Generalized Heisenberg ferromagnet and the Gross-Neveu model

P. P. Kulish and N. Yu. Reshetikhin

Lebedev Branch of the V. A. Steklov Mathematical Institute, Academy of Sciences of the USSR


We use the quantum-mechanical inverse scattering method to diagonalize the Hamiltonian of a many-component quantum system on a one-dimensional lattice. We construct the generating functional of the commuting integrals of motion and the corresponding eigenfunctions and eigenvalues. As an application of the formalism developed here we evaluate the spectrum in the Gross-Neveu model. We enumerate a number of quantum-mechanical systems for which one can obtain an exact solution by this method.

PACS numbers: 75.10.Jn, 75.10.Dg

The development of a quantum-mechanical inverse scattering method enables one to combine in a single scheme the known exact solutions of many different quantum-mechanical systems and to indicate new exactly soluble models (see the review article by Faddeev and Refs. 2-10). Yang's generalization of the Bethe Ansatz to the case of the many-component non-linear Schrödinger equation which describes the interaction of a Bose-field multiplet has been analyzed from the point of view of the quantum-mechanical inverse scattering method (QISM). One of the central objects of the QISM is the R-matrix which determines the commutation relations between the generating functional of the quantum-mechanical integrals of motion and the creation and annihilation operators. The R-matrix satisfies the Yang-Baxter relation (equation 1)' and is the R-matrix which determines the interaction of a Bose-field multiplet. For those cases we evaluate the spectrum of the elementary excitations and their S-matrix.

The revival of interest in the use of the Bethe Ansatz for solving relativistically invariant sine-Gordon models and the massive Thirring model led to interesting results for asymptotically free models with an isotropic fermion multiplet (number of colors $N_c = 2$) and with a four-fermion interaction (modification of the Gross-Neveu and Vaks-Larkin model).

Ref. 16. We show that the equations for the quasi-momenta, which determine the eigenfunctions of the Hamiltonians of these models, are the same as in the inhomogeneous term, as the corresponding equations of the generalized Heisenberg ferromagnet. This enables us to evaluate the spectrum of the asymptotic states of the model, which is the same as the quasi-classical answer in Ref. 16.

The plan of the paper is as follows. In section 1 we construct the monodromy matrix of the auxiliary linear problem, we evaluate the commutation relations between its elements, and we find the generating functional for the integrals of motion $Q$. In section 2 we diagonalize the trace of the transition matrix, and give the transcendental equations for the quasi-momenta which parametrize the eigenstates $|\psi\rangle$ for a finite number of sites $N$ on the lattice. We go to the limit as $N \to \infty$ in the ferromagnetic case $c = -1$ and calculate the S-matrix of the excitations, and realize the Zamolodchikov algebra in section 3. In section 4 we consider the more complicated case of the antiferromagnetic state $c = 1$, when the eigenstates of the Hamiltonian (2) are described by integral equations as $N \to \infty$. In the last section the formalism developed here is applied to the Gross-Neveu model and we list a number of quantum models the exact solution of which can be obtained by this method.

Ansatz. We apply the quantal inverse scattering method to construct a generating functional for the integrals of motion $Q$. We get the commutation relations between the $Q$ and the creation and annihilation operators of the eigenstates of the Hamiltonian (2), and we recover the transcendental equation for $(M - 1)$-st set of quasi-momenta that determine the eigenstates for finite $N$. In the limit as $N \to \infty$ the system will be a ferromagnet ($c = -1$) or an antiferromagnet ($c = 1$). For those cases we evaluate the spectrum of the elementary excitations and their S-matrix.

1. As in the classical variant of the inverse scattering method, the basic object of study for the QISM\cite{11} is the auxiliary linear problem connected with the non-linear equation (in the present case—the Heisenberg equations for quantum operators). We shall give later the Heisenberg equations generated by the Hamiltonian (2), but we shall write down as follows. We construct in a chain of N sites:

\[ \Phi(x_1, x_2, \ldots, x_N) \]

The operator \( L_{x} \) acts in the space \( \mathcal{C} \otimes \mathcal{H} \) [see (1)] and we can represent it as an \( M \times M \) matrix in \( \mathcal{C} \) (the auxiliary space) with matrix elements—operators in \( \mathcal{H} \) (in the space of the quantum states). The operator \( L_{x} \) which interests us can be written as follows:

\[ L_{x} = (a(x)b(x) + b(x)a(x))P_{0} \]

where \( P_{0} \) is the permutation operator in \( \mathcal{C} \otimes \mathcal{H} \), \( \lambda \) is the spectral parameter, \( a(x) = b(x) = \lambda + i \mu \), and \( \mu \) is a real number.

The analog in the QISM of the transition from the potential of the linear problem to the scattering data is the transition from the local operators that act non-trivially only in \( \mathcal{H} \) to the elements of the transition matrix \( T_{x} \) operators in \( \mathcal{H} \). For Eq. (4) \( T_{x} \) is given as a product:

\[ T_{x} = T_{x_1} T_{x_2} \ldots T_{x_N} \]

The central role in the exact solution of the quantum equations is played by the commutation relations between the matrix elements of \( T_{x} \). These relations can be found because of the special form of the operator \( L_{x} \). Indeed, \( L_{x} \) satisfies the Yang-Baxter relations, which we write down as follows. We consider the product of three spaces \( \mathcal{C} \otimes \mathcal{C} \otimes \mathcal{C} \) (the lower indices are used to distinguish operators which act non-trivially in the corresponding spaces). We introduce three operators:

\[ L_{x} = (a(x)b(x) + b(x)a(x))P_{0} \]

and similarly \( L_{y} \) and \( L_{z} \). In that case the following relation holds:

\[ L_{y} L_{z} L_{x}(a(x)b(x) + b(x)a(x))P_{0} = L_{z} L_{x} L_{y}(a(x)b(x) + b(x)a(x))P_{0} \]

We now consider two transition matrices \( T_{y} \) and \( T_{z} \) in the space \( \mathcal{C} \otimes \mathcal{C} \otimes \mathcal{H} \) which act trivially in \( \mathcal{C} \), \( \mathcal{C} \), and \( \mathcal{H} \), respectively. Using the fact that \( L_{x} \) and \( L_{y} \) commute when \( x \neq y \), we get

\[ L_{y} L_{z} L_{x} = L_{z} L_{x} L_{y} \]

and the following equation holds:

\[ L_{y} L_{z} L_{x}(a(x)b(x) + b(x)a(x))P_{0} = L_{z} L_{x} L_{y}(a(x)b(x) + b(x)a(x))P_{0} \]

This is the equation which determines the commutation relations between the elements of \( T_{x} \).

To write Eq. (9) more compactly it is convenient to rewrite it, using the tensor product

\[ (a(x)b(x) + b(x)a(x))P_{0} \]

where \( R(x) = \delta(x) + \mathcal{E}(x) \), while \( F \) is the permutation operator in \( \mathcal{C} \otimes \mathcal{C} \). An obvious consequence of Eq. (10) is that the trace of \( T_{x} \) as matrix in \( \mathcal{C} \) commutes for different \( \lambda \) with itself:

\[ (a(0)b(0) + b(0)a(0))P_{0} = (a(0)b(0) + b(0)a(0))P_{0} \]

This enables us to consider \( F(x) \) as a generating functional of the higher integrals of motion. Indeed, the operators

\[ H_{\lambda} = \frac{d}{dt} \ln \langle f(t) | f(t+\lambda) \rangle \]

are local for \( I < N \) and by virtue of (11) commute with one another. The first two of them have the form

\[ H_{0} = \sum_{i=1}^{N} (\mathcal{E}(i) + \mathcal{E}_N(i)) \]

The operator for a cyclic permutation \( f(0), f(1), \ldots, f(N-1) \)

is, in the limit \( N \rightarrow \infty \), connected with the momentum operator, and it is natural to call the logarithm of its eigenvalue the momentum of the state.

The linear problem (4) is connected as follows with the Heisenberg equations of motion. We rewrite the Hamiltonian (2) in terms of the basis matrices \( s_{\pm} \):

\[ H = \sum_{i} \left( x_{i} s_{+} - y_{i} s_{-} \right) \]

The Heisenberg equations of motion for the \( u^{\lambda} \),

\[ \frac{du^{\lambda}}{dt} = \sum_{\gamma} (x_{\gamma} u^{\gamma} s_{+} - y_{\gamma} u^{\gamma} s_{-}) \]

where the operator-valued matrix \( M_{\gamma} \) is given by the equations

\[ H M_{\gamma} = \sum_{\gamma} (x_{\gamma} u^{\gamma} s_{+} - y_{\gamma} u^{\gamma} s_{-}) M_{\gamma} \]

The proof of this statement follows easily from the Heisenberg equations of motion for \( L_{x} \) and the Yang-Baxter relations.

2. To make the formulae clearer and simpler we shall consider the case \( M = 3 \) (we give the final answers for arbitrary \( M \)). The 3 \( \times \) 3 matrix \( L_{x} \) can be written as follows:

\[ L_{x} = \begin{pmatrix} a(x) + b(x) & a(x) + b(x) & c(x) + b(x) \\ a(x) + b(x) & a(x) + b(x) & c(x) + b(x) \\ c(x) + b(x) & c(x) + b(x) & a(x) + b(x) \end{pmatrix} \]

where \( L_{x}^{\alpha \beta} L_{x}^{\gamma \delta} = \delta_{\alpha \gamma} \delta_{\beta \delta} \) are the basic matrix-operators in \( \mathcal{H} \). There exists in \( \mathcal{H} \) a \( \alpha \)-independent vector \( | \alpha \rangle \), action on which (element-by-element) makes \( L_{x} \) (\( x \)-dependent) triangular:

\[ L_{x} | \alpha \rangle = \begin{pmatrix} a(x) | \alpha \rangle \\ b(x) | \alpha \rangle \\ c(x) | \alpha \rangle \end{pmatrix} \]

We construct in \( \mathcal{H} \) the vector \( | \alpha \rangle \) which is the product of the vectors \( | \alpha \rangle \) for all sites (bare vacuum, ferromagnetic state),

P. P. Kulish and N. Yu. Reshetikhin

Operating on such a vector all $\frac{\partial}{\partial \lambda}$ are triangular and, hence, the transition matrix $T_\lambda(A)$ is triangular:}

$$T_\lambda(A) = \left( \begin{array}{c} \phi(A) \\ \frac{\partial}{\partial \lambda} \phi(A) \\ \frac{\partial^2}{\partial \lambda^2} \phi(A) \end{array} \right).$$

In correspondence with these results it is natural to write $T_\lambda(A)$ in block form:

$$T_\lambda(A) = \left( \begin{array}{c} \phi(A) \\ \frac{\partial}{\partial \lambda} \phi(A) \\ \frac{\partial^2}{\partial \lambda^2} \phi(A) \end{array} \right).$$

The action of the introduced operators on the generating vector $|0\rangle$ is as follows:

$$A_\lambda |0\rangle = |0\rangle, \quad B_\lambda |0\rangle = 0.$$ 

The vectors $|0\rangle$ and $|\lambda\rangle$ are linearly independent, so that it is natural to call $C_\lambda$ creation operators and $B_\lambda$ annihilation operators.

We get from (10) the commutation relations (we write down those which are of most interest for what follows):

$$[A_\lambda, B_\lambda] = C_\lambda, [B_\lambda, C_\lambda] = 0, [C_\lambda, A_\lambda] = 0.$$ 

The difference from the commutation relations for the non-linear Schrödinger equation and the spin-1/2 chain is connected with the appearance in (17) of the matrix $r(\lambda)$:

$$r(\lambda) = b(\lambda)I + a(\lambda)P,$$ 

where $P$ is the permutation operator in $C^2 \otimes C^2 (\text{in } \otimes^{M-2} C^2 \text{ for the general case } M > 3).$

In correspondence with the algebraic explanation of the Bethe Ansatz for the construction of the eigenvectors $\lambda(A)$—the trace of the monodromy matrix—we must act upon $|0\rangle$ with the creation operators $C_\lambda$ and, using Eqs. (17) and (18), obtain transcendental equations for the quasi-momenta $\lambda_i$. We consider the vector

$$|\lambda_i, \ldots, \lambda_i, F\rangle = C_{\lambda_i}(|0\rangle) \ldots C_{\lambda_i}(|0\rangle) |0\rangle F \ldots |0\rangle F,$$ 

where $F$ is the permutation operator in $C^2 \otimes C^2 (\text{in } \otimes^{M-2} C^2 \text{ for the general case } M > 3).$

In order that this is an eigenvector for $\lambda(A)$ it is necessary and sufficient that $F$ be an eigenvector for the linear operator $T^{(\lambda)}(A)$ and that the quasi-momenta satisfy the equation

$$e^{i(\lambda_i) F} = e^{i(\lambda) F}.$$ 

The operators which appear here

$$T^{(\lambda)}(A), \quad T^{(\lambda)}(A) = A_\lambda - B_\lambda, \quad T^{(\lambda)}(A) = C_\lambda, \quad T^{(\lambda)}(A) = \frac{\partial}{\partial \lambda},$$ 

completely reproduce our initial construction with the single difference that now we have a chain of $n$ sites and the linear operator $T^{(\lambda)}(A)$ in $C^2 \otimes C^2 \otimes \ldots \otimes C^2$ depends through $\lambda_i$ on the site number (inhomogeneous chain). The latter, however, does not prevent a relation which is analogous to (10) for $T^{(\lambda)}(A)$ from being satisfied for $T^{(\lambda)}(A)$:

$$r(\lambda) = T^{(\lambda)}(A) \cdot T^{(\lambda)}(A) = T^{(\lambda)}(A) \cdot T^{(\lambda)}(A) = T^{(\lambda)}(A).$$

Hence it follows that

$$|\lambda_i, \lambda_i(\lambda)\rangle = 0,$$

and Eq. (21) and the requirement that the vector $F$ be an eigenvector of $\lambda(A)$ are therefore consistent. In our case ($M = 3$) $T^{(\lambda)}(A)$ is an operator in $C^2 \otimes C^2$, i.e., a $2 \times 2$ matrix and the matrix elements are operators in $C^2 = \sum_{\lambda=0}^{\infty} T^{(\lambda)}(A) \cdot T^{(\lambda)}(A)$. Writing $T^{(\lambda)}(A)$ in the form

$$T^{(\lambda)}(A) = \left( \begin{array}{cc} c_{\lambda} & d_{\lambda} \\ e_{\lambda} & f_{\lambda} \end{array} \right),$$ 

we can construct the eigenvector $F$ in the same way as in the preceding case:

$$F = c_{\lambda} \varphi_{\lambda}(\lambda) \ldots c_{\lambda} \varphi_{\lambda}(\lambda) |0\rangle \ldots |0\rangle F.$$ 

The commutation relations between $A^{(1)}(\lambda)$, $B^{(1)}(\lambda)$, $C^{(1)}(\lambda)$, and $D^{(1)}(\lambda)$ which follow from (23) and (24) enable us to find the eigenvalues $\lambda_i$ of the operator $\lambda(A)$ and the equations for the auxiliary set of quasi-momenta $\lambda_i^{(1)}(\lambda)$:

$$\lambda_i(\lambda) = \prod_{i=1}^{n} \lambda_i^{(1)}(\lambda) = 0.$$ 

The required eigenvector $\lambda_i(\lambda)$ is thus determined by

$$\text{Sov. Phys. JETP} 53(1), \text{Jan. 1981}$$

P. P. Kulish and N. Yu. Reshetikhin

110
the \((M-1)\)-st set of quasi-momenta

\[ \{ \mathbf{a}^{(M)} \} \text{, } k = 1, \ldots , M-1 , \]

satisfying the following equations:

\[ \prod_{i=1}^{M} \frac{1}{e^{ \lambda_i + \mathbf{a}^{(M)} \cdot \mathbf{A}_i } - 1} = \prod_{i=1}^{M} \frac{1}{e^{ \lambda_i - \mathbf{a}^{(M)} \cdot \mathbf{A}_i } - 1} . \]  

\[ \prod_{i=1}^{M} \frac{1}{e^{ \lambda_i + \mathbf{a}^{(M)} \cdot \mathbf{A}_i } - 1} = \prod_{i=1}^{M} \frac{1}{e^{ \lambda_i - \mathbf{a}^{(M)} \cdot \mathbf{A}_i } - 1} . \] (30)

The eigenvalue \( \nu (\lambda) \) for given sets \( \{ \mathbf{a}^{(M)} \} \) is the following:

\[ \nu (\lambda) = \prod_{i=1}^{M} \frac{1}{e^{ \lambda_i + \mathbf{a}^{(M)} \cdot \mathbf{A}_i } - 1} \] (31)

It is convenient to introduce the variables \( \mu_i^{(M)} = \lambda_i + \frac{1}{2} i \lambda \) and then, taking the logarithm of the obtained set (30), we arrive at the equations from Sutherland's paper:

\[ 2 \nu (\lambda) = - \nu (\mu^{(M)}) + \sum_{i=1}^{M} \left( \sum_{j \neq i}^{M} \theta (\mu_i^{(M)} - \mu_j^{(M)}) - \sum_{j=1}^{M} \theta (\mu_i^{(M)} - \mu_j^{(M)}) \right) . \]

\[ 2 \nu (\lambda) = - \nu (\mu^{(M)}) + \sum_{i=1}^{M} \left( \sum_{j \neq i}^{M} \theta (\mu_i^{(M)} - \mu_j^{(M)}) - \sum_{j=1}^{M} \theta (\mu_i^{(M)} - \mu_j^{(M)}) \right) . \] (32)

Here the \( \mu_i^{(M)} \) are integers or half-integers (depending on whether the number \( n_1 - n_2 + n_3 - n_4 \) is even or odd), and \( \theta (x) = -2 \operatorname{arctan} x \).

We note also that if we use the formalism developed here we can diagonalize intrinsically the transition matrix \( \tilde{T} (z) \) from (6). To prove that it is sufficient to note that it can be written in the form of the trace of a new transition matrix:

\[ \tilde{T} (z) = \tilde{T} (z) = \sum_{j=0}^{M-1} \theta (\mu_j^{(M)} - \mu_j^{(M)}) \] (33)

such that the renormalized energy operator \( H + N \) has positive excitation energies as \( N \to \infty \).

Proceeding as in the spin-2 case \((M=2)\) and the case of the non-linear Schrödinger equation14.5, i.e., regularizing the operator \( \tilde{T} (z) \) by using its vacuum average

\[ \langle \tilde{T} (z) \rangle = \frac{1}{2} \left( \begin{array}{cc} \mu_j^{(M)} & 0 \\ 0 & 1 \end{array} \right) . \]

we are led to the following result: the generating function of the integrals of motion as \( N \to \infty \) becomes \( D (\lambda) \).

The commutation relations take the form

\[ D (\lambda) (\mu_j^{(M)}) D (\lambda) (\mu_i^{(M)}) = D (\lambda) (\mu_i^{(M)}) D (\lambda) (\mu_j^{(M)}) \] (34)

Let us say a few words about the region in which the quasi-momenta \( \lambda_j \) vary. As \( N \to \infty \) the first of the equations of the set (33) splits off and the following variants are possible. If there are no identical numbers among the \( \lambda_j \), all \( \lambda_j \) lie on the line \( \text{Im} \lambda_j = \frac{1}{2} \). However, some of the numbers \( \lambda_j \) are the same, the \( m \) quasi-momenta \( \lambda_j \) corresponding to them tend, as \( N \to \infty \), to the values

\[ \lambda_j = \frac{1}{2} \left( \lambda_k + \frac{1}{2} \right) + \frac{1}{2} \left( \cos \theta M \right) , \quad k = 1, \ldots , M . \]

The eigenvalues of the Hamiltonian \( H + N \) and of the momentum are equal to

\[ \nu (\lambda) = e^ {\mu_j^{(M)} - \frac{1}{2} \left( \lambda_k + \frac{1}{2} \right) + \frac{1}{2} \left( \cos \theta M \right) , \quad k = 1, \ldots , M . \]

If we consider the operators \( A_j (\mu) \),

\[ A_j (\mu) = C (\mu) D (\mu) \] (35)

we get, using the commutation relations (34),

\[ A_j (\mu) A_k (\mu) = e_{j \mu} (\mu) A_k (\mu) A_j (\mu) \] (36)

The operators \( A_j (\mu) \) realize a Zamolodchikov algebra10 for the magnons and \( S \) is the magnon scattering operator (one can verify this also directly by studying the coordinate representation of the wave function).

We note that we can diagonalize the Hamiltonian (2) when we add to it the operator \( H_1 \):

\[ H = \sum \left( \lambda_j + \frac{1}{2} \right) + \sum \left( \sum_{j \neq i}^{M} \theta (\mu_j^{(M)} - \mu_j^{(M)}) \right) . \]

The parameters \( \lambda_j \) describe \( M-1 \) "magnetic fields."

4. We consider now the limit \( N \to \infty \) in the case \( \varepsilon = 1 \). The vector \( |0\rangle \) corresponds to the largest eigenvalue and we are not interested in it. We denote by \( |0\rangle \) the vector corresponding to the lowest eigenvalue ("handed vacuum", antiferromagnetic state).
Following Sutherland, one can show that \( |H \rangle \) is characterized by the following occupation numbers \( n_k \) and quantum numbers \( J^0_k \):

\[
\alpha_k = \frac{N - k}{M}, \quad H^{J^0_k}_k |H \rangle = \hbar J^0_k + 1; \quad k = 1, 2, \ldots, M - 1. \tag{41}
\]

In the limit as \( N \to \infty \) the numbers \( n_k \to \infty \) and the quasi-momenta densely fill the axis. We change to a continuous characteristic of the quasi-momenta—their occupation density:

\[
p_k(h) = \lim_{\Delta \to 0} \frac{1}{\Delta} \int_{h - \Delta / 2}^{h + \Delta / 2} \frac{d\lambda}{\Delta \sqrt{\hbar^2 - \lambda^2}}, \quad \hbar^2 = \lambda.
\]

For the \( p_k(h) \) which characterize the ground state \( |1 \rangle \) we get from the set (32) a set of integral equations:

\[
p_k(h) = \sum_{l} p_l (h - k/2) e^{-i \int_{h - k/2}^{h + k/2} d\lambda \sqrt{\hbar^2 - \lambda^2}}, \quad k = 1, 2, \ldots, M - 1. \tag{43}
\]

The Fourier transform \( \rho_b(k) \) of the solution \( p_k(h) \) has the form

\[
\rho_b(k) = \int \frac{d\lambda}{\Delta} p_k(h) e^{-i \lambda k}, \quad k = 1, 2, \ldots, M - 1. \tag{45}
\]

We show that in the limit \( N \to \infty \) the ground state \( |1 \rangle \) is characterized by a continuous occupation density-

\[
\rho_b(k) = \int \frac{d\lambda}{\Delta} p_k(h) e^{-i \lambda k}, \quad k = 1, 2, \ldots, M - 1. \tag{45}
\]

The function \( f_l(h) \) obtained here is the generating function of the excitations \( C_l(\lambda) \) and the operators \( a_b(\lambda) \) have the form

\[
a_b(\lambda) = \rho_b(\lambda) - \rho_b(\lambda - \hbar/2), \quad \hbar^2 = \lambda. \tag{51}
\]

The Hamiltonian of this model

\[
H = \sum \delta(z) (\delta^+(\lambda) - \delta^+(\lambda) \sigma^-(\lambda)), \quad \hbar^2 = \lambda.
\]

was diagonalized in Refs. 14, 15 by means of the coor-

\[
F(z) = \sum \delta(z) (\delta^+(\lambda) - \delta^+(\lambda) \sigma^-(\lambda)).
\]

The operators \( \delta^+(\lambda), \delta^-(\lambda) \) do not only a spinor index \( a, b = 1, 2, \ldots, M 
\]

\[
\delta(\lambda) = \int \frac{d\lambda}{\Delta} \delta(\lambda) e^{-i \lambda k}, \quad k = 1, 2, \ldots, M - 1. \tag{45}
\]

The Hamiltonian of this model

\[
H = \int \delta(z) (\delta^+(\lambda) - \delta^+(\lambda) \sigma^-(\lambda)), \quad \hbar^2 = \lambda.
\]

There are thus \( M - 1 \) excitation branches.

We now turn to the scattering of excitations. In the limit \( N \to \infty \) we are unable to study the coordinate representation of the wave function. However, the right-hand sides of the periodicity conditions, Eqs. (32), have the meaning of the scattering phase of a magnon of type \( I \) with momentum \( \lambda^0_I \) by the other magnons. Proceeding as in Korepin's paper, we get for the phase shift of the scattering of the \( I \)-th excitation by the \( m \)-th

\[
\phi_m(\lambda, k) = \int \frac{d\lambda}{\Delta} (\delta(\lambda) - \delta(\lambda - \hbar/2)), \quad \hbar^2 = \lambda.
\]

For the corresponding S-matrix we can also construct a Zamolodchikov algebra [28].

\[
A(\lambda) = \int \frac{d\lambda}{\Delta} (\delta(\lambda) - \delta(\lambda - \hbar/2)), \quad \hbar^2 = \lambda.
\]

\[
5. \quad a_1, a_2, \ldots, a_M \]
The eigenstates of $H$ are parameterized by the momenta $k \in (0, L)$, the isotropic vector $\mathbf{F}$ which will be defined below. The eigenstates are constructed from the non-physical vacuum $\mid \Omega \rangle$ and the isotopic vector $\mathbf{F}$ for which we remove the cut-off. $g-\Omega$.

We are interested in the field-theoretic limit: $N, L, A - \infty$. In that limit the ground state is described by the densities $\rho_j(k)$ of the isotopic momentum $J_j^{(1)}$. We can find this from the set of integral equations:

$$
\rho_j(k) = \int_{-L}^{L} \frac{dk'}{2\pi} \frac{e^{iM}}{N} \left(\frac{k}{2\pi}\right) \frac{1}{\sin(kL/2)} \frac{1}{d(k-M)} \left(\frac{1}{d(k-M+j)}\right) \rho_j(k'),
$$

As in the case of an antiferromagnet, one possible kind of excitation is connected with the variation of some set of numbers $J_j^{(1)}$, and this leads in the limit considered to the appearance of a $\delta$-function in the homogeneous term in Eq. (60). The difference between the quasi-momentum density in the excited and the ground state is, as before, determined by Eq. (46), the resolvent in which we can evaluate explicitly:

$$
\rho_j(k) = \int \frac{d^3k'}{2\pi^3} \frac{e^{iM}}{N} \left(\frac{k}{2\pi}\right) \frac{1}{\sin(kL/2)} \frac{1}{d(k-M)} \left(\frac{1}{d(k-M+j)}\right) \rho_j(k').
$$

The energy and momentum of the excitation are determined as the difference between their values in the excited and the ground states:

$$
\rho_j(k) = \frac{N}{2} \int_{-L}^{L} \frac{dk'}{2\pi} \frac{e^{iM}}{N} \left(\frac{k}{2\pi}\right) \frac{1}{\sin(kL/2)} \frac{1}{d(k-M)} \left(\frac{1}{d(k-M+j)}\right) \rho_j(k'),
$$

The model studied is asymptotically free, i.e., when we remove the cut-off and in the spectrum of the model there appear massive excitations (mass generation). A and $L$.

$$
\rho_j(k) = \frac{N}{2} \int_{-L}^{L} \frac{dk'}{2\pi} \frac{e^{iM}}{N} \left(\frac{k}{2\pi}\right) \frac{1}{\sin(kL/2)} \frac{1}{d(k-M)} \left(\frac{1}{d(k-M+j)}\right) \rho_j(k'),
$$

there is also a massless branch corresponding to non-isotopic excitations.

The phases for the scattering of the massive excitations with one another are, as before, given by Eq. (49):

$$
\psi_j(k) = \psi_j(k') = \int \frac{d^3k''}{2\pi^3} \frac{e^{iM}}{N} \left(\frac{k''}{2\pi}\right) \frac{1}{\sin(k'L/2)} \frac{1}{d(k''-M)} \left(\frac{1}{d(k''-M+j)}\right) \rho_j(k''),
$$

Here $\theta = \delta_j - \delta_j$ is the difference in the speeds of the colliding particles. When $L = 1$, $M = 1$ the answer is the same as the one obtained earlier by the method of factorizing the $S$-matrix.

In conclusion it seems to us to be appropriate to consider the case of arbitrary $M$.

For finite $L$ and $N$ we shall have $M - 1$ sets of quasi-momenta $J_j^{(1)}$, $j = 1, 2, \ldots, N - 1$, which satisfy the set of Eqs. (32) with small changes in the first equations:

$$
\delta_j \sin(kL/2) \sum_{i=1}^{N} \left(\sin(kL/2) \sum_{i=1}^{N} \frac{1}{d(k-M)} \left(\frac{1}{d(k-M+j)}\right) \rho_j(k)
\right)
$$

Using this set to determine (for given $L$, $N$, $\sigma_j$, $\eta_j$, $J_j^{(1)}$) the momenta $k_j$ we find the energy and momentum of the state:

$$
E = \sum_{j=1}^{M} \epsilon_j k_j,
$$

To get a Hamiltonian which is semi-bounded from below we introduce an ultra-violet cut-off $\Lambda (< N)$ and we define the physical vacuum as the state with the lowest energy with this cut-off. The number of fermions in the ground state will then be connected with $\Lambda$ and $L$: $N < \Lambda$. We are interested in the field-theoretic limit: $N, L, \Lambda - \infty$. In that limit the ground state is described by the densities $\rho_j(k)$ of the quasi-momentum $J_j^{(1)}$. We can find this from the set of integral equations:

$$
\rho_j(k) = \int_{-L}^{L} \frac{dk'}{2\pi} \frac{e^{iM}}{N} \left(\frac{k}{2\pi}\right) \frac{1}{\sin(kL/2)} \frac{1}{d(k-M)} \left(\frac{1}{d(k-M+j)}\right) \rho_j(k'),
$$

As in the case of an antiferromagnet, one possible kind of excitation is connected with the variation of some set of numbers $J_j^{(1)}$, and this leads in the limit considered to the appearance of a $\delta$-function in the homogeneous term in Eq. (60). The difference between the quasi-momentum density in the excited and the ground state is, as before, determined by Eq. (46), the resolvent in which we can evaluate explicitly:

$$
\rho_j(k) = \int \frac{d^3k'}{2\pi^3} \frac{e^{iM}}{N} \left(\frac{k}{2\pi}\right) \frac{1}{\sin(kL/2)} \frac{1}{d(k-M)} \left(\frac{1}{d(k-M+j)}\right) \rho_j(k'),
$$

The energy and momentum of the excitation are determined as the difference between their values in the excited and the ground states:

$$
\rho_j(k) = \frac{N}{2} \int_{-L}^{L} \frac{dk'}{2\pi} \frac{e^{iM}}{N} \left(\frac{k}{2\pi}\right) \frac{1}{\sin(kL/2)} \frac{1}{d(k-M)} \left(\frac{1}{d(k-M+j)}\right) \rho_j(k'),
$$

The model studied is asymptotically free, i.e., when we remove the cut-off and in the spectrum of the model there appear massive excitations (mass generation). A and $L$.

$$
\rho_j(k) = \frac{N}{2} \int_{-L}^{L} \frac{dk'}{2\pi} \frac{e^{iM}}{N} \left(\frac{k}{2\pi}\right) \frac{1}{\sin(kL/2)} \frac{1}{d(k-M)} \left(\frac{1}{d(k-M+j)}\right) \rho_j(k'),
$$

there is also a massless branch corresponding to non-isotopic excitations.

The phases for the scattering of the massive excitations with one another are, as before, given by Eq. (49):

$$
\psi_j(k) = \psi_j(k') = \int \frac{d^3k''}{2\pi^3} \frac{e^{iM}}{N} \left(\frac{k''}{2\pi}\right) \frac{1}{\sin(k'L/2)} \frac{1}{d(k''-M)} \left(\frac{1}{d(k''-M+j)}\right) \rho_j(k''),
$$

Here $\theta = \delta_j - \delta_j$ is the difference in the speeds of the colliding particles. When $L = 1$, $M = 1$ the answer is the same as the one obtained earlier by the method of factorizing the $S$-matrix.

In conclusion it seems to us to be appropriate to consider the case of arbitrary $M$.
enumerate relatively simple generalizations of the models considered to which one can apply completely the formalism developed above.

1. The generalized Heisenberg ferromagnet with fermions. The operator of the linear problem on the lattice will be

\[ L(\lambda) = \sum_{\alpha} \langle \phi(\alpha) | H_1 | \phi(\alpha) \rangle \],

where \( \phi \) is the permutation operator in the \( \mathbb{Z}_2 \)-spaces \( C^2 \otimes C^2 \). The first \( M-1 \) elements of the base in \( C^2 \) are even and the last element is odd, i.e., the gauge is \((M-1,1)\).

2. The non-linear matrix Schrödinger equation with gauge (see also Ref. 5)

\[ i\dot{\psi} = -\frac{\lambda}{\hbar} \psi \text{ with } \psi(x) = \prod_{i=1}^{n} \phi(x_i) \quad \text{where } \psi \text{ is an } n \times m \text{ matrix, all rows of which are even and a number } p \text{ of columns is odd (i.e., those columns consist of anti-commuting elements). The } R \text{-matrix has the usual form (10), but } P \text{ is the permutation operator in } C^2 \oplus C^2 \text{ where } M = n + m, \text{ the gauge of the space is } (n + m - p, p). \text{ Apart from the initial momenta } \dot{\psi}, \text{ the eigenfunctions of the trace of the monodromy matrices are parametrized by two sequences of quasi-momenta.}

3. The matrix generalizations of the Belavin84 and Andrej and Lowenstein86 models:

\[ \mathcal{F} = \sum_{\alpha} \langle \phi(\alpha) | H_1 | \phi(\alpha) \rangle \quad \text{where } \psi(\alpha) \text{ are operators with respect to the Lorents group in the } 1 + 1 \text{ space and are } n \times m \text{ matrices, and the matrices } \mathcal{A}^\alpha \text{ are generators of the fundamental representation of the } SU(n) \text{ group. The Lagrangian is invariant under the transformations } \psi = U \psi V^\dagger, \quad V \in SU(m). \text{ The conditions for the factorization of the two-particle } S \text{-matrix, guaranteeing the possibility of applying the formalism developed here, are fulfilled if one of the equations: } 1) \psi_1 = \psi_2 = 0; \quad 2) \psi_1 = \psi_2 = \psi = 0; \quad 3) \psi_1 = \psi_2 = \psi_3 = 0. \]

We are deeply grateful to L. D. Faddeev for stimulating discussions. We are thankful to V. E. Korepin, A.G. Reiman, E.K. Sklyanin, and S.A. Tseytlin, who have helped us to get this paper appear.

\[ \text{References are given in the text.} \]

\[ \text{We do not dwell on a description of the general scheme of the quasit inverse scattering method which has been expounded in detail in Faddeev's review.} \]

\[ \text{Translated by D. ter Haar} \]