The interaction between phonons and degenerate centers (spin, pseudospin)

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Submitted 14 June 1980

The problem is considered as to how dynamic susceptibility arises in an isolated degenerate system as a result of interaction with phonons. This susceptibility consists of a narrow Lorentz peak and a smooth background. In the lowest approximation with respect to degenerate-center concentration, the centers clad with phonons lead to an anacoustically strong (in a broad temperature range) and temperature-independent decay of phonons of the viscosity type. One consequence of such a decay is that the heat conductivity below the Debye temperature is proportional to the temperature. It is also found that the interaction between the phonons and the centers becomes strong at sufficiently low temperatures. The results are practically independent of the nature of the degenerate center and are of quantum origin.

PACS numbers: 66.70. + f, 63.70. + h

1. INTRODUCTION

Two questions are considered in this work. First is the dynamics of an individual center, which possesses an internal degenerate degree of freedom by virtue of the fact that the acoustic phonons produce transitions between the degenerate levels. The second is the re- action of such phonon-clad centers on the field of the acoustic phonons. Here the specific nature of the degenerate center is almost of no consequence to us. This can be either the spin $S$ or the total angular momentum of the ion $\mathbf{J} = (J, S, \mathbf{J})$ in the absence of a degeneracy-removing constant field, or the same thing in a field which only partially removes the degeneracy. Further, this can be different cases of degenerate or almost degenerate orbital motion, combined below in the term "pseudospin." For example, a dynamic Jahn-Teller system or else a tunnel state in the case of interaction with phonons. This susceptibility consists of a narrow Lorentz peak and a smooth background.

In all cases, only one thing is of importance to us, namely that the phonons produce transitions between levels in the first order in the interaction, and not merely modulate the levels. In other the interaction with the phonons must not be written down in the form of a single term that is the product of some function of $J_\alpha$ with the phonon operator, but rather the presence is necessary of several terms which depend on different components of the vector $J_\alpha$ and which do not commute with one another. Thus, the considered phenomena have a purely quantum nature and disappear in the classical limit $J_{\alpha} \rightarrow \infty$.

This is the essential difference of the results obtained below from the usually considered effects connected with phonon broadening of local levels (see, for example, the work of Kravtov et al.1-4 and the work of Duke and Mahan5). Along with this, a very similar formulation of the problem is contained in the paper of Ivanov and Fishman, where the absorption of sound by a dynamic Jahn-Teller system is considered.

We now discuss qualitatively the picture that arises in the case in which an isolated center is clad by a field of phonons. First of all, we note that the results are practically identical for centers of different nature. The entire difference reduces to the form of the coefficients that enter into the answer, and depend on the nature of the center and the specific form of its interaction with the phonons. For the sake of convenience, we shall frequently speak of spins in what follows.

As is well known, a free spin has in a zero external field only a static susceptibility $\chi^{(0)}$ of the Debye type, while the dynamic susceptibility $\chi^{(\omega)}$ is identically equal to zero. Interaction with phonons leads to the result that each spin state survives a finite time and therefore should feel an external field of the corresponding frequency. Here the decisive role is played by the frequency dependence of the phonon field acting on the spin.

As is well known, in the low-frequency limit the spin interacts with the field of the strain tensor (for example, see the book of Al'tshuler and Kotov6). As a result, it turns out that the spectral density of the phonon field acting on the spin is proportional to $\omega^2$, which leads to the result that a Lorentzian line of finite width is not formed in lowest order perturbation theory, and the dynamic susceptibility $\chi^{(\omega)}$ has the form of a smooth background that is temperature-independent. The width of this background is of the order of the Debye frequency $\Theta$, with $\text{Im}\chi^{(\omega)} \sim \omega^{-4}$. A Lorentzian peak of finite width $\Gamma$ arises only upon taking into account the effects of the next higher perturbation theory, and it turns out that in the static limit $\omega \to 0$ the susceptibility is somewhat less than the static susceptibility of the free spin, although, as before, its smooth part is proportional to $\omega^{-4}$. The difference between the adiabatic and isothermal susceptibilities introduced by Kubo is, however, completely absent here.

It should be emphasized that such a coincidence of the two static susceptibilities does not always occur (see, for example, the work of Lazuta7). The general form of the function $\text{Im}\chi^{(\omega)}$ is shown in Fig. 1. It should be remarked that the described picture of the frequency dependence of the susceptibility is a consequence of the weakness of the spin-phonon interaction as $\omega \to 0$. Such a weak coupling takes place only in three-dimensional systems. In the case of a lower dimensional, the coupling is much greater; however, the discussion of the spin susceptibility in such sys-
where we now turn.

2. CHOICE OF MODELS; DIAGRAM TECHNIQUE

arises an anomalous temperature dependence of the temperature. Second, a term proportional to $T$, while at $T \gg 1$ it does not depend on the temperature. In the case of a Jahn-Teller system, the damping has been discussed by Ivanov and Fishman$^4$ from a somewhat different point of view. This damping leads to very important physical consequences. First, there emerges an anomalous temperature dependence of the thermal conductivity: if we neglect other mechanisms, then at $T \gg 1$ the thermal conductivity turns out to be proportional to $T$, while at $T \rightarrow 0$ it does not depend on the temperature. Second, a term proportional to $w$ appears in the spectral density of the phonon field that acts on the spin, thanks to which the spin-phonon interaction becomes strong at sufficiently low temperatures and the need arises of solving the self-consistent problem of the dynamics of the spin-phonon system. We propose to discuss the problem of this interaction in the following.

2. CHOICE OF MODELS; DIAGRAM TECHNIQUE

As was already noted in the Introduction, the nature of the degenerate center is almost of no significance to us. However, it is nevertheless convenient in what follows to carry out the entire analysis while keeping in mind some specific models, to the description of which we now turn.

1. The spin (total angular momentum) in the field of elastic oscillations is in the isotropic case,

\[ H_{\text{el}} = \frac{1}{2} \sum_{J} \epsilon_{J} J_{+} J_{-}, \]

where $J_{+}$ and $J_{-}$ are the coordinates of the spin tensor.

2. As is known, the crystalline field partially removes the degeneracy, and then, in place of the $(2J+1)$-fold degeneracy of the levels, there arise several series of levels of lower multiplicity.$^9$ If we are interested in the frequency of the phonons and in temperatures that are greater than corresponding splitting, then this splitting is unimportant and, as before, we can use the Hamiltonian (1). In the inverse limiting case, we are interested only in the lowest multiplet. It is only necessary that there be transitions between its components in the first order in the interaction. We shall not discuss the possible situations. We only note that there is no splitting in a field of cubic symmetry and in place of (1) we have

\[ H_{\text{el}} = \frac{1}{2} \sum_{J} \left( J_{+} J_{-} + J_{-} J_{+} \right) + \frac{1}{2} \sum_{J} \left( J_{+} J_{+} + J_{-} J_{-} \right) \]

(2)

3. In the case of the dynamic Jahn-Teller effect or of the tunnel state (when we can neglect in it splitting of the levels), in the case of twofold degeneracy, the Hamiltonian takes the form

\[ H_{\text{el}} = \frac{1}{2} \sum_{J} \left( X J_{+} J_{+} + X J_{-} J_{-} + Z J_{+} J_{-} \right), \]

(3)

where $X$ and $Z$ are numerical matrices.

In what follows, it is necessary that the phonons not only modulate the degenerate levels, but also produce transitions among them. For this, there should not be a direction $n$ in the $zx$ plane such that $(\theta \cdot n)$ commutes with $H_{0}$. It is not difficult to establish the fact that the corresponding commutator differs from zero in the case in which $X$ is not proportional to $Z$, i.e., if $X = \gamma Z$. In the Jahn-Teller case, the matrices $Z$ and $X$ are of the same order; there is a more detailed discussion of this, for example, in Ref. 4. In the case of the tunnel state, the matrix $X$ is small in comparison with $Z$, since it should be proportional to the penetrability of the barrier.

For what follows, it is convenient to write down the interaction (1)-(3) in unified fashion:

\[ H_{\text{int}} = \sum_{\alpha} \left( \sum_{R} \sigma_{\alpha} \epsilon_{\alpha} R_{\alpha} \right), \]

(4)

where $Q_{\alpha}$ are operators acting in the space of the functions of the degenerate state [the combinations $J_{J}$ in (1) and (2) or $\sigma_{\alpha}$ and $\epsilon_{\alpha}$ in (3)], while $\epsilon_{\alpha}$ are the combinations of components of the tensor $\epsilon_{J}$.

For calculation of the dynamic susceptibility below, we shall use the technique of Abrikosov, proposed by him for the study of the Kondo effect.$^{19}$ We shall now formulate this technique briefly in a form that is convenient for us (see also the work of Walker$^{20}$). In place of the Hamiltonian (4) we introduce a new Hamiltonian, which describes an ensemble of infinitely heavy particles situated in the immediate vicinity of the point $R_{0}$ and interacting with the phonons:

\[ H_{\text{int}} = \sum_{\alpha} \left( \sum_{R} \sigma_{\alpha} \epsilon_{\alpha} R_{\alpha} \right) + \sum_{Q} \left( Q_{\alpha} \omega_{Q} R_{\alpha} \right) R_{\alpha}, \]

(5)

Here $Q_{\alpha}$ and $\omega_{Q}$ are the creation and annihilation operators of the particles (for definiteness, fermions) at the point $R_{0}$ and $M$ is a quantum number that distinguishes the states inside the multiplet (for example,
the projection of the angular momentum. If the mass of the particle approaches infinity, then its orbital motion becomes classical and statistical are applicable for its description. Here \( \lambda \to 0 \) in (5) is the chemical potential, which tends to minus infinity, and the averages of the occupation numbers \( \langle \phi_n^+ \phi_m \rangle = \langle n \rangle \) are proportional to \( \exp(-\lambda/T) \), i.e., they tend to zero.

The ordinary temperature diagram technique is applicable to the Hamiltonian (5) (see, for example, the book of Abrikosov, Gor'kov and Dzyaloshinskii). Since \( \lambda \to -\infty \) each closed fermion loop is proportional here to \( \exp(-\lambda/T) \), i.e., it is exponentially small. Therefore, in the calculation of the susceptibility, it is necessary to consider only a single closed loop, and it is not necessary to take into account diagrams that have a large number of such loops and correspond to the interaction of two or more particles located at the point \( R_{\alpha} \). In addition, since a single particle is located at the point \( R_{\alpha} \) the susceptibility should be orthonormalized in a suitable way. As a result, we obtain the following for the susceptibility that describes the reaction to the action conjugate to the operator \( P \):

\[
\chi_\alpha = 2 \sum_{\omega} \exp(-\omega T) \langle P(\omega), P(0) \rangle.
\]

The first few perturbation-theory diagrams for \( \chi \) are shown in Fig. 2, where the lines with arrows correspond to the Green's functions of the particles \( G_{\alpha\beta}(\omega, \omega') \), and the wavy lines are the Green's functions of the particles \( \phi_{\omega}(\omega, \omega') \).

In the case of magnetic susceptibility, \( P \) is the spin or total angular momentum, while in the case of reaction to a strain it is one of the operators \( \delta \) determined by the equations (1)-(4). We emphasize once again that in the case of reaction to the interaction of two or more particles located at the point \( R_{\alpha} \), the Green's functions are also known, continuation of the function \( \delta \) for all the centers.

The first part of this program in our case is very simple. We choose the frequencies of the wave lines \( \omega \) and the complete vertex part \( \Gamma \), and the following expression exists for the retarded susceptibility:

\[
\chi_{\alpha\beta}(\omega) = \frac{i}{\pi} \frac{\delta \rho_{\alpha\beta} \exp(-\omega T)}{\omega^2 + \rho_{\alpha\beta} \exp(-\omega T)}.
\]

It is seen from the specific analytic form of the expressions corresponding to the diagrams of Fig. 3 for \( \Gamma(\omega, \omega') \), that the vertex part is an analytic function of two independent variables: \( \omega \) and \( \omega' \), on each of which it has a cut along the real axis. The general proof of this assertion can be found in the work of the author. Since the analytic properties of the Green's functions are also known, a continuation of \( \chi_{\alpha\beta}(\omega) \) in the plane of complex \( \omega \) is easily carried out (see Ref. 13 and the work of Ginsburg) and the following expression exists for the retarded susceptibility:

\[
\chi_{\alpha\beta}(\omega) = \frac{i}{\pi} \frac{\delta \rho_{\alpha\beta} \exp(-\omega T)}{\omega^2 + \rho_{\alpha\beta} \exp(-\omega T)}.
\]

Here \( G \) is the retarded Green's function, the plus and minus signs on \( \Gamma \) indicate the signs of the imaginary part of the corresponding arguments (for example, \( \Gamma^{-}(\omega, \omega') = \Gamma(\omega, \omega' + i\epsilon) \)).

The trace is taken over the projections of \( M \). In the derivation of this formula, a shift of all the energies by an amount \( \Delta \) was carried out. As a result, the factor \( \exp(-\Delta/T) \) is multiplied by the integral, which appears upon replacement of the sum over \( \omega \) by an integral.

\[ \text{S. V. Maleev} \]

FIG. 2.

FIG. 3.
was replaced by \(\exp(-x/(s\lambda/\gamma^2))\).

It remains to discuss the properties of the function \(\text{Im}\Delta(\omega)\). For noninteracting phonons it has the form

\[
\text{Im}\Delta(\omega) = -\frac{\pi}{\omega}\sum_{\alpha} \frac{\hat{d}_{ij}(\alpha)}{(2\pi)^3} \times \left(\Delta(u_{\alpha}) - (\Delta(u_{\alpha}) - \delta(u_{\alpha})).\right)
\]

where \(\alpha\) is the mass of the elementary cell, \(u_{\alpha}\) is its volume, \(\alpha\) is the index of polarization. Obviously, \(\text{Im}\Delta\) can be represented in the following fashion:

\[
\text{Im}\Delta(u_{\alpha}) = -\omega(\alpha)^{n+1}(\alpha)\text{Im}\Delta(u_{\alpha}).
\]

where \(\alpha\) is the characteristic phonon frequency, which we shall call the Debye frequency, \(s\) is the mean sound speed, defined by the equation \(3s^2 = 2\omega^2 + s^2\); \(\Delta(\omega)\) is a cutoff factor, equal to unity at \(\omega = 0\) and decreasing rapidly with increase in \(\omega\) at \(\omega = u_{\alpha}\) and if \(\alpha\) is a frequency-independent tensor, for which, in the Debye approximation, we have

\[
\Delta = \frac{1}{10} \left[ \left(\Delta^{ij}\right) \left(\Delta^{ij}\right) \right] + \frac{1}{20} \left[ \left(\Delta^{ij}_d\right) \left(\Delta^{ij}_d\right) \right]. \quad (9)
\]

3. SUSCEPTIBILITY

We shall use perturbation theory in the calculation of the susceptibility. This is valid if the dimensionless spin-phonon coupling constant \(s^2F^2(3\hbar\omega^2)^{1/4}\) is small.

Here, however, it is impossible to limit ourselves only to the first order perturbation theory, and it is necessary to take into account also that part of the contribution of second order (proportional to \(f^4\)) which leads to a finite width of the Lorentz line.

We need to know the Green's function \(g\) and the vertex part \(\Gamma_V\), for the calculation of the susceptibility. We shall first discuss the matrix structure of these quantities. We begin with the Green's function. Since \(g_{\alpha}(\omega)\) does not depend on the temperature, it can be included in \(\lambda\). Then it is not difficult to write the Green's function of first order in the form

\[
g^{(1)}(\omega) = \left\{g^{(0)}(\omega) + \frac{\partial g^{(0)}(\omega)}{\partial \omega}\right\} \frac{\partial A(\omega)}{\partial \omega} + \left\{g^{(0)}(\omega) - \frac{\partial g^{(0)}(\omega)}{\partial \omega}\right\} \frac{\partial A(\omega)}{\partial \omega}, \quad (13)
\]

We see that in first order no finite damping arises in the Green's function, i.e., as before, there is a pole at \(\omega = 0\); here, however, the residue at the pole becomes less than unity and a non-pole background increment \(g^{(1)}(\omega)\) arises. It is also evident that the retention of terms of order \(f^3\) in the denominator is an exaggeration of the accuracy.

Finite damping of \(g\) does arise only in the next order of perturbation theory, i.e., when account is taken of the second and third diagrams of Fig. 3. The contribution of the second diagram is easiest to obtain by substituting (12) in (10) and separating the terms proportional to \(f^3\), while the correction of the third diagram must be calculated by using the functions \(g_{\alpha}\). As a result, we obtain

\[
s^{(3)}(\omega) = \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \left\{g^{(0)}(\omega) - \frac{\partial g^{(0)}(\omega)}{\partial \omega}\right\} \frac{\partial A(\omega')}{\partial \omega} + \left\{g^{(0)}(\omega) + \frac{\partial g^{(0)}(\omega)}{\partial \omega}\right\} \frac{\partial A(\omega)}{\partial \omega}, \quad (14)
\]

\[
\lambda^{(3)}(\omega) = \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \left\{g^{(0)}(\omega) - \frac{\partial g^{(0)}(\omega)}{\partial \omega}\right\} \frac{\partial A(\omega')}{\partial \omega} + \left\{g^{(0)}(\omega) + \frac{\partial g^{(0)}(\omega)}{\partial \omega}\right\} \frac{\partial A(\omega)}{\partial \omega}.
\]

In the function \(s^{(3)}(\omega)\) we can separate out the terms that vanish in the limit \(\omega \to 0\); they give a contribution of order \(f^3\) to the renormalization of the residue \(Z\) and the background part of the Green's function and are of no interest in what follows. The part that is finite in the limit \(\omega \to 0\) leads first, to an additional shift in the energy zero and is also of no importance in what follows, and second, to a finite damping. As a result, the part of \(s^{(3)}(\omega)\) that is of interest to us can be written
in the form
\[ n(w) = \left( \frac{\beta}{\pi \theta^2} \right) \frac{1}{2 \pi^2} \int_0^{\infty} dw_n \Theta(w_n a) \mathcal{N}(w_n) \mathcal{L}(a \cdot w_n) \left( \frac{1}{a + x_1 \theta} - \frac{1}{a + x_2 \theta} \right), \]
where the integral in the first term is taken in the sense of the principal value.

As a result, we get for the Green's function
\[ \mathcal{G}_{11} = \frac{Z \mathcal{G}_1 \mathcal{G}_2}{\theta}, \]
where \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \) are determined by the expressions \((13a)\).

The function \( \mathcal{G}_1(w) \) has a pole with residue \( Z \) at the point \( w = -\frac{\pi}{\theta} \), where
\[ \Gamma = 2\pi \left( \frac{\theta}{\pi} \right) \sum_{\nu} \int dx \mathcal{E}(x) \psi(x) \psi^*(x) + \int dx \mathcal{E}(x) \psi(x) \psi^*(x). \]

It is clear from the physical requirement \( \Gamma > 0 \) that \( \beta \) is positive. This can be established with the help of actual calculations, using the interactions \((1) - (3)\), and a general proof can also be given, based on the definition of \( \beta \). However, we shall not concern ourselves with this. The functions \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \) possess an important property:

In the order \( g^2 \) in which we are interested, the interaction does not change the normalization in \((7)\), i.e.,
\[ N = \mathcal{G}_1 \exp(-X/T), \]
where \( \mathcal{G}_1 \) is the multiplicity of the degeneracy of the considered system. Actually, we represent the number of particles \( N \) in the form of an integral of the Green's function:
\[ N = \int dx \mathcal{E}(x) \psi(x) \psi^*(x) = \int dx \mathcal{E}(x) \psi(x) \psi^*(x). \]

The solution of this equation has the form
\[ \mathcal{G}_2 = \frac{\partial}{\partial w} \left( \mathcal{G}_1 \exp \left( \frac{-X}{T} \right) \right). \]

The situation is more complicated in the case \( \Gamma > 0 \), since at \( w = 0 \) there is an intermediate state in which \( |\mathcal{G}_2|^2 \) is under the integral sign. Here the poles of the functions \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \) contract the contour of integration and the corresponding integrals diverge as \( \Gamma \to 0 \). As a result, the integral equation for \( \mathcal{G}_2 \) takes in our approximation the form
\[ \Gamma_{12}^{0}(x + u, x) + \frac{\partial}{\partial w} \int dx_n \mathcal{N}(w_n) \mathcal{G}_1(x + u, x) \mathcal{G}_2(x + u, x) \quad (19) \]
\[ = - \mathcal{G}_1(x + u, x) \mathcal{G}_2(x + u, x). \]

We attempt to solve this equation by iteration. It is easy to show that a divergence develops in the second term at \( w \to 0 \) only in the second iteration, and it is of the same order as the divergence of the first iteration of the third term. This means that in the solution of Eq. \((21)\), the first iteration of the second term must be taken into account by perturbation theory, while the second iteration is used for reconstruction of the kernel in the third term, so that this kernel takes into account not only the diagram with the cross (fourth diagram of Fig. 3) but also the two rungs of the ladder (third diagram). As a result, Eq. \((21)\) must be rewritten in the form
\[ \Gamma_{12}^{0}(x + u, x) + \frac{\partial}{\partial w} \int dx_n \mathcal{N}(w_n) \mathcal{G}_1(x + u, x) \mathcal{G}_2(x + u, x) \quad (22) \]
\[ = - \mathcal{G}_1(x + u, x) \mathcal{G}_2(x + u, x). \]

This equation is easily solved if we note that at \( w \ll T, 6, \) the important region of integration over \( x \) in the third term is close to the poles of the last two Green's functions. Since the remaining factors under the integral change little in the case of such \( x \), we can set \( x = x - x_3 \) in them and, moreover, neglect the dependence on \( \phi \). As a result, we obtain the following equation:
\[ \Gamma_{12}^{0}(x + u, x) + \frac{\partial}{\partial w} \int dx_n \mathcal{N}(w_n) \mathcal{G}_1(x + u, x) \mathcal{G}_2(x + u, x) \]
\[ = - \mathcal{G}_1(x + u, x) \mathcal{G}_2(x + u, x). \]

The situation is more complicated in the case \( \Gamma > 0 \), since at \( w = 0 \) there is an intermediate state in which
The temperature dependence of $\Gamma_\nu$ is obviously the same as of $\Gamma_\nu$. The same dependence is possessed by the paramagnetic-resonance linewidth due to the Raman scattering of the phonons (see the work of Altshuler and Kozyr'ev14). This is not accidental, since we have actually taken into account the very same process in the limit, when the resonance frequency is equal to zero.

Everything said above about $\gamma_\nu$ can be said about $\gamma_\nu$. The substitution of (2), (13), (16) and (20) in (7) leads, after long calculations (see the Appendix), to the following rather complicated expression for the susceptibility:

$$\chi_{\nu}(\omega) = -\frac{\Pi}{\omega} \left[ \frac{\alpha^2}{2} \frac{d^{3}x}{d^{3}p} \right]^{3/2} \frac{d^{3}p'}{d^{3}p} \int d\omega \frac{dx\Lambda}{\omega - \omega - i\epsilon} \int d\omega_{1} \frac{dx\Lambda_{1}}{\omega_{1} - \omega - i\epsilon} \int d\omega_{2} \frac{dx\Lambda_{2}}{\omega_{2} - \omega - i\epsilon}$$

$$\int d\omega_{3} \frac{dx\Lambda_{3}}{\omega_{3} - \omega - i\epsilon}$$

In this expression, the first term is the usual Lorentz peak with width $\Gamma_\nu$ and with an amplitude somewhat changed by the interaction in comparison with the usual value $\frac{\Pi}{\omega}$ for the magnetic susceptibility, $\frac{\Pi}{\omega} + \frac{\delta}{3} + 1/3$. The second term is a Lorentz peak with width $\Gamma_\nu$ and with a small amplitude proportional to $\frac{\Pi}{\omega}$. Both these terms are calculated under the assumption that $\omega = \omega_{1}$ at high frequencies; allowance for them is an exaggeration in the accuracy, since they are proportional to $\frac{\Pi}{\omega}$. Finally, the last term is the background part of the susceptibility, which begins to fall off only at $\omega >> \omega_{1}$. The nature of the background is very simple. This is the amplitude of the elastic scattering of a quantum of the field conjugate to the operator $P$, accompanied by the emission of a virtual phonon. The ordinary energy dependence of this amplitude is due to the fact that, because of the degeneracy, the energy denominators corresponding to phonon-free intermediate states are equal to the energy of the quantum $\omega$. In the next section, in the discussion of phonon damping, we shall return to this question.

The following expression is obtained from (25) for the static susceptibility:

$$\chi_{\nu}(0) = -\frac{\Pi}{\omega} \left[ \frac{\alpha^2}{2} \frac{d^{3}x}{d^{3}p} \right]^{3/2} \frac{d^{3}p'}{d^{3}p} \int d\omega \frac{dx\Lambda}{\omega - \omega - i\epsilon}$$

$$\int \frac{dx\Lambda_{1}}{\omega_{1} - \omega - i\epsilon} \int \frac{dx\Lambda_{2}}{\omega_{2} - \omega - i\epsilon} \int \frac{dx\Lambda_{3}}{\omega_{3} - \omega - i\epsilon}$$

We see that this susceptibility is somewhat smaller than for the free system, and the difference increases with increase in the temperature. This is not surprising, since the phonons always generate transitions between levels (spin flip) and the intensity of this process obviously increases with the temperature.

The expression (27) is identical with the ordinary isothermal susceptibility. This can be established if we check the fulfillment of the sum rule (see, for example, Ref. 7):

$$\int d\omega \frac{dx\Lambda}{\omega - \omega - i\epsilon}$$

The corresponding calculations are given in the Appendix. For what follows, it is necessary to neglect insignificant small increments in (25) and use the following simple expression for $\chi_{\nu}(\omega)$:

$$\chi_{\nu}(\omega) = -\frac{\Pi}{\omega} \left[ \frac{\alpha^2}{2} \frac{d^{3}x}{d^{3}p} \right]^{3/2} \frac{d^{3}p'}{d^{3}p} \int d\omega \frac{dx\Lambda}{\omega - \omega - i\epsilon}$$

$$\int \frac{dx\Lambda_{1}}{\omega_{1} - \omega - i\epsilon} \int \frac{dx\Lambda_{2}}{\omega_{2} - \omega - i\epsilon} \int \frac{dx\Lambda_{3}}{\omega_{3} - \omega - i\epsilon}$$

We see that the imaginary part of the background susceptibility does not depend on the temperature and is proportional to $\omega$ at $\omega >> \omega_{1}$. Comparing the imaginary parts of the background and peak, and taking (17) and (24) into account, it is not difficult to verify that the background gives the principal contribution to the susceptibility if the condition

$$\omega = \omega_{1} = \frac{1}{\omega}$$

is satisfied. Since the sign of $\Im \chi_{\nu}$ should be the same as the sign of $\omega$, the constant $\omega_{1}$ is positive; we can say about the same that was said above about $\beta$.

The widths $\Gamma_\nu$ and $\Gamma_\nu$ as well as the amplitude of the background $\nu_\nu$ have a purely quantum nature, i.e., they vanish in the classical limit. It is easiest to understand this through the example of the interaction (1). Actually, the energy of the interaction is finite as $\Gamma_\nu \to 0$, only if $\Gamma_\nu \to 0$. But here it follows immediately from (14), (24), and (25) that $\Gamma_\nu$ and $\nu_\nu$ are proportional to $\Gamma_\nu$, and that $\Gamma_\nu \to 0$.

4. THE EFFECT OF DEGENERATE CENTERS ON THE PHONONS

Thus, we have obtained the expression (28) for the susceptibility of a center clad with phonons. If the concentration of such centers is finite, they lead to a change in the phonon spectrum. We now consider this phenomenon and the physical effects connected with it in the case in which the quantity $n\nu$ is small, where $n$ is the atomic concentration of the centers. Taking the expression (4) into account, we obtain

$$D_{\nu}(k, \omega) = \Gamma_{\nu}(k, \omega) = \frac{\alpha^2}{2} \frac{d^{3}x}{d^{3}p} \frac{d^{3}p'}{d^{3}p} \int \frac{dx\Lambda}{\omega - \omega - i\epsilon} \frac{dx\Lambda_{1}}{\omega_{1} - \omega - i\epsilon} \frac{dx\Lambda_{2}}{\omega_{2} - \omega - i\epsilon} \frac{dx\Lambda_{3}}{\omega_{3} - \omega - i\epsilon}$$

We shall now assume that the phonons are described by the Debye model and limit ourselves to consideration of the isotropic case, in which

$$\chi_{\nu} = \frac{3}{5} \frac{\alpha^2}{2} \frac{d^{3}x}{d^{3}p} \frac{d^{3}p'}{d^{3}p} \int \frac{dx\Lambda}{\omega - \omega - i\epsilon} \frac{dx\Lambda_{1}}{\omega_{1} - \omega - i\epsilon} \frac{dx\Lambda_{2}}{\omega_{2} - \omega - i\epsilon} \frac{dx\Lambda_{3}}{\omega_{3} - \omega - i\epsilon}$$

for the phonon Green's function $D$ in this approximation. In the derivation of this expression, we have taken into account the symmetry of $\chi_{\nu}$ in $pq$ and $im$.

In this expression (1), the isotropy certainly exists, and $\chi_{\nu,\nu} = 0$, and therefore, $\chi_{\nu} = \chi_{\nu}/15$.
\( \chi = \chi_{0}/10 \), where \( \chi_{0} = \chi_{\infty} \). In the general case, there is no isotropy but this obviously has no influence on the basic physical results obtained below.

We first consider the region of high frequencies, where the condition (39) is satisfied and the susceptibility is determined by the background. In this case, the phonon damping of the \( x \)-th branch \( (\kappa = i, j) \) and the renormalized sound speed \( s' \) have at \( \omega \to 0 \) the form
\[
\gamma' = \frac{\epsilon \omega^{2}}{2 \kappa^{2}} \left( 1 - \frac{\epsilon \omega^{2}}{2 \kappa^{2}} \right),
\]
(33)
where \( \kappa \) is the combination of the products \( (\Omega^{2})^{\gamma} \), entering in the expression for \( D_{\nu} \).

Thus, in this region of frequencies, the damping does not depend on the temperature and is proportional to \( \omega^{4} \). As was pointed out in the Introduction, such a damping for a Jahn-Teller system has been obtained in Ref. 4. One can connect the temperature-independent "spin viscosity" coefficients with the damping (33)
\[
\frac{\omega_{\nu}}{F(\omega)} \text{ at } \omega \ll \omega_{\nu},
\]
where \( \omega \) is the density of the body.

It follows from (33) that the scattering cross section of a phonon by a center is proportional to \( \omega^{4} \). The same frequency dependence exists also for resonance scattering of a phonon by a multi-level system if the frequency of the phonon is significantly greater than the energy of the resonances which is natural, since in our case these energies are equal to zero.\(^{11}\)

We now consider the reverse limiting case, in which the background can be neglected. If \( \omega \ll \Gamma \) the renormalized sound speed has the form
\[
\gamma' = \frac{s'^{2}}{s^{2}} \left( 1 - \frac{s'^{2}}{s^{2}} \right),
\]
(35)
where \( Q^{2} \) is the corresponding combination of the constants \( Q^{2} \). We see that upon a decrease in the temperature, one of the sound speeds vanishes and, consequently, the system loses stability. Below this temperature, a state ought to develop with nonzero mean values of the operators \( Q^{2} \). It should be emphasized that such a phase transition, due to interaction with phonons, could be realized only if the remaining forms of interaction (for example, exchange) were significantly weaker. Somewhat later we shall see that the simple theory developed above for the dynamics of an isolated center, which does not take into consideration the change in phonon damping by interaction with such centers, becomes inapplicable before the phase transition occurs. Therefore, the problem of the phase transition requires additional analysis beyond the framework of the present research.

The phonon damping at \( \omega \ll \Gamma \) also has a viscous character, but it depends strongly on the temperature. The following expression holds for it:
\[
\gamma = \frac{p_{0}^{2} m^{3} \omega^{3}}{4 \pi^{2} T},
\]
(36)
We shall not write out the simple formulas for the range of frequencies \( \Gamma < \omega < \omega_{\nu} \).

Equation (9) is the basis of the theory developed above. According to this equation, the damping \( \Im \Delta \sim \omega^{6} \) at small \( \omega \) such a frequency dependence was obtained for ideal harmonic phonons and can obviously change if allowance is made for their damping. However, the damping of the ordinary type, due to scattering by static inhomogeneities and anharmonism, depends on a high power of the phonon frequency (see, for example, the book by Kvitovskii\(^{20}\) and therefore does not change this dependence.\(^{21}\) Along with this, the contribution of the background part of the susceptibility to the damping turns out to be very important. In order to understand this, we must substitute \( \Im \Delta \) in place of the combination of \( \Delta \) functions multiplied by \( \Im A_{\gamma} \). As a result, as is not difficult to establish, the following increment appears at \( T = 0 \) and \( \omega > \omega_{\nu} \) in addition to the contribution to \( \Im \Delta \) in the form (8):
\[
\Im A_{\nu} = \frac{\omega_{\nu}^{4}}{\Delta(\omega_{\nu}^{2})},
\]
(37)
where the constant \( \Delta \) depends on the form of the phonon spectrum at large \( \Delta \) and on what combination of atomic displacements replaces the tensor \( c_{ij} \) in (4) in the microscopic treatment (see, for example, Ref. 4).

It thus turns out that at \( \omega < \omega_{\nu} \text{ at } F(\omega) \sim \omega^{3} \)
(38)
the dependence of \( \Im \Delta \) on the frequency is not cubic but linear. This means that the theory developed above is not applicable at such frequencies. In the calculation of the damping, a range of frequencies of the order of the temperature was important (see [13]). The situation if \( T > \omega_{\nu} \) but \( \Gamma < \omega_{\nu} \) and the Lorentz peak is calculated correctly, only the expression for the background turns out to be incorrect at \( \omega < \omega_{\nu} \). However, at \( T < \omega_{\nu} \) all the obtained results are inapplicable and it is necessary to solve the entire problem of the self-consistent spin-phonon system. We propose to do this later in another work. We also note that if \( F \) is not too small, the frequency \( \omega_{\nu} \) is greater than that temperature at which, according to (35), the sound speed vanishes.

It remains for us to discuss the question as to how the interaction of the phonons with the degenerate centers affects the thermal properties. The corresponding calculations are rather cumbersome and we shall give them in the Appendix. It is shown there that the basic contribution to the heat capacity is made by the background and has the form
\[
\frac{m C}{\omega^{2}} \left( \frac{\Lambda(\omega) \omega^{4}}{\lambda^{2} + \theta^{2}} \right)
\]
(39)
where the constants \( \Lambda_{\gamma} \) and \( \Lambda_{\nu} \) are given below by the formulas (A.15). We see that at all temperatures the degenerate centers lead to small corrections to the known temperature dependences for the lattice heat capacity.

The result for the heat capacity turns out to be the most interesting. As has already been mentioned above, the background part of the susceptibility leads to the result that the cross section for phonon scattering by the center is proportional to \( \omega^{4} \). This means that the mean free path length of the phonon at \( T \to 0 \)
is proportional to $T^{-4}$ and does not depend on the temperature at $T \gg 0$. This easily permits us to estimate the heat capacity if we use the well-known formula $C = \frac{1}{3}k_{B}T^{3}$. As a result, we obtain $\theta^{*} = \frac{\hbar^{2}c}{k_{B}T}$. A further discussion, with account of (33),

$$
\frac{\theta^{*}}{\omega_{0}} \ll 1,
$$

where the constants $\theta^{*}$ and $\omega_{0}$ are of the order of unity.

They can be calculated by the Kubo formula (see, for example, the book of Lifshitz and Pis'kovski15). This is done in the Appendix; their values are given there.

Formula (40), of course, is applicable only in that range of values of the parameters in which the scattering by the degenerate centers determines the free path length of the phonons. We again emphasize that the expressions (39) and (40) are valid only if $T > \omega_{0}$.

5. DISCUSSION OF THE RESULTS. SOME GENERALIZATIONS

The obtained results are based on an assumption that $f^{*}$ is small; more exactly, the deviation of $Z$ (13a) from unity should be small, which at high temperatures means smallness of the quantity $f^{*}T^{3}$. The question arises as to what values of $f^{*}$ are encountered in nature, Here, of course, too small $a_{0}$, $f^{*}$ would be uninteresting, since the considered effects become unobservable.

We first discuss the case of rare earth ions. The spin-phonon interaction of such ions has been studied experimentally (see, for example, the works of Baker and Carroll16 and Coster et al.17). It was found in these works that the value of $g$ changes in order of magnitude from hundreds of an electron volt to several electron volts. If we use the definition (11) for $a_{0}$ and $f^{*}$ and assume the atomic weight of the elementary cell to be of the order of one hundred then to this range of values of $g$ there corresponds a change of $f^{*}$ from $10^{-4}$ to several units. In particular, this means that the characteristic energy $\omega_{0}$, defined by (38) and limiting the regions of applicability of the theory from below, can be found to be very large, lying in a range easily reached by experiment. Thus, if $f^{*} \ll 1$ and $\gamma \ll 1$, then $\omega_{0} \gg 0.03\gamma$.

It should be noted that if $f^{*} \ll 1$, then the effects considered above, which are associated with the influence of the centers on the phonon, take place even for concentrated systems ($a_{0}^{-1}$) if the characteristic energy $O(\hbar)$ of interaction of the centers through the phonon is greater than the exchange energy and the energy of the magnetic-dipole forces.

In the case of Jahn-Teller systems, the value of the coupling constant $g$ is limited by the requirement that the effect be dynamic. Roughly speaking, this means that $g \ll \omega_{0}$. But here $f^{*}$ turns out to be of the order of $10^{-4}$ to $10^{-5}$ and, consequently, the considered effects much less pronounced.

We now discuss the case of a tunnel state. Obviously, the developed theory is applicable only at frequencies that are greater than the separation of levels in the neighboring wells. In the interaction (3) the second term describes the transitions induced by phonons between the potential wells, and is proportional to the amplitude of transmission through the barrier $\Delta$, which has the same exponential smallness as the tunnel splitting of the levels. In the case of interaction with phonons, according to (24) and (25), the observed quantities—the peak width and the background amplitude—are expressed in terms of the commutator and are therefore proportional to $\hbar^{2}$ and $\hbar$. Here, if the quantity $\omega_{0}$ turns out to be much greater than the level splitting, the thermal properties described above should change strongly even before this splitting begins to appear.

The quantity $g$ can be of the order of a single electron volt (see, for example, the work of Black and Halperin11) to which corresponds $f^{*} \ll 1$. Therefore, if the splitting is less than one degree, the effects connected with $a_{0}$ should be taken into account at $\hbar \ll 10^{-4}$ if $\gamma \ll 10^{-4}$. We have seen that there are cases in nature in which $f^{*} \ll 1$. Strictly speaking, our theory is inapplicable there. However, we now advance arguments of virtue of which it can be thought that the results are valid at $T = 0$ and $\omega = 0$, with some changes, also at $f^{*} \ll 1$. We begin with the width of the Lorentz peak. Account of diagrams with three phonon lines obviously leads to a contribution of the order of $\hbar^{3} / T^{3} \approx \omega^{3} / T^{3}$ and if $f^{*} / T^{3} \ll 1$, it can be neglected. Obviously one can also neglect the contribution from the more complicated diagrams. So far as the background and the amplitude of the Lorentz peak are concerned, account of the more complicated diagrams as $T \to 0$ should lead to a renormalization of the corresponding amplitudes, so that they will be functions of $f^{*}$ and of the order of unity. The contribution linear in $\omega$ to the imaginary part of the background susceptibility must be regarded as the first term of the low-energy expansion of this quantity in $\omega$.

All the results obtained above are the consequence of formula (9), according to which $\text{Im} a_{0} \propto C$. Such a dependence of $\text{Im} a_{0}$ on $\omega$ takes place only for three-dimensional systems. In the case of systems of lower dimensionality (two dimensional or quasi-one dimensional), this dependence is significantly weaker and, correspondingly, the spin-phonon interaction is much stronger at low frequencies. The dependence of $\text{Im} a_{0}$ on $\omega$ in low-temperature systems is easily determined by means of well-known formulas for the phonon frequencies of such systems (see the book by Landaau and Lifshitz18). Thus, in two-dimensional systems or quasi-two-dimensional, if we neglect layer interaction, $\text{Im} a_{0} \propto C(\omega)$ for the components of the tensor $\omega$ parallel to the layer, where $C$ is the sign function, and $\text{Im} a_{0} \propto C(\omega)$ for the mixed components. In quasi-one-dimensional systems, in the frequency range in which filament interaction is insignificant, $\text{Im} a_{0} \propto C(\omega)$, which is the mixed component. Obviously, each of the enumerated cases of behavior of $\text{Im} a_{0}$ requires special analysis. As an illustration, we note that if $\text{Im} a_{0} \propto C$, then the theory becomes logarithmic in order $f^{*}$.

In conclusion, the author expresses his thanks to A. V. Lazuta for numerous discussions of the problem of spin dynamics, as a result of which the idea for this
1. According to (20) and (24), the susceptibility can be separated into three terms; the first, \( \chi_1 \), is obtained if we replace the vertex in (7) by unity, the second, \( \chi_2 \), if we substitute in (7) terms from (20) and (24) that are linear in \( \alpha \), and, finally, the third, \( \chi_3 \), is obtained by substitution of the last term from (24) in (7).

The expression for \( \chi_1 \) can be reduced to the form

\[
\chi_1 = \sum_{n=1}^{\infty} \frac{1}{n^2} \sin(n\omega) \sin(n\omega_0) \sin(n\omega_0 - \omega).
\]

Thanks to (19), there are no increasing exponents under the integral sign in the case of negative \( x \). Taking into account that at \( \omega \approx 0 \), the susceptibility can be calculated to be proportional to \( \frac{1}{\omega^2} \), and therefore account of the contour integral, recognizing that now the function becomes small, we get

\[
\chi_1 = \frac{2\pi i}{\omega_0} \int dz \frac{g(z)}{z^2 - \omega_0^2},
\]

where we use (25) for this, since its terms proportional to \( \alpha \) calculated satisfies the sum rule (27). It is not possible to use (25) for this, since its terms proportional to \( \alpha \). After this, the terms containing \( \alpha \) vanish, since \( \Phi(0) = 0 \), and we finally get for \( \chi_2 \)

\[
\chi_2 = \frac{2\pi i}{\omega_0} \int dz \frac{g(z)}{z^2 - \omega_0^2} \left[ -\frac{N}{\omega_0^2} + \frac{1}{\omega_0} - \frac{1}{\omega_0^2} \right].
\]

Here the integrand with respect to \( x \) and \( x_0 \) has in the upper half plane, besides the poles of the functions \( g^\ast(x) \) and \( g^\ast(x_0) \), only distant singularities, connected with \( L \) and \( \pi/2 \), etc. (\( x \)); which lie above the real axis at distances of the order of \( L \) and \( T \). We raise the contour of integration to these singularities. As a result, we obtain

\[
\chi_2 = \frac{2\pi i}{\omega_0} \int \frac{dz}{z^2 - \omega_0^2} \left[ \frac{N}{\omega_0^2} + \frac{1}{\omega_0} - \frac{1}{\omega_0^2} \right].
\]

Since we replace the vertex in (7) by unity, the second, \( \chi_2 \), if we substitute in (7) terms from (20) and (24) that are linear in \( \alpha \), and, finally, the third, \( \chi_3 \), is obtained by substitution of the last term from (24) in (7). The function \( g \) can be replaced on the contour of integration \( g^\ast (x) \), after which the integrand turns out to be proportional to \( \alpha \), and therefore account of the contour integral is an exaggeration of accuracy. We consider \( l \) more detail. Just as before, we raise the contour of integration, recognizing that now the residue at the pole of \( g^\ast(x) \) is equal to zero. We then replace \( g \) on the contour by \( g^\ast (x) \), expand in \( \Gamma_T / T \); an expression is obtained in which there are no singularities up to the real axis, to which we and therefore one can return along it. As a result, we obtain

\[
\chi_3 = \frac{2\pi i}{\omega_0} \int \frac{dz}{z^2 - \omega_0^2} \left[ \frac{N}{\omega_0^2} + \frac{1}{\omega_0} - \frac{1}{\omega_0^2} \right] g^\ast (x) g^\ast (x_0).
\]

Substituting this expression in (A.3) and taking into account that when \( \omega \ll T \) and \( \omega \ll 0 \) we have

\[
g^\ast (x + iT) = (\omega + iT)^{-3/2}.
\]

we arrive at the expression

\[
\chi_3 = \frac{2\pi i}{\omega_0} \int \frac{dz}{z^2 - \omega_0^2} \left[ \frac{N}{\omega_0^2} + \frac{1}{\omega_0} - \frac{1}{\omega_0^2} \right] g^\ast (x) g^\ast (x_0).
\]

Here the integrand with respect to \( x \) and \( x_0 \) has in the upper half plane, besides the poles of the functions \( g^\ast(x) \) and \( g^\ast(x_0) \), only distant singularities, connected with \( L \) and \( \pi/2 \), etc. (\( x \)); which lie above the real axis at distances of the order of \( L \) and \( T \). We raise the contour of integration to these singularities. As a result, we obtain

\[
\chi_2 = \frac{2\pi i}{\omega_0} \int \frac{dz}{z^2 - \omega_0^2} \left[ \frac{N}{\omega_0^2} + \frac{1}{\omega_0} - \frac{1}{\omega_0^2} \right] g^\ast (x) g^\ast (x_0).
\]

2. We now prove that the susceptibility that we have calculated satisfies the sum rule (27). It is not possible to use (25) for this, since its terms proportional to \( \alpha \) at large \( w \) are calculated incorrectly. Therefore, it is necessary to carry out the proof in general form, using (7), (20), and (24). It follows from (7) that

\[
\frac{\partial}{\partial \omega} \chi_3 = 2\pi i \int \frac{dz}{z^2 - \omega_0^2} \left[ \frac{N}{\omega_0^2} + \frac{1}{\omega_0} - \frac{1}{\omega_0^2} \right] g^\ast (x) g^\ast (x_0).
\]

Here \( G(\omega) \) is a function that is regular in the upper half plane. Obviously, (27) is rewritten in the form

\[
\frac{\partial}{\partial \omega} \chi_3 = 2\pi i \int \frac{dz}{z^2 - \omega_0^2} \left[ \frac{N}{\omega_0^2} + \frac{1}{\omega_0} - \frac{1}{\omega_0^2} \right] g^\ast (x) g^\ast (x_0) \left( 1 + x / x_0 \right).
\]

Here the remaining part of \( G^\ast \) falls off like \( \omega^\alpha \) as \( \omega \to \infty \) and therefore the corresponding integral is equal to zero. It is easy to verify this by writing it down in the form

\[
\frac{\partial}{\partial \omega} \chi_3 = 2\pi i \int \frac{dz}{z^2 - \omega_0^2} \left[ \frac{N}{\omega_0^2} + \frac{1}{\omega_0} - \frac{1}{\omega_0^2} \right] g^\ast (x) g^\ast (x_0) \left( 1 + x / x_0 \right).
\]

The zero is obtained here because of the fact that the
contour of integration in each of the components can be deformed into an infinitely distant semicircle.

3. For the calculation of the heat capacity we use the formula for the correction that must be introduced in the thermodynamic potential \( \phi \) when the interaction is turned on. This formula, after replacement of the sum over the frequencies by an integral, and with account of the expressions (32), can be transformed to the form

\[
\Delta \phi = \int \frac{dx}{\pi n} \left( \sum_p \frac{1}{\pi} \frac{dx}{x} \right) \frac{dx}{x} \frac{dx}{x},
\]

(A.13)

Since we are interested in effects of order \( n \), we have replaced the total Green's function of the phonon in this expression by the free Green's function. Making use of the sum rule (27) and discarding the terms known to be temperature independent, it is not difficult to obtain the following form

\[
\Delta \phi = \int \frac{dx}{\pi n} \left( \sum_p \frac{1}{\pi} \frac{dx}{x} \right) \frac{dx}{x} \frac{dx}{x},
\]

(A.13)

If we substitute the expression for the Lorentz peak in place of \( \text{Im}\phi \) in this expression, then, because of the factor \( x^2 \) in the numerator, we do not have to take into account the damping \( \tau \), in the denominator. As a result, it turns out that this part of \( \Delta \phi \) is proportional to \( x \) and is negligibly small. The contribution from the background can be written in the form

\[
\Delta \phi = \int \frac{dx}{\pi n} \left( \sum_p \frac{1}{\pi} \frac{dx}{x} \right) \frac{dx}{x} \frac{dx}{x},
\]

(A.13)

Note added in proof (Oct. 3, 1980). In the paper of V.V. Kokshenev [J. Low Temp. Phys. 20, 373 (1979)], for the case of a mixture of ortho-hydrogen in solid para-hydrogen, a viscous-type damping of the photons was obtained, similar to (32), and the corresponding expression for the thermal conductivity is in satisfactory agreement with the experimental data.
ERRATUM


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PACS numbers: 74.50.+r, 73.40.Gk, 99.10.+g

In expression for \( I_{tot} \) in Eq. (42), the coefficient \( 16/3 \) should be replaced by \( 8/3 \). In Eq. (45) (last two lines), \( \varepsilon \) and \( \varepsilon' \) all the terms with the exception of \( V/R \) must be multiplied by \( \sqrt{3} \). The expression for \( \alpha(V) \) at \( T < \Delta \) should take the form

\[
\alpha(V) = 2 \left( \frac{1}{2} \frac{V}{kT} \right)^{2} \left( \frac{2}{3} \left[ \frac{V}{2} \right]^{2} - \frac{1}{2} \right)^{1/2} \left( \frac{V}{2} \right)^{1/2} \]

The value of \( \alpha_{c}(T) \) is \( 2\pi \) at \( T < \Delta \) and \( (1 + 4\Delta/3\pi)^{-1} \) at \( T \approx \Delta \).

Translated by J. G. Adamski