifetime (45) is much longer than the characteristic time $\sim R/v$ for the motion of a fluid particle along a line of force.

5. CONCLUSION

The equations of magnetohydrodynamics thus admit of an exact solution which describes a localized topological soliton. This kind of solution has already been met with in the physics of the condensed state (see, e.g., Refs. 16, 17). We note here some difference between the magnetohydrodynamic soliton and, say, a soliton in a ferromagnet.¹⁷ In a ferromagnet the mapping $S^3 \rightarrow S^2$ is realized by the order parameter—the magnetization vector $\mathbf{m}(\mathbf{x})$. Here the sphere S^2 has a direct physical meaning, namely, it is the configuration space of the vector \mathbf{m} . At the same time the map of a point from S^2 has no special physical meaning—it is the line on which $\mathbf{m}(\mathbf{x})$ takes a constant value of the Hopf invariant characterizes the linking of such lines. In magnetohydrodynamics there is no ordering parameter and the sphere S^2 has a completely arbitrary character: its points merely “number” the magnetic lines of force and the correspondence between the lines of force and the points on S^2 is established by the Hopf mapping $S^3 \rightarrow S^2$. This mapping is not realized in such an intuitive manner as in the case of a soliton in a ferromagnet, but now the maps of the points of S^2 have a direct physical meaning—they are the magnetic lines of force, and the Hopf invariant characterizes their linking.

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¹ We shall use in this section the convenient formalism of the theory of external forms (see, e.g., Refs. 10, 11).

² These formulae can also be obtained from the condition that the covariant derivatives in R^3 and S^3 are the same (Ref. 12).

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