Loosely bound particle with nonzero orbital angular momentum in an electric or magnetic field

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(Submitted 23 March 1981)

The effect of a magnetic or electric field on a loosely bound particle in a potential well with nonzero orbital angular momentum is investigated. A consistent analysis of the Schrödinger equation inside the well is replaced by a boundary condition for the wave function on the surface of a sphere. The radius of the sphere and the binding energy of the particle in the absence of a field are regarded as phenomenological parameters.

Outside the bounding sphere, the wave function is constructed by differentiating the Green's function for a particle in an electric or magnetic field. The energy shift in weak and strong magnetic fields is calculated for a particle with rectangular potential well.

1. INTRODUCTION

In our preceding papers1–3 we considered the effect of an electric and magnetic fields on a particle with low binding energy in a potential well in the s-state. The Schrödinger equation within the well was replaced by a boundary condition for the wave function in the center, in analogy with the approach in which a zero-radius potential is used. However, this approach cannot be literally applied to the case of a nonzero orbital angular momentum. If the radius of a potential well capable of retaining the particle in a bound state with a given orbital angular momentum is allowed to tend to zero, and the depth is allowed to tend to infinity (with the position of the energy level unchanged), then the wave function everywhere outside the well vanishes. Therefore the boundary condition that replaces the Schrödinger equation inside the well must be imposed on a sphere of finite radius $r_r$.

Thus, in place of a single phenomenological parameter (the binding energy) used in the zero-radius potential model, in the case of a nonzero orbital angular momentum it is necessary to have to parameters: the binding energy and the radius $r_r$ of the bounding sphere. We set $r_r$ equal to the effective radius employed in the theory of slow-particle scattering.4 The radius of the bounding sphere can be expressed in terms of the normalization coefficient in the asymptotic wave function.

By virtue of the known connection between these coefficients and the residue of the scattering amplitude at its pole, both definitions turn out to be equivalent (see Appendix I).

We shall assume $r_r$ to be sufficiently small and take into account, wherever possible, only the values of the lowest order in $r_r$.

In Sec. 2 we formulate the boundary condition for an arbitrary angular momentum $l$. In Sec. 3, the boundary condition is used to consider the influence of the magnetic field on a weakly bound particle in the $p$-state, while in Sec. 4 it is used to consider the effect of the electric field. Certain calculations are relegated to the Appendix.

2. BOUNDARY CONDITION

We consider a particle with rectangular potential well of radius $r_w$, having an energy $e$ and orbital angular momentum $l$. We use a system of units with $\hbar = M = 1$ (for a charged particle we put also $e = 1$) and represent the energy in the form $e = -a^{3/2}$. We assume that $ar_r < 1$.

We write the wave function in the form

$$\psi = T_0(0, \theta, \phi)$. \tag{2.1}$$

The radial wave function $R_\ell(r)$ outside the sphere of radius $r_r$ is proportional to the spherical Hankel function $H_\ell^0(\alpha r)$. We put

$$R = a_0 e^{i\ell \phi} H_\ell^0(\alpha r)$. \tag{2.2}$$

The factor $i^{\ell \phi}$ is introduced to make the normalization factor $a_0$ real. The factor of $e^{i\ell \phi}$ is introduced to obtain in the limit as $a \to 0$ a finite wave function of zero energy (belonging to the discrete spectrum). The normalization factor $a_0$ depends little on energy at small $a$. Since we are interested in a state with low binding energy, we can use the limiting value of $a_0$ corresponding to $a \to 0$. This value is determined from the normalization condition and is equal to (see Appendix I)

$$a_0 = \frac{\sqrt{2 \ell + 1}}{\sqrt{2 \ell + 1}}$$

At $a \gg 1$ we obtain from (2.2)

$$R = e^{-a r} \exp \left(-\frac{a}{r} \right).$$

where

$$R = e^{-a r} \exp \left(-\frac{a}{r} \right). \tag{2.4}$$

We use relation (2.4) also in the case of a short-range potential well of arbitrary shape. Then (2.4) serves as a definition of $r_r$, according to which $r_r$ is expressed
in terms of the coefficient $B_2$ in the asymptotic form of the wave function. We consider now the value of $R_i$ at $r=r_o$. Recognizing that $\alpha r_o=1$, we expand $h^2/(\alpha r_o)$ in a series and retain only the terms that matter most for the calculation, and neglect the small corrections to them.

To clarify the structure of this expansion, we use the relation

$$ h^{1/L}(\alpha r_0) + a_0 \alpha r_0. $$

The first two terms of the expansion of $n_i (z)$ take the form $-(2L-1)!/[\pi^{1/2}(2L-1)!2^L]$. The third term is of order $\alpha^{-1}$. The expansion of the function $f_j$ begins with $\alpha^L$. If we retain only two terms in the expansion of $n_i$, then we must discard $j$ at all $l \geq 2$, since such as these $\alpha^L$ is of higher order of smallness than $\alpha^{-1}$. We therefore put at $l=2$

$$ R_i = n_i (2L-1)! \left( \frac{1}{\pi^{1/2}(2L-1)!2^L} \right)^{a_0 \alpha r_0.} \quad (2.5) $$

At $l=1$, allowance for the first term in the expansion of $f_j$ is legitimate, for in this case $\alpha^{-1}$ is of higher order of smallness than $\alpha^L$. Allowance for $f_j$ leads in fact only to a small insignificant correction in many cases. However, for example in the calculation of the diamagnetic energy shift, we shall see that this term turns out to be significant. We therefore write out fully the limiting value of $R_i$, including the term $-\alpha^L$:

$$ R_i = n_i (2L-1)! \left( \frac{1}{\pi^{1/2}(2L-1)!2^L} \right)^{a_0 \alpha r_0.} \quad (2.6) $$

We shall find it convenient to change over to a different normalization, in which the coefficient of $\alpha^{-1}$ is equal to unity. Substituting (2.5) or (2.6) in (2.1), leaving out the factor $n_i (2L-1)!$ and denoting the result by $\phi$, we obtain

$$ \phi = n_i (2L-1)! \left( \frac{1}{\pi^{1/2}(2L-1)!2^L} \right)^{a_0 \alpha r_0.} \quad (2.7) $$

$$ \int \phi^2 dr_x, \quad \alpha r_0. \quad (2.8) $$

Relations (2.7) and (2.8) will be regarded as the boundary condition that replaces the consistent analysis of the Schrödinger equation in the field of a short-range well of arbitrary shape. We shall use these conditions also in the case when a magnetic or electric field is present in addition to the short-range well. The function $\phi$ is then a suitable exact solution of the Schrödinger equation for a particle in a homogeneous electric or magnetic field. Of course, the angle and radial variables do not separate in this solution.

3. EFFECT OF A MAGNETIC FIELD ON A LOOSELY BOUND PARTICLE IN THE p-STATE

The problem is to find a solution $\phi_p$ for the Schrödinger equation for a charged particle in a homogeneous magnetic field directed along the $z$ axis. The solution must satisfy on the surface of a sphere of small radius $r_o$ the boundary condition

$$ \phi_p = 0 \quad \text{on} \quad r_o \quad (3.1) $$

We denote by $a^2/2$ the binding energy in the absence of the magnetic field. The sought solution is expressed in terms of the derivatives, with respect to the cylindrical coordinates, of the Green's function $G(r, \rho)$ for a homogeneous magnetic field at $r' = 0$.

$$ \Phi = (2 \pi) \frac{a^2}{2} \quad (3.2) $$

Here $\omega$ is the Larmor frequency, equal in ordinary units to $a \hbar / 2$. The function $G$ is of order smallness than $a^2/2$.

The function $a^2/2$ is satisfied in obvious fashion the Schrödinger equation in a homogeneous field. The fact that $\Phi_p (3.2)$ also satisfies the Schrödinger equation is somewhat less obvious, but can also be proved (see Appendix II). The function $G$ can be represented in the form of the integral

$$ G = \frac{a^2}{2} e^{ixz}. $$

The kernel $K$ for a particle in a homogeneous electric or magnetic field was constructed by Feynman and Hibbs.

In the case of a magnetic field, it is convenient to replace $e$ by the integration variable $x = i \omega t$. Then $G (0, 0)$ takes the form

$$ G = \frac{a^2}{2} e^{ixz}. $$

The expansion of the function $G$ begins with $x = i \omega t$. We therefore put at $t = 0$

$$ G = \frac{a^2}{2} e^{ixz}. $$

The position of the boundary of the continuous spectrum at $m = -1$ is also equal to $x$, and at $m = 1$ it is $x = 0$.

Differentiating (3.3) with respect to $p$ and $\rho$ and substituting in (3.3), we obtain

$$ \Phi_p = a^2 \left( \frac{a^2}{2} e^{ixz} \right) \left( \frac{\partial}{\partial \rho} \right) \left( \frac{\partial}{\partial \rho} \right) \frac{a^2}{2} e^{ixz}. $$

We consider now $\Phi_p$ and $\Phi_p$ on a bounding sphere of radius $r_o$. The term $(1/2)(\partial^2 / \partial \rho^2) \ln (\alpha r_o)$ in the argument of the experimental does not exceed $(1/2)(\partial^2 r_o^2)$. The quantity $w r_o^2$ is the square of the ratio of the well radius to the magnetic length $a^2/\hbar^2$ and is assumed to be small (otherwise the fixed boundary condition cannot be used). We therefore neglect this term. Next, to separate the singular part in the wave function, we add and subtract the quantity $1/2$ under the integral sign in (3.4) and $1/a^2$ in (3.5). The singular part is expressed in terms of the spherical Hankel function $h^2/(\alpha r_o)$, in which we take into account three terms of the series expansion. In the remaining regular part we retain the terms that do not depend on $r_o$ and the terms proportional to $r_o$ and discard all others. Substituting the approximate expressions obtained in this manner for $\Phi_p$ and $\Phi_p$ in (3.1), we obtain equations for the determination of $\alpha$. We consider now $\Phi_p$ and $\Phi_p$ on a bounding sphere of radius $r_o$. The term $(1/2)(\partial^2 / \partial \rho^2) \ln (\alpha r_o)$ in the argument of the experimental does not exceed $(1/2)(\partial^2 r_o^2)$. The quantity $w r_o^2$ is the square of the ratio of the well radius to the magnetic length $a^2/\hbar^2$ and is assumed to be small (otherwise the fixed boundary condition cannot be used). We therefore neglect this term. Next, to separate the singular part in the wave function, we add and subtract the quantity $1/2$ under the integral sign in (3.4) and $1/a^2$ in (3.5). The singular part is expressed in terms of the spherical Hankel function $h^2/(\alpha r_o)$, in which we take into account three terms of the series expansion. In the remaining regular part we retain the terms that do not depend on $r_o$ and the terms proportional to $r_o$ and discard all others. Substituting the approximate expressions obtained in this manner for $\Phi_p$ and $\Phi_p$ in (3.1), we obtain equations for the determination of $\alpha$.
We consider two limiting cases.

1. Weak fields, \( \omega \ll a_0^2/2 \). In this case \( a \) is close to \( a_0 \). In the integrals \( A_1 \) and \( A_2 \), the main contribution is made by the region of small \( x \), and the values of the integrals can be estimated by replacing the pre-exponential factor by the first term of the expansion in powers of \( x \). We obtain

\[
\frac{a}{2} - \frac{1}{2} \omega x + \frac{a_0}{2} \omega x.
\]

The equations (3.6) are solved by successive approximations. In the first-order approximation we neglect all the quantities proportional to \( a_0 \), and retain only the terms independent of \( a_0 \). In the second approximation we find the correction proportional to \( a_0 \), in which we retain the terms not higher than \( a_0^2 \). We obtain

\[
\frac{a}{2} - \frac{1}{2} \omega x + \frac{a_0}{2} \omega x.
\]

Changing from the quantity \(-a_0^2/2\) to the energy \( E \) (with allowance for the position of the boundary of the continuous spectrum), we obtain

\[
E = \omega x - \frac{a_0^2}{2}.
\]

The term \( \omega x \) is the paramagnetic energy shift. The terms proportional to \( a_0^2 \) are the diamagnetic energy shift and agree (in first order in \( a_0 \)) with the result of the perturbation-theory calculation.

2. Strong fields, \( \omega \gg a_0^2/2 \). In this case the terms \( (1/3)a_0^3 r_0 \) and \( a_0^2 \omega x \) are small corrections and can be neglected. Equations (3.6) take the form

\[
a = - \frac{1}{2} \omega x + a_0 \omega x, \quad m = 0.
\]

To estimate the integral \( A_1 \), we can take the limit as \( a_0^2/\omega \to 0 \) in the argument of the exponential. We obtain

\[
A_1 = \frac{a_0}{2} - \frac{1}{2} \omega x + \frac{a_0}{2} \omega x.
\]

As for the integral \( A_2 \), the limit as \( a_0^2/\omega \to 0 \) cannot be taken, since the integral then diverges. Replacing by way of estimate the pre-exponential factor by its limiting value as \( x \to \infty \) and calculating the integral, we have

\[
A_2 = 0.
\]

Finally, we consider the possibility of appearance of a level in the case when there is no level in the absence of a field, that is, \( a_0^2/2 \leq \omega \). We designate \( \omega_0 \) by \( \omega \).

Neglecting in (3.15) \( a_0^2/2 \) compared with \( E_0 \), we obtain an estimate for \( a_0 \):

\[
\omega_0 = (3/2)\omega_0/2\pi.
\]

Accordingly, the binding energy \( W \) of the level, which appears under the influence of the field, is equal to

\[
W = 2a_0^2/2\pi.
\]

4. EFFECT OF ELECTRIC FIELD ON A LOOSELY BOUND PARTICLE IN THE \( \phi \)-STATE

In the presence of an electric field of intensity \( \mathcal{E} \), the energy becomes complex. The imaginary part, as is well known, characterizes the probability of penetrating through the potential barrier. Of greatest physical interest is the case when the ratio of the imaginary and real parts of the energy is small. This case is realized when the dimension of the potential barrier along the field, which is of the order of \( a_0^2/\mathcal{E} \), greatly exceeds the characteristic dimension \( a_0^2 \) (the "radius" of the wave function, i.e., when \( a_0^2/\mathcal{E} \ll 1 \). In this case \( a_0^2/\mathcal{E} \) will be all the smaller. Under these conditions, the term \( 1/3a_0^3r_0 \), in the boundary condition (2.8) is a small correction which we shall disregard.

The problem consists of finding a suitable singular solution of the Schrödinger equation \( \Phi_\phi \) in an electric field \( \mathcal{E} \) directed along the \( z \) axis and satisfying the boundary condition at \( r = r_0 \). Just as in the case of magnetic field, \( \Phi_\phi \) is expressed in terms of derivatives of the Green's function \( G \), which satisfies

\[
(\mathcal{E} + m)\Phi_\phi = 0.
\]

The derivative \( (\mathcal{E} + m)\Phi_\phi \) satisfies in obvious fashion the Schrödinger equation. We therefore put

\[
\Phi_\phi = \Phi \cos \theta - \Phi \sin \theta.
\]

The method of calculating the integral in the right-hand side of (4.3) is indicated in Appendix III. We write out

\[
\Phi_\phi = \int \frac{d^3r}{(2\pi)^3} e^{iEit}. \]

We consider now \( \Phi \) on a bounding sphere of radius \( r_0 \). Assuming that \( \Phi_\phi \sim \Phi \), we neglect the term \( \Phi_\phi \) in the argument of the integrand. Then

\[
\Phi_\phi = \frac{1}{4\pi} \frac{1}{r_0} \int \exp \left[ i \left( \frac{E}{2} - \frac{m}{2} \right) \right] \frac{d^3r}{(2\pi)^3}.
\]
which follows from the boundary condition, is of the form

\[ \phi_{m} = \frac{1}{2} \left( \frac{\phi'}{2} + \phi'' \frac{1}{24} + \phi''' \frac{1}{24^3} \right) Y_{m}. \]  

We have left out of (4.4) a corrections of higher order in \( r_{2} \). Substituting (4.4) in the boundary condition, we obtain

\[ \phi_{m} = \frac{1}{2} \left( \frac{\phi'}{2} + \phi'' \frac{1}{24} + \phi''' \frac{1}{24^3} \right) Y_{m}. \]  

The real part of the energy shift \( \Delta \) is equal to

\[ \Delta = \frac{\phi''}{2} \]  

and coincides with the value obtained by perturbation theory neglecting the corrections of higher order in \( r_{2} \). The imaginary part of the shift is half the level width \( \Gamma \). As follows from (4.5),

\[ \Gamma = \frac{\phi''}{2}. \]  

We turn now to the case \( m = 0 \) (\( s \)-state). It is easy to verify that the expression \( \phi'' \) satisfies the Schrödinger equation for a particle in a homogeneous electric field. Starting from this, we put

\[ \phi_{0} = (\hbar n) \left[ \frac{1}{2} \lambda + \frac{2 \pi n}{\lambda} \right]. \]  

Substituting here (4.1), we obtain

\[ \phi_{0} = (\hbar n) \left[ \frac{1}{2} \lambda + \frac{2 \pi n}{\lambda} \right] \exp \left( \frac{\hbar n}{2} + \frac{\pi \hbar n}{4} \right) \frac{1}{\lambda}. \]  

In the argument of the exponential in the first integral of the right-hand side of (4.6) we neglect, just as in (4.3), the quantity \( \hbar \beta_{y} \cos \theta \). As for the second integral, we retain two terms of the series expansion of \( \exp(\hbar \beta_{y} \cos \theta \cos \phi) \) in powers of \( \lambda \), i.e., we replace this exponential by \( 1 - \hbar \beta_{y} \cos \theta + \lambda \frac{\hbar \beta_{y} \cos \theta}{2} \). We obtain

\[ \phi_{0} = (\hbar n) \left[ \frac{1}{2} \lambda + \frac{2 \pi n}{\lambda} \right] \exp \left( \frac{\hbar n}{2} + \frac{\pi \hbar n}{4} \right) \frac{1}{\lambda}. \]  

The term containing \( Y_{0} \) makes no contribution whatever to the boundary condition, by virtue of the orthogonality of the spherical functions.

The integrals in (4.10) are calculated by the method indicated in Appendix III. The expression for \( \alpha = \frac{\phi''}{2} \), which follows from the boundary condition, is of the form

\[ \alpha = \frac{1}{2} \left( \frac{\phi'}{2} + \phi'' \frac{1}{24} + \phi''' \frac{1}{24^3} \right) Y_{0}. \]  

The real part of the energy shift is

\[ \Delta = \frac{\phi''}{2}, \]  

which also agrees with the perturbation-theory result (accurate to higher powers of \( r_{2} \)) for the polarizability.

For the width we obtain

\[ \Gamma = \frac{\phi''}{2}. \]  

The term \( \phi'' \) in the parentheses is a small correction to the principal term.

In conclusion we examine the dependence of the real part of the energy shift \( \Delta = -\phi'' \phi_{0} \) on \( \alpha_{0} \) and \( \beta_{y} \) at different \( \lambda \). We confine ourselves to the case of a maximum value of the projection of the angular momentum \( m = 1 \), for which the calculation reduces to the determination of the \( \lambda \)-th derivative of the Green's function

\[ \backslash \text{in the same approximation as before}. \]  

With the aid of the boundary condition (2.8) we find that the polarizability \( \beta_{y} \) is proportional to the quantity \( r_{2}^{5/2}(\alpha_{0}/2)! \) for the perturbation theory (neglecting the corrections of higher order in \( r_{2} \)). We confine ourselves to the case of a maximum value of the projection of the angular momentum \( m = 1 \), for which the calculation reduces to the determination of the \( \lambda \)-th derivative of the Green's function

\[ \lambda = (\partial / \partial \phi_{0}) \phi_{0}. \]  

Since \( r_{2} \ll 1 \), we can replace \( H_{2m}^{(2)}(r_{2}) \) by the first term of the series expansion.

At \( \lambda = 1 \) and 2, the first term is proportional to \( (\alpha_{0})^{m+1} \), so that

\[ \beta = \alpha = \frac{1}{2}. \]  

At \( \lambda = 1 \) we obtain the already-known result. At \( \lambda = 2 \)

\[ \beta = \alpha = \frac{1}{2}. \]  

At \( \lambda \geq 3 \), however, the first term of the expansion of \( H_{2m}^{(2)}(r_{2}) \) is proportional to \( (\alpha_{0})^{m+2} \). In this case

\[ \beta = \alpha \]  

and is independent of \( \alpha \).

5. COMPARISON WITH THE RESULTS OF OTHER CALCULATIONS

Dalidovich and Smirnov have considered an electron in the field of several zero-radius potentials in the presence of an external homogeneous electric field. They simulated the \( p \)-state by an odd wave function in the field of two centers separated by a distance \( R \) under the condition \( \alpha_{0} \ll 1 \). However, the question of how the distance \( R \) is connected with the radius \( r_{2} \) in the potential well for which the two-center model is constructed, remained open in their paper. Comparing the formulas (4.6) and (4.12) for the energy shift with the results of the calculation of the shift in Ref. 6, 1, we arrive at the conclusion that \( R = 2r_{2} \). In Ref. 6 is calculated also the level width. The final expressions (30) and (31) of that reference contain errors in the numerical coefficients. If \( \Gamma \) is recalculated using the general Eq. (39) of Ref. 6, but with \( R = 2r_{2} \), full agreement is obtained with our expressions (4.7) and (4.13).

We note also Refs. 7 and 8, in which the approximation of several zero-radius wells in an external field and perturbation theory were also considered.

An expression for the width was derived also by Smirnov and Chibisov for a more general case, by another method that made it possible to obtain only the first term of the pre-exponential series. A comparison with our results has revealed errors in Ref. 9. One error is that the expression for \( \sin \alpha \) in terms of the parabolic coordinates \( x = r + z \) and \( \eta = r - z \) at small
the results of calculation by means of (5.1) agree with
$Z=\text{rect}$ expression for the electron-detachment probabil-
ity per unit time:
$$
W_{\text{e}} = \frac{2\pi^2 m^2 (\mu_{14} - \mu_{11} - \mu_{12})}{\mu_{14}} \left( \frac{2m^2}{\mu_{14}} \right)^{3/2} \exp \left( -\frac{2Z_0^2}{\mu_{14}} \right) \equiv \left( \text{rect} \right) (5.1)
$$

In the notation of the present paper, $\gamma = \alpha_{11}, F = \text{rect}, W = \text{rect}$, and $m$ means $m_{14}$. In addition, we must put $Z_0 = 0$, inasmuch as in our case there is no Coulomb field. If we use the expression (2.4) for $B_l$, then the results of calculation by means of (5.1) agree with (4.7) and (4.13) (without the second term in the paren-
theses).

The indicated corrections allow us to resolve the con-
tradiction in Ref. 10, where, when the coefficients $B_l$ were determined by comparing the theory with experi-
ment for the $\pi$ and $\sigma$ states of He $^+$, a difference by a factor of 1.5 was obtained. After making the correc-
tions and including in $B$ the additional factor $2^{1/2}$ the two coefficients agree within the limits of experimental error.

6. REGION OF APPLICABILITY OF THE APPROXIMATION

We discuss now the region of applicability of the ap-
proximation considered here. Actually in both con-
sidered problems (magnetic and electric fields) there are three independent parameters with the dimension of length $(l)$: the effective radius $a_\alpha^1$; 3) the "wave-function radius" $a^2$; 3) the characteristic scale $R$ of the exter-
nal field. For the magnetic field this is the magnetic length $R = a^2/\hbar$, for the electric field this is the width of the potential barrier $R = a^2/\hbar^2$. The theory is valid if $a_{\alpha l} \ll 1$ and $r_{\alpha l}/R \ll 1$, and we take into account below only the lowest terms in the expansion in these param-
eters. At the same time, $a_{\alpha l}$ can be either larger than unity (weak fields) or less than unity (strong fields), as well as of the order of unity, although in real cases we are dealing usually with weak fields (even at a negative-
ion binding energy 0.01 eV we have $a_{\alpha l}^2 = 2 \cdot 10^{-11}$ cm, $a_{\alpha l}^2 = 1$ at $F = -5 \cdot 10^5$ V/cm or $R = 2 \cdot 10^{-5}$ G = 200 T). In addition, at $a_{\alpha l} - 1$ the level width $\Gamma$ in an electric field becomes large, and it is practically impossible to ob-
serve the corresponding state. We therefore assume $(aR)^{1/2}$ in Eqs. (3.10) and (4.13) and expand the sought quantities in powers of this parameter, taking into account the necessary number of terms, this being an additional and generally speaking not obligatory ap-
proximation.

The second assumption with which the relation between the effective radius and the asymptotic normalization factor is connected is that the potential well and the exter-
nal region are separated by a centrifugal potential barrier. If in addition there is a comparable potential barrier of a different type (non-centrifugal) at the edge of the well, then this relation is violated. A similar

violation results, for example, from a Feshbach-like state of a strongly bound system (detachment of electron as a result of a two-electron transition in the presence of a weak dynamic coupling between the electrons in a negative ion). In all these cases, introduction of an ef-
factive radius is also possible but calls for a more de-
tailed examination of the problem.

Finally, we have used throughout implicitly the as-
sumption that an interaction takes place only with states with a given angular momentum, i.e., there is no "ac-
idental" degeneracy at low energies of states with dif-
fert $l$, while for the considered potential other bound
and quasi-stationary states lie substantially farther from the
origin on the complex energy plane, on both the physical and unphysical sheets, with the same relation prev-
iously also in the presence of external fields. This
assumption is perfectly natural for negative ions (a, to a lesser degree, for scattering of slow neu-
trons by nuclei (as a result of the complicated struc-
ture of the low-energy scattering for many nuclei). It
should be noted that the specific symmetry of the atomic
potential $^{133}I$ can lead to an almost simultaneous ap-
pearance of weakly bound states with different $l$ when
the nuclear charge $Z$ is increased, but a more accurate
allowance for the polarization and other interactions
makes the splitting of these states sufficiently large,
so that apparently this interaction can be disregarded
for most real negative ions in really attainable fields.

It must be emphasized that although the results de-
pend little on the shape of the potential well (except for the restrictions noted above), in contrast to the $s$-state
they depend substantially on the spherical-symmetry assumption and, correspondingly, on the $(2l+1)$-fold
degeneracy of the levels in the absence of an external field.
Nonspherical small-radius potential wells and the
parameterization with the aid of boundary conditions
(all for a more detailed analysis. This is precisely
why simulation of spherically potential well with the aid of
a system of zero-radius potential must be carried out
with caution.

7. CONCLUSION

The results of the present paper, jointly with Ref. 3,
demonstrate clearly that it is natural to use a two-par-
parameter approximation, similar to the two-parameter
approximation for $s$-states (Ref. 5, § 133), for the de-
scription of weakly bound states with an arbitrary angular
momentum in a short-range force field. One of these
parameters $a_\alpha^2$ is connected with the energy of the
bound state $[a^2 < 0]$ of with the position of the quasi-
stationary state_resonance in the scattering $[a^2 < 0]$. The
force parameter, on the other hand, characterizes
the effective radius of the forces and is correspondingly
connected with the relative contribution made to the
normalization integral by the regions of space inside
and outside the potential well. By the same token, it
can be expressed in terms of the normalization coeffi-
cient of the wave function of the bound state outside the
well. The effective radius is convenient $a^2$ a second
parameter in view of its clarity, of the connection with the
analogous parameter for the s-states, and of its
suitability for use for both bound and quasistationary states.

The behavior of a weakly bound \( p \)-state in a magnetic field can be explained qualitatively by starting from simple physical considerations. Indeed, when the magnetic field is turned on, the problem becomes quasi-one-dimensional since the magnetic field prevents the particle from moving away in directions perpendicular to the field. For the \( p \) states there is, in addition, a nodal surface at \( z = 0 \), and we obtain a quasi-one-dimensional problem on a semi-infinite interval. The bound state is formed in this case only after the potential well reaches a certain depth, as is in fact observed. For the \( ps \) states, the plane \( z = 0 \) is not a nodal surface, the wave function is symmetric with respect to the replacement of \( z \) by \(-z\), and then there is a bound state in the one-dimensional problem in any potential well. In this case, however, the wave function vanishes on the \( z \)-axis, the influence of the potential well is weaker than for the \( ps \) state considered in Ref. 2, and the binding energy is proportional not to the second power of the field (as for the \( ps \) case) but to the fourth power. Such states can be apparently observed in semiconductors with low effective electron or hole masses in superstrong magnetic fields at low temperatures, but this is a much more complicated task than the observation of analogous \( ps \) states.

For the \( p \) state in an electric field, the most interest- ing in the lifetime and the width of the level \( \Gamma \), inasmuch as in really attainable fields we can destroy the weakly bound negative ions at a binding energy less than 0.1 eV (Ref. 10) and when an accelerator's magnetic field (a Lorentz force equivalent to the electric field) is used this can be done at energies up to \(-1 \) eV. Since \( \Gamma \) depends very strongly on the binding energy \( E_0 \), these measurements make it possible to determine \( E_0 \), quite reliably and the effective radius somewhat less reliably (field spectroscopy of weakly bound states).

In (4.7) and (4.13) the argument of the exponential is the effective radius, and contains an additional power of the field (a Lorentz force equivalent to the electric field) is used this can be done at energies up to \(-1 \) eV, quite reliably and the effective radius somewhat less reliably (field spectroscopy of weakly bound states).

In (4.7) and (4.13) the argument of the exponential is quite trivial and is equal to the phase integral for a trivial and is equal to the phase integral for a triangular potential barrier. However, the pre-exponential factor is not trivial, is proportional in our approximation to the effective radius, and contains an additional power of the small parameter \( F_0^2x_0^2 \) for the \( s \) state, so that to reach the same accuracy in the calculation of \( \Gamma \), it is necessary to retain two terms in the pre-exponential factor.

The quantity \( \Gamma = (1/3)\Gamma_s^2(2/3)\Gamma_p^2 \), averaged over the \( s \) and \( p \) states is meaningful when the time of stay of the ion in the field is short compared with the half-lives, so that the decay exponential can be expanded in a series and only the terms linear in \( \Gamma \) retained. Under real conditions allowance for \( \Gamma \), and for the second term in \( \Gamma \), can change \( \Gamma \) by up to 15–20%.

Quantities such as the polarizability, diamagnetism, and others, which are not connected with tunnel transitions, do not depend on the binding energy at all at sufficiently large \( l \) in this simplest approximation. The centrifugal potential clamps the particle in this case to the edge of the potential well, so that we obtain the rigid-rotator model, and the below-barrier part of the wave function both inside and outside the well becomes negligible. This transition can take place at different \( l \) for different values; for the polarizability, as we have already found, it takes place at \( l = 3 \).

Inasmuch as for all the negative ions the electron-affinity energy is much less than the polarization potential (the energy of the detachment of the next electron), the single-electron approximation used here and earlier is quite satisfactory; the collective effects are important only inside the atom, where the behavior of the wave function has little influence on the phenomena considered here.

In some cases the effective radius may not enter in the final formulas. This is precisely the situation in the calculation of the spectrum of electrons emitted in slow collisions between negative ions and atoms, when this process is regarded as "pushing out" of a bound state with a specified \( l \) into the continuous spectrum from a small-radius nonstationary potential well.

We note finally that most monatomic negative ions have a weakly bound electron precisely in the \( p \)-state, so that the case \( l = 1 \) considered here is particularly important from the point of view of possible applications.

APPENDIX I

We determine the coefficient \( a_i \) in (2.3) at \( \alpha = 0 \) for a rectangular well. Inside the well, the radial wave function is of the form

\[
\psi(r) = \psi_0(r) = \sqrt{2/(2\pi)} e^{-\alpha r},
\]

where \( V_0 \) is the depth of the potential well. Outside the wall, at \( \alpha = 0 \) we have from (2.3)

\[
\psi(r) \sim \sqrt{\frac{2}{\pi \alpha}} e^{-\alpha r}.
\]

From the normalization condition

\[
\int_0^\infty \psi^* \psi \, dr = 1,
\]

it follows that

\[
\psi(r) \sim \sqrt{\frac{2}{\pi \alpha}} e^{-\alpha r}.
\]

Using the relation

\[
\int_0^\infty \psi^* \psi \, dr = \frac{\sqrt{\pi}}{2} \left[ 1 + (\psi(0) - \psi_0(0)) e^{-\alpha r} \right],
\]

taking into account the condition for the appearance of a level with angular momentum \( l \)

\[
l + (\psi(0) - \psi_0(0)) e^{-\alpha r} = 0,
\]

and the continuity of \( \psi \) at \( r = r_0 \), we obtain

\[
a_i = \left( \frac{2l + 1}{2l + 3} \right) \frac{\sqrt{\pi \alpha}}{2 (2\pi \alpha)^{l+1}}.
\]

We derive also Eq. (2.4) for \( R_1 \) by considering the residue of the scattering amplitude at the pole. According to Ref. 3, the scattering amplitude in the effective-radius approximation is of the form


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The Green's function for an electron in a magnetic field can be represented in the form of the series

$$G = \sum_{n} R_{n}(\mu) R_{n}(\nu) \delta_{\mu, \nu},$$

where $R_{n}$ is the normalized radial wave function of the electron in a magnetic field at zero projection of the angular momentum on the field direction. It is equal

$$R_{n}(\mu) \exp(-\mu \sigma_{\text{R}}),$$

From this we get expression (2.4).

APPELLIX II

The Green's function for an electron in a magnetic field can be expressed in the form of the series

$$G = \sum_{n} R_{n}(\mu) R_{n}(\nu) \exp(-\mu \sigma_{\text{R}}),$$

where $R_{n}$ is the normalized radial wave function of the electron in a magnetic field at zero projection of the angular momentum on the field direction. It is equal to

$$R_{n}(\mu) \exp(-\mu \sigma_{\text{R}}) \exp(-\mu \sigma_{\text{R}}).$$

Here $F$ is a confluent hypergeometric function.

To prove that $\exp(-1/3) \exp(3\mu^{2}/4)$ satisfies the Schrödinger equation in a magnetic field at an angular-momentum projection $\sigma_{\text{R}}$, it suffices to show that $dF/d\mu = -\mu F$, is the radial wave function for the case of an angular-momentum projection $\sigma_{\text{R}}$. Differentiating $R_{n}$ with respect to $\mu$, we obtain

$$dR_{n}/d\mu = (2\mu \sigma_{\text{R}} + 1) R_{n}(\mu) \exp(-\mu \sigma_{\text{R}})$$

Using the known relations

$$F(a, b, \nu) = \frac{\Gamma(a)}{\Gamma(b)} F(a, b, \nu) + \frac{\Gamma(a+b)}{\Gamma(a)} F(a+b, a, \nu),$$

we obtain

$$dR_{n}/d\mu = (2\mu \sigma_{\text{R}} + 1) R_{n}(\mu) \exp(-\mu \sigma_{\text{R}}).$$

According to Ref. 5, this expression is apart from the normalization, exactly the required radial wave function.

APPELLIX III

To investigate the integrals in (4.3) and (4.10) it is advisable to introduce first of all a new integration variable $r = \sigma_{\text{R}}^{2}/2$. The integrals obtained are of the form

$$J = \int_{r} \exp \left[ \frac{(\sigma_{\text{R}})^{2}}{4r} \right] \frac{K_{\mu-1/2}(2\sigma_{\text{R}}^{2}/r)}{2r^{3/2}}.$$

and contain two dimensionless parameters $\sigma_{\text{R}}$ and $\beta = \sigma_{\text{R}}^{2}/2$. We apply to this integral the artifice employed in Ref. 6. We deform the integration contour so that it follows the lower imaginary semi-axis of the complex plane to the stationary-phase point $r_{0}$, and then a straight line parallel to the real axis to $r_{0}$. From this we get expression (2.4).

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