[6]A. G. Doroshkevich and I. D. Novikov, Zh. Eksp. Teor. Fiz. **74**, 3 (1978) [Sov. Phys. JETP **47**, 1 (1978)].
[7]M. Simpson and R. Penrose, Int. J. Theor. Phys. **7**, 183 (1973).
[8]J. M. McNamara, Proc. R. Soc. London Ser. **A358**, 499 (1978); **A364**, 121 (1978).
[9]Y. Gürsel, V. D. Sandberg, I. D. Novikov, and A. A. Starobinsky, Phys. Rev. D **19**, 413 (1979).
[10]V. F. Shvartsman, Zh. Eksp. Teor. Fiz. **60**, 881 (1971) [Sov. Phys. JETP **33**, 475 (1971)].
[11]R. Ruffini and J. R. Wilson, Phys. Rev. D **12**, 2959 (1975).
[12]M. A. Markov and V. P. Frolov, Teor. Mat. Fiz. **3**, 1 (1970).
[13]G. W. Gibbons, Commun. Math. Phys. **44**, 245 (1975).
[14]T. Damur and R. Ruffini, Phys. Rev. Lett. **35**, 463 (1975).
[15]D. N. Page, Phys. Rev. D **16**, 2402 (1977).
[16]A. B. Migdal, Zh. Eksp. Teor. Phys. **62**, 1621 (1972) [Sov. Phys. JETP **35**, 845 (1972)].
[17]A. I. Nikishov, Zh. Eksp. Teor. Fiz. **57**, 1210 (1969) [Sov. Phys. JETP **30**, 660 (1970)].
[18]V. B. Berestetskiĭ, E. M. Lifshitz, and L. P. Pitaevskiĭ, Relyativistskaya kvantovaya teoriya, Part 1, Nauka, Moscow (1968); English translation: Relativistic Quantum Theory, Oxford (1971).
[19]E. M. Lifshitz and L. P. Pitaevskiĭ, Relyativistskaya kvantovaya teoriya (Relativistic Quantum Theory), Part 2, Nauka, Moscow (1971).
[20]Ya. B. Zel'dovich, in: Magic without Magic: John Archibald Wheeler (ed. J. Klauder), W. H. Freeman and Co., San Francisco (1972), p. 277.
[21]L. Parker and J. Tiomno, Astrophys. J. **178**, 809 (1972).
[22]Y. Gürsel, V. D. Sandberg, I. D. Novikov, and A. A. Starobinsky, Phys. Rev. D **20**, 6 (1979).
[23]I. D. Novikov, Pis'ma Zh. Eksp. Teor. Fiz. **3**, 223 (1966) [JETP Lett. **3**, 142 (1966)].
[24]V. de la Cruz and W. Israel, Nuovo Cimento **A51**, 744 (1967).
[25]J. M. Bardeen, Bull Am. Phys. Soc. **13**, 41 (1968).

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Isotropic cosmological models determined by vacuum quantum effects

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The evolution of cosmological models with scalar or spinor quantized fields is studied. In the class of spatially homogeneous isotropic models, all self-consistent models are found in which the metric is determined by vacuum quantum effects of massless fields. It is shown that the obtained results are also valid for massive fields.

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1. INTRODUCTION

One of the main problems of cosmology is the description of the evolution of the Universe near the cosmological singularity. As follows from the Penrose–Hawking theorems,[1] if the dominant energy conditions are satisfied for the matter that determines the metric it is impossible to avoid the occurrence of singularities in classical general relativity. At the same time, it is known that allowance for quantum effects leads to violation of the dominant energy conditions.[2] At the present time, a completed theory of quantization of the gravitational field does not yet exist, and it is therefore expedient to consider the part played by quantum effects in the framework of a semiclassical scheme, in which the gravitational field is classical but the fields of particles are second quantized. Such a scheme corresponds to the single-loop approximation to a fully quantized theory.

On dimensional grounds, one can conclude that the semiclassical approach is valid for a gravitational field characterized by a curvature that is small compared with the Planck curvature, $\rho \ll G^{-1/2} \sim 10^{33}\ \mathrm{cm}^{-1}$ (G is the gravitational constant; we use a system of units in which $\hbar = c = 1$). If it is found that certain quantization effects of the matter fields are sufficient to eliminate the singularity, it can be assumed that this result is also true when quantization of the gravitational field is taken into account.

In the present paper, we solve the self-consistent problem of the evolution of isotropic cosmological models with quantized scalar or spinor fields. The condition of self-consistency takes the form that the external gravitational field produces a vacuum energy density and pressure of the quantized scalar or spinor fields that are required for the creation of this gravitational field in accordance with the Einstein equation. As will be shown below, in the class of homogeneous isotropic cosmological models there exist models that are self-consistent in this sense and do not possess singularities. This makes it possible to interpret the occurrence of the Universe as a manifestation of an instability of the vacuum state of a quantized field.

In Sec. 2 of the present paper, we formulate the equations of self-consistency of the cosmological models with scalar or spinor quantized fields. In Sec. 3, the self-consistency equations are solved in the case of massless fields, and we find all isotropic models determined by vacuum quantum effects. These models include de Sitter models of Planck dimensions and models that coincide asymptotically with Milne's model. In

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Sec. 4, we find the corrections to the results of Sec. 3 that arise when allowance is made for a mass of the field. We show that for realistic masses, satisfying the condition $Gm^2 \ll 1$, the obtained models remain approximately self-consistent.

2. SELF-CONSISTENCY CONDITIONS FOR ISOTROPIC COSMOLOGICAL MODELS

We consider a second-quantized scalar or spinor field in a homogeneous isotropic space with metric

$$ds^2 = g_{ik}dx^i dx^k = a^2(\eta)\,[d\eta^2 - \gamma_{\alpha\beta}dx^\alpha dx^\beta], \tag{1}$$

where $\gamma_{\alpha\beta}$ is the metric tensor of the three-space of constant curvature $\varkappa = +1, 0, -1$, and η is the conformal time, which is related to the synchronous proper time t by $\eta = \int dt/a$.

Suppose that at the time η_0 the quantized field is in the vacuum state $|0\rangle$. Because the metric (1) is nonstationary, for $\eta \neq \eta_0$ the state $|0\rangle$ will no longer be a vacuum state. In Refs. 3–5, by means of various methods of regularization, mutually consistent finite expressions were obtained for the expectation values of the operator of the energy–momentum tensor of massless quantized fields with spin σ with respect to the state $|0\rangle$: $\langle T_{ik}^{(\sigma)}\rangle_{ren}^{(m=0)}$. These expressions have a local nature and are interpreted as polarization of the vacuum by the gravitational field. In Ref. 6, by means of the method of n-wave regularization proposed in Ref. 7, expressions were obtained for the total energy–momentum tensor $\langle T_{ik}^{(\sigma)}\rangle_{ren}$ of quantized fields with mass, these expressions containing both local terms identical to those obtained in Refs. 3–5 as well as nonlocal terms describing particles produced by the gravitational field.

In accordance with what we have said in the Introduction, the self-consistent cosmological models must satisfy the equation

$$R_{ik} - {}^1/_2 R g_{ik} = -8\pi G \langle T_{ik}^{(\sigma)}\rangle_{ren}. \tag{2}$$

Here it is assumed that the metric of space-time is entirely determined by the vacuum quantum effects of the scalar or spinor fields (by vacuum polarization and particle production from the vacuum). In their turn, these quantum effects are entirely produced by the gravitational field corresponding to the given metric. An attempt to solve Eq. (2) in the quasi-Euclidean case for a scalar field ($\sigma = 0$) was made in Ref. 8. However, because of the uneliminated indeterminate form on the right-hand side of (2) a contradictory result was obtained in Ref. 8.

As is well known, for fixed metric (1) the vacuum expectation values of the energy–momentum tensor have the form

$$\langle T_{ik}^{(\sigma)}\rangle_{ren} = \langle T_{ik}^{(\sigma)}\rangle' + \langle T_{ik}^{(\sigma)}\rangle''. \tag{3}$$

Here, $\langle T_{ik}^{(\sigma)}\rangle'$ does not depend on the mass of the field and is calculated in Refs. 3–6 for arbitrary expansion law $a(\eta)$:

$$\langle T_{ik}^{(\sigma)}\rangle' = \frac{1}{1440\pi^2}(\alpha_\sigma\, {}^{(3)}H_{ik} + \beta_\sigma\, {}^{(1)}H_{ik} + \delta_{\varkappa,-1}\gamma_\sigma J_{ik}), \tag{4}$$

where we have introduced the notation

$$\begin{gathered}
{}^{(3)}H_{ik} = R_i^l R_{lk} - {}^2/_3 R R_{ik} - {}^1/_2 (R_{lm}R^{lm} - {}^1/_2 R^2) g_{ik},\\
{}^{(1)}H_{ik} = 2(\nabla_i\nabla_k R - g_{ik}\nabla_l\nabla^l R) + 2R(R_{ik} - {}^1/_4 R g_{ik}),\\
\alpha_0 = 1, \quad \beta_0 = -{}^1/_6, \quad \alpha_{1/2} = {}^{11}/_2, \quad \beta_{1/2} = -{}^1/_2,\\
\gamma_0 = -6, \quad \gamma_{1/2} = -{}^{51}/_2.
\end{gathered} \tag{5}$$

By J_{ik} we denote the tensor that in the metric (1) has the components

$$J_{ik} = a^{-2}\begin{pmatrix} 1 & 0 \\ 0 & -{}^1/_3\gamma_{\alpha\beta}\end{pmatrix}.$$

The second term in (3) contains, in general, terms that are nonlocal with respect to η and vanish for $m = 0$.

As can be seen from (5), ${}^{(1)}H_{ik}$ contains derivatives of third and fourth orders in g_{ik}. Substitution of such terms in the right-hand side of Eqs. (2) radically changes the nature of the solutions, rendering them unphysical.[9] Therefore, we eliminate from (4) the term containing ${}^{(1)}H_{ik}$ by means of a finite renormalization of the constant in front of R^2 in the unrenormalized Lagrangian of the gravitational field. This is possible, since

$${}^{(1)}H_{ik} = \frac{2}{\sqrt{-g}}\frac{\delta}{\delta g^{ik}}\int d^4x\sqrt{-g}\,R^2.$$

At the same time, the tensor ${}^{(3)}H_{ik}$ introduced in Ref. 10 cannot be obtained in the general case by variation of a local action and is conservative only in conformally flat spaces.

Going over to the self-consistent problem (2) and using (3), (4), and the explicit expression for $\langle T_{ik}^{(0)}\rangle''$ (Ref. 11), we obtain the 00 component of the Einstein equations ($\sigma = 0$):

$$\begin{aligned}
3\left(\frac{\dot a^2}{a^2} + \varkappa\right) = 8\pi G\Bigg\{&\frac{1}{480\pi^2 a^2}\left(\frac{\dot a^4}{a^4} + 2\varkappa\frac{\dot a^2}{a^2} + \varkappa^2 - 2\delta_{\varkappa,-1}\right)\\
&+ \frac{m^2}{2\pi^2 a^2}\int_0^\infty d\lambda\,\lambda^2\left[\int_{\eta_0} d\eta' a(\eta')\dot a(\eta')\left(|g_\lambda(\eta')|^2 - \frac{1}{\omega(\eta')}\right) - \frac{m^2 a^2 \dot a^2}{8\omega^5}\right]\Bigg\}.
\end{aligned} \tag{6}$$

Here, the dot denotes the derivative with respect to η, $\omega^2 = \lambda^2 + m^2 a^2$, λ is the dimensionless momentum, and $g_\lambda(\eta)$ is a solution of the oscillator equation

$$\ddot g_\lambda(\eta) + \omega^2 g_\lambda(\eta) = 0, \tag{7}$$

which is obtained from the Klein–Gordon–Fock equation with conformal coupling after separation of the spatial variables. The function $g_\lambda(\eta)$ is fixed by the requirement of a positive frequency at the initial time η_0. This requirement is equivalent to the condition of diagonality of the Hamiltonian of the quantized field $H(\eta_0)$, whose ground state is the vacuum $|0\rangle$ (Ref. 12). For $\varkappa = +1$, the integration in (6) can be replaced by summation over $\lambda = 1, 2, \ldots$.

We do not require the spatial components of (2), since they reduce by virtue of the fact that the energy–momentum tensor (4) is conservative to the equations that are obtained from (6) by differentiation with respect to η.

For a spinor field, using (4) and the explicit expressions for the nonlocal terms,[6,13] we write the 00 components of Eqs. (2) in the form

$$3\left(\frac{\dot a^2}{a^2}+\varkappa\right)=8\pi G\left\{\frac{11}{960\pi^2}\left(\frac{\dot a^4}{a^4}+2\varkappa\frac{\dot a^2}{a^2}+\varkappa^2-\frac{17}{11}\delta_{\varkappa,-1}\right)\right.$$

$$\left.+\frac{m}{\pi^2a^2}\int_0^\infty d\lambda\left[\left(\lambda^2-\frac{\varkappa}{4}\right)\int_{\eta_0}^{\eta}d\eta'\dot a(\eta')\left(|f_{\lambda+}(\eta')|^2-1+\frac{ma(\eta')}{\omega(\eta')}\right)-\frac{m\dot a^2\lambda^4}{8\omega^5}\right]\right\}. \tag{8}$$

The functions $f_{\lambda\pm}$ satisfy the equations

$$\dot f_{\lambda\pm}+i\lambda f_{\lambda\mp}\pm ima(\eta)f_{\lambda\pm}=0, \tag{9}$$

which are obtained from the generally covariant generalization of the Dirac equation after separation of the spatial variable. In the case $\varkappa=+1$, the integration in (8) is replaced by summation over $\lambda=3/2, 5/2, \ldots$.

As can be seen from (6) and (8), the self-consistency conditions are integrodifferential equations for the unknown scale factor $a(\eta)$, containing it both explicitly and implicitly (the functions f and g depend on it). We shall seek solutions to (6) and (8) in two stages. First, we consider the case $m=0$, when nonlocal effects are absent, as a result of which (6) and (8) are transformed into ordinary differential equations for $a(\eta)$. We then calculate the corrections that arise for $m\neq 0$.

3. MASSLESS FIELDS

For $m=0$ and $\varkappa=0,+1$, the self-consistency conditions (6) and (8) take the form

$$C^2+\varkappa=\alpha_\sigma\tilde a^{-2}(C^2+\varkappa)^2, \tag{10}$$

where $\tilde a=a(180\pi/G)^{1/2}$, $C=(\ln\tilde a)$, and α_σ are defined in (5).

For $\varkappa=0$, Eq. (10) has not only the trivial solution $a(\eta)=$ const corresponding to Minkowski space-time but also the nontrivial

$$\tilde a(\eta)=\alpha_\sigma^{1/2}/\eta. \tag{11}$$

The solution (11) describes a de Sitter space of the first kind in orispherical coordinates[1] with curvature

$$R=2160\pi/G\alpha_\sigma=\text{const}>0.$$

For $\varkappa=+1$, the solution of (10) is

$$\tilde a(\eta)=\alpha_\sigma^{1/2}/\cos\eta.$$

This metric corresponds to the same de Sitter space (with spherical spatial sections).

It is obvious that the obtained models have sizes of the order of the Planck length. This means that they can be significantly changed when allowance is made for quantization of the gravitational field.

We now consider the case $\varkappa=-1$. The self-consistency conditions (6) and (8) take the form

$$C^2-1=\alpha_\sigma\tilde a^{-2}[(C^2-1)^2+\gamma_\sigma/3\alpha_\sigma], \tag{12}$$

or

$$\left|\frac{\dot{\tilde a}}{\tilde a}\right|=\frac{1}{\sqrt2}\left[2+\frac{\tilde a^2}{\alpha_\sigma}\pm\left(\frac{\tilde a^4}{\alpha_\sigma^2}+b_\sigma\right)^{1/2}\right]^{1/2}, \tag{13}$$

where $b_\sigma=4\gamma_\sigma/3\alpha_\sigma$.

The solutions of Eq. (13) for the + and − signs in front of the radical are

$$\eta=\frac{\sqrt2}{b_\sigma^{1/4}}\left[\frac{1}{(c_\sigma+1)^{1/2}}\operatorname{Arth}\left(\frac{c_\sigma+1}{c_\sigma+u_+}\right)^{1/2}\right.$$

$$\left.-\frac{1}{c_\sigma^{1/2}}\operatorname{Arth}\left(\frac{c_\sigma}{c_\sigma+u_+}\right)^{1/2}+\frac{1}{(1-c_\sigma)^{1/2}}\operatorname{arctg}\left(\frac{1-c_\sigma}{c_\sigma+u_+}\right)^{1/2}\right], \tag{14}$$

$$\eta=\frac{\sqrt2}{b_\sigma^{1/4}}\left[\frac{1}{c_\sigma^{1/2}}\operatorname{Arth}\left(\frac{c_\sigma-u_-}{c_\sigma}\right)^{1/2}-\frac{1}{(c_\sigma+1)^{1/2}}\operatorname{Arth}\left(\frac{c_\sigma-u_-}{c_\sigma+1}\right)^{1/2}\right.$$

$$\left.-\frac{1}{(1-c_\sigma)^{1/2}}\operatorname{arctg}\left(\frac{1-c_\sigma}{c_\sigma-u_-}\right)^{1/2}+\frac{\pi}{2(1-c_\sigma)^{1/2}}\right], \tag{15}$$

where $c_0=1/\sqrt2$, $c_{1/2}=\sqrt{11/17}$ and we have introduced the notation

$$u_\pm=(x^2+1)^{1/2}\pm x,\quad x=\tilde a^2/\alpha_\sigma b_\sigma^{1/2}.$$

The constants of integration are chosen such that the time η varies from 0 to $+\infty$. (The case of vector field is considered in Ref. 14.)

The asymptotic behavior of the solutions (14) for small and large η is given by

$$\tilde a\sim\begin{cases}\alpha_\sigma^{1/2}/\eta, & \eta\ll1\\ a_\sigma\exp(-k_\sigma\eta), & \eta\gg1\end{cases}, \tag{16}$$

where $k_0=1.554$, $k_{1/2}=1.498$, $a_0=3.458$, $a_{1/2}=7.798$.

The corresponding values of the scalar curvature are

$$\frac{G}{180\pi}R\sim\begin{cases}12/\alpha_\sigma, & \eta\ll1\\ 3b_\sigma^{1/2}/\tilde a^2, & \eta\gg1\end{cases}. \tag{17}$$

It can be seen from (16) and (17) that models with the scale factors (14) for $\eta=0$ have sizes of the order of the Planck length and contract to a point as η increases to $+\infty$. The effects of quantization of the gravitational field ignored here must have an appreciable influence on the evolution of such models.

The asymptotic behavior of the solution (15) is described by the formulas

$$\tilde a\sim\begin{cases}d_\sigma+e_\sigma\eta^2, & \eta\ll1\\ f_\sigma\exp\eta, & \eta\gg1\end{cases}, \tag{18}$$

where $d_0=1$, $e_0=1/6$, $d_{1/2}=\sqrt3$, $e_{1/2}=0.107$, $f_0=0.245$, $f_{1/2}=0.211$.

The behavior of the scalar curvature is determined by the expression

$$\frac{G}{180\pi}R\sim\begin{cases}-r_\sigma, & \eta\ll1\\ -h_\sigma/\tilde a^6, & \eta\gg1\end{cases}, \tag{19}$$

where $r_0=4$, $r_{1/2}=1.571$, $h_0=48$, $h_{1/2}=4769$.

As follows from (18) and (19), the models defined by formula (15) expand without limit as η varies from 0 to $+\infty$, beginning at $\eta=0$ with a size of the order of the Planck length. It can be assumed that for such models when $\eta\gtrsim1$ the effects of quantization of the gravitational field are unimportant. According to (18), for $\eta\gg1$ (in fact, it is sufficient that $\eta\gtrsim3$) the obtained nonsingular models are described by the Milne metric [in terms of the synchronous proper time $a(t)=t$]. Thus, at large η the model approaches locally to Minkowski space.

4. VACUUM QUANTUM EFFECTS OF MASSIVE FIELDS

To take into account the corrections to the results obtained above that arise when $m\neq0$, we consider first a massive scalar field in the de Sitter space with the metric (1) and the scale factor

$$a(\eta)=a_0/\eta \tag{20}$$

(a_0 is for the time being assumed to be an arbitrary parameter). We shall specify the vacuum state of the field by the requirement that the Hamiltonian be diagonal as $\eta\to+\infty$; this choice of the vacuum agrees with the one adopted in Refs. 15–17.

To calculate the nonlocal terms on the right-hand side of (6), it is necessary to solve the oscillator equation (7) with the scale factor (20). Its solution, determined by the positive-frequency requirement as $\eta\to+\infty$, is

$$g_\lambda(\eta)=(\pi\eta/2)^{1/2}H_\nu^{(1)}(\lambda\eta), \tag{21}$$

where $H_\nu^{(1)}(z)$ is a Hankel function of the first kind, and $\nu=[(1/4)-\mu^2]^{1/2}$, $\mu\equiv ma_0$.

The terms in (3) that do not depend on the mass of the field take the following form in the metric with the scale factor (20):

$$\langle T_{ik}^{(0)}\rangle'=\frac{1}{96\pi^2}\frac{1}{5a_0^4}g_{ik}. \tag{22}$$

We now calculate $\langle T_{ik}^{(0)}\rangle''$ [the terms with the integral over λ in (6)]. Substituting in (6) the explicit expression (21) for the solutions $g_\lambda(\eta)$ and reversing the order of integration with respect to λ and η', we obtain

$$\langle T_{00}^{(0)}\rangle''=\frac{m^2}{8\pi^2\eta^2}\left[\int_0^\infty dt\,t^2\varphi(t)-\frac{1}{6}\right]\equiv\frac{m^2}{8\pi^2\eta^2}\left(I-\frac{1}{6}\right), \tag{23}$$

where

$$\varphi(t)=\frac{\pi}{2}H_\nu^{(1)}(t)H_\nu^{(2)}(t)-(t^2+\mu^2)^{-1/2}. \tag{24}$$

The integrals of each of the terms in (24) separately diverge. Using a procedure analogous to dimensional regularization, for the integral I in (23) we find

$$I={}^1/_4\mu^2[\psi(-{}^1/_2+\nu)+\psi(-{}^1/_2-\nu)-\ln\mu^2], \tag{25}$$

where $\psi(z)$ is the logarithmic derivative of the Γ function.

Substituting (25) in (23) and determining the three-space components of the energy–momentum tensor from the condition of its conservativeness, we obtain, using (3) and (22),

$$\langle T_{ik}^{(0)}\rangle_{ren}=\frac{m^4}{32\pi^2}\left[\psi\left(-\frac{1}{2}+\nu\right)+\psi\left(-\frac{1}{2}-\nu\right)-\ln\mu^2-\frac{2}{3}\mu^{-2}+\frac{1}{15}\mu^{-4}\right]g_{ik}, \tag{26}$$

where g_{ik} denotes the metric tensor of the de Sitter space. As can be seen from (26), the geometrical structure of $\langle T_{ik}^{(0)}\rangle_{ren}$ reflects the symmetry of the four-space. The result (26) was obtained earlier in Ref. 15 by the generalized ζ function method and in Ref. 16 by the method of covariant point splitting. For scalar field with minimal connection a result analogous to (26) was obtained in Ref. 17 by the method of adiabatic regularization.

We now consider a spinor field. The solutions of Eqs. (9) in the metric (1), (20) corresponding to our choice of the vacuum have the form

$$f_{\lambda\pm}(\eta)=\pm i\left(\frac{\pi\lambda\eta}{2}\right)^{1/2}\exp\left(\mp\frac{\pi\mu}{2}\right)H^{(1)}_{1/2\pm i\mu}(\lambda\eta). \tag{27}$$

Making a calculation using (27) analogous to the scalar case, for the quantity in the curly brackets (8) we have

$$\langle T_{ik}^{(1/2)}\rangle_{ren}=\frac{m^4}{16\pi^2}\left\{(1+\mu^{-2})[\ln\mu^2-2\,\mathrm{Re}\,\psi(i\mu)]+\frac{\mu^{-2}}{6}+\frac{11}{60}\mu^{-4}\right\}g_{ik}. \tag{28}$$

We consider limiting cases of the results (26) and (28). For spaces with small scalar curvature $R=12/a_0^2\ll m^2$ (the radius of curvature large compared with the Compton length) $\mu\gg1$ and

$$\langle T_{ik}^{(\sigma)}\rangle_{ren}\approx-A_\sigma\frac{m^4}{\pi^2(ma_0)^6}g_{ik}, \tag{29}$$

where $A_0=1/1260$ and $A_{1/2}=31/20160$.

The self-consistent solution (11) corresponds, in contrast, to a very large curvature $R\gg m^2$, since for the masses of all known elementary particles $Gm^2\ll1$. In this case, we obtain

$$\langle T_{ik}^{(\sigma)}\rangle_{ren}\approx B_\sigma R^2g_{ik}/\pi^2, \tag{30}$$

where $B_0=1/69120$ and $B_{1/2}=11/138240$.

Since the result (30) is exact for fields with $m=0$, we may conclude that allowance for mass does not change the conclusion drawn in the preceding section concerning the self-consistency of the de Sitter models determined by the vacuum quantum effects of the massless fields.

We now consider the influence of the production of massive particles on the self-consistent solutions with $\varkappa=-1$. With regard to the solution (14), since $Gm^2\ll1$ for all known elementary particles, it is clear in advance that the corrections associated with nonzero mass are quite negligible. Let us consider in more detail the corrections to the solution (15).

For $t\lesssim t_m=m^{-1}$, calculation of the corrections to $\langle T_{ik}^{(\sigma)}\rangle_{ren}$ in accordance with the method of Refs. 11 and 13 and using the asymptotic behaviors (18) gives

$$\langle T_{00}^{(\sigma)}\rangle''\sim\begin{cases}m^2\eta^2, & 0<\eta\leqslant1\\ m^2, & 1\leqslant\eta\leqslant50\end{cases}. \tag{31}$$

At the same time, the local terms of the energy–momentum tensor on the right-hand sides of (6) and (8) have the order

$$\langle T_{00}^{(\sigma)}\rangle'\sim\begin{cases}G^{-1}, & 0<\eta\leqslant1\\ G^{-1}\exp(-2\eta)=(Gt^2)^{-1}, & \eta\geqslant1\end{cases}. \tag{32}$$

It is obvious that for $t\ll t_m$ the contribution of the quantities (31) can be ignored compared with (32).

As is shown in Refs. 11 and 13, the production of massive particles occurs most intensively at $t\sim t_m$ and for $t\gtrsim t_m$

$$\langle T_{00}^{(\sigma)}\rangle''=\delta_\sigma m/a(t), \tag{33}$$

where $\delta_\sigma=\mathrm{const}$. We shall assume that all the massive particles are produced at the time $t=t_m$ and that from this time their energy–momentum tensor (33) begins to influence the expansion of the Universe. Then Eq. (12) is replaced by

$$C^2-1=\alpha_\sigma\tilde{a}^{-2}[(C^2-1)^2+\gamma_\sigma/3\alpha_\sigma+\theta_\sigma ma], \tag{34}$$

where $\theta_\sigma=480\pi^2\delta_\sigma/\alpha_\sigma$. Solving this equation, we find asymptotically in the limit $t\gg t_m$

$$a(t)\approx t+a_1 Gm\ln(mt), \tag{35}$$

and the constant satisfies $a_1\sim 1$.

As can be seen from (35), the departure of the metric from the Milne metric due to the nonvanishing mass is in this case too negligibly small. Thus, all the models we have found remain self-consistent for massive fields as well.

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[1]S. W. Hawking and G. F. R. Ellis, The Large Scale Structure of Space-Time, CUP (1978) (Russian translation published by Mir, Moscow (1977)).
[2]Ya. B. Zeldovic and L. P. Pitaevskii, Commun. Math. Phys. **23**, 185 (1971).
[3]P. C. W. Davies, S. A. Fulling, S. M. Christensen, and T. S. Bunch, Ann. Phys. **109**, 108 (1977).
[4]T. S. Bunch, Phys. Rev. **D18**, 1844 (1978).
[5]T. S. Bunch, J. Phys. **A11**, 603 (1978).
[6]S. G. Mamayev and V. M. Mostepanenko, Phys. Lett. **A67**, 165 (1978).
[7]Ya. B. Zel'dovich and A. A. Starobinskiĭ, Zh. Eksp. Teor. Fiz. **61**, 2161 (1971) [Sov. Phys. JETP **34**, 1159 (1972)].
[8]R. M. Wald, Ann. Phys. **110**, 472 (1978).
[9]G. T. Horowitz and R. M. Wald, Phys. Rev. **D17**, 414 (1978).
[10]V. L. Ginzburg, D. A. Kirzhnits, and A. A. Lyubushin, Zh. Eksp. Teor. Fiz. **60**, 451 (1971) [Sov. Phys. JETP **33**, 242 (1971)].
[11]S. G. Mamaev, V. M. Mostepanenko, and A. A. Starobinskii, Zh. Eksp. Teor. Fiz. **70**, 1577 (1976) [Sov. Phys. JETP **43**, 823 (1976)].
[12]A. A. Grib, S. G. Mamayev, and V. M. Mostepanenko, Gen. Relat. Grav. **7**, 535 (1976).
[13]S. G. Mamaev and V. M. Mostepanenko, Yad. Fiz. **28**, 1640 (1978) [Sov. J. Nucl. Phys. **28**, 845 (1978)].
[14]V. M. Mostepanenko, Yad. Fiz. **31**, No. 3 1980 [*sic*].
[15]J. S. Dowker and R. Critchley, Phys. Rev. **D16**, 3390 (1977).
[16]T. S. Bunch and P. C. W. Davies, Proc. R. Soc. London Ser. **A360**, 117 (1978).
[17]N. D. Birrel, Proc. R. Soc. London Ser. **A361**, 513 (1978).

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Properties of a pion condensate in a magnetic field

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A study is made of the behavior in a magnetic field of a pion condensate that is either homogeneous (with characteristic wave vector $k_0\approx 0$) or inhomogeneous (the physically interesting case of a pion condensate in a nucleon medium has $k_0\sim p_F$, where p_F is the nucleon Fermi momentum). An expression is obtained for the spatial distribution of the pion and magnetic fields in a medium with a pion condensate in an external homogeneous magnetic field H. It is shown that the pion condensate is a superconductor of the second type with Ginzburg-Landau parameter $\varkappa\gg 1$. The structure of the mixed state of the system is studied. For a homogeneous condensate, it is the same as for a metallic superconductor of the second type. For an inhomogeneous condensate in the range of variation of the external magnetic field $H_{c1}<H<H'_{c2}$ (where $H_{c1}\sim H_c/\sqrt{\varkappa}$ is the lower critical field, and $H'_{c2}\sim H_c$, where H_c is the thermodynamic critical field) plane layers of the normal phase arise. These layers are parallel to the plane $(\mathbf{k}_0, \mathbf{H})$ $(\mathbf{k}_0 \perp \mathbf{H})$. At values of the magnetic field in the region $H'_{c2}<H<H_{c2}$, where H_{c2} is the upper critical field, the structure of the mixed state for an inhomogeneous condensate is the same as for a homogeneous condensate. It is shown that the value of H_{c2} for an inhomogeneous condensate is finite, irrespective of the amplitude of the condensate field at $H=0$. The magnetic susceptibility χ of the system is found. It is shown that the qualitative picture of the phenomena that occur does not depend on the actual choice of the model of the pion-nucleon interaction but only on whether the condensate is homogeneous or inhomogeneous.

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INTRODUCTION

The phenomenon of rearrangement of a boson vacuum in strong fields of various types—scalar, electric, nuclear—was first investigated by Migdal in 1971.[1] He showed that in a sufficiently strong field forming for a particle a potential well an instability arises that leads to rearrangement of the ground state of the system, i.e., to a phase transition with the formation of a Bose condensate. The formation of the condensate stabilizes the system and leads to a reduction of its energy. The lightest bosons, for which the instability occurs earlier than for the other particles, are pions. Nuclear matter is a potential well for pions, whose depth increases with increasing density of the nuclear matter. Therefore, at a sufficiently high density n a pion condensate must be formed in a nucleon medium.[1]

In Refs. 2 and 3, and then in Ref. 4, a method was developed for finding the spectrum of pion excitations in nuclear matter with number of neutrons N approximately equal to the number of protons Z, and also in a neutron medium with $Z\ll N$. It was found that in both cases the instability leading to the formation of the pion

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