Skin effects in metals in a perpendicular magnetic field

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The behavior of the surface impedance of anisotropic metals in a perpendicular magnetic field in the radiofrequency range is investigated theoretically and experimentally. It is found that below the helicon threshold there are two regions in which the skin effect has a different character. In weak fields, the skin effect is anomalous. However, the surface resistance $R_s$ and reactance $X_s$ vary rapidly with the magnetic field. In stronger fields, there is a skin effect of a novel type. Here the change in the impedance of the bulk metal with changing field is non-monotonic. The results of the calculations are in qualitative agreement with the experimental data for silver, copper, and aluminum.

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INTRODUCTION

The effect of a constant field $H$ on the surface impedance of a metal under conditions of the anomalous skin effect was considered in Ref. 1. It follows from the results of Ref. 1 that the value of the impedance in weak fields differs little from its value at $H = 0$, i.e., that the presence of a magnetic field has practically no effect on the anomalous skin effect. This conclusion was confirmed by Alg. 13 who made a numerical calculation of the dependence of the impedance of an alkali metal on the value of a magnetic field perpendicular to the surface. According to Ref. 2, in the range of fields much weaker than the threshold field of the helicon, the surface resistance $R_s$ and reactance $X_s$ change slowly with $H$ both for specular and for diffuse reflection of the electrons from the surface. This result also follows from the analytic expression obtained in Ref. 3 for the impedance.

It is shown in the present paper that the situation can be different in uncompensated metals with anisotropic Fermi surfaces. In the case of certain orientations of the vector $H$ relative to the crystallographic axes, a number of such metals reveal a rapid change in the surface impedance in fields much smaller than the threshold field of the helicon. This pertains to the impedance of the bulk metal for both circular polarizations. In order that the impedance be strongly dependent on the value of $H$ in the region of weak fields, the Fermi surface should possess the following feature: The principal group of carriers should have such a dispersion law that their displacements along the magnetic field in one cyclotron period are greater than a certain minimum value. This occurs if the function $S_S/\nu_p$ has a broad minimum $S(S)$ is the cross-sectional area of the Fermi surface cut by the plane $p_0 = \text{const}$, $z$ axis parallel to $H$. We note that in the total absence of electrons with small displacements there would be no collisionless wave absorption to cause the anomalous skin effect. In real metals, there is always a group of carriers with small displacements. However, it frequently happens that the number of such carriers is very small. These carriers determine the dissipative part of the nonlocal conductivity, the value of which in weak magnetic fields is practically independent of $H$. The principal group of carriers with large displacements makes a contribution proportional to $H$ to the nondissipative part of the nonlocal conductivity. Therefore, the region of weak fields, in which the dissipative conductivity of the small group is predominant and the anomalous skin effect takes place, turns out to be significantly narrower than in alkali metals, and the impedance changes sharply with magnetic field. In stronger fields, the nondissipative component becomes dominant and the character of the skin effect changes. The damping of the radiofrequency field in the skin layer in this region is principally exponential, while the thickness of the skin layer is proportional to $(\omega H)^{-1/2}$, where $\omega$ is the frequency of the exciting field. We emphasize that such a skin effect is absent in alkali metals, in which the dissipative component of the conductivity is important up to the helicon threshold.

We shall carry out the calculation below for a comparatively simple model that accounts for the significant properties of real metals, demonstrate the features of the skin effect mentioned above, and make a comparison with the data of our experiments for silver, copper, and aluminum.

THEORY

1. The nonlocal conductivity of a metal with a spherical Fermi surface in a perpendicular magnetic field $B_z |H| |S|$ is described by the expression

$$\sigma_{ns} = e^2 m^* n v_F \left( 1 - \frac{e H}{m^* v_F} \right), \quad (1)$$

where $e$ is the absolute value of the charge, $m^*$ is the effective mass of the conduction electrons, $n$ is the concentration, $v_F$ is the Fermi velocity, $H$ is the magnetic field, $S$ is the cross-sectional area of the Fermi surface, and $v_F$ is the frequency of collisions.

2. We are interested in the case of frequencies $\omega$ that
are small in comparison with the frequency of collisional
nu and such magnetic fields in which B >> v. Upon satis-
faction of these conditions, the nonlocal conductivity
\( \sigma_n \) does not depend on \( \omega \) and the value of \( q \) reduces to the
ratio of the maximum displacement of the electrons per cyclotron period to the electromagnetic wavelength,
\( q = \omega B / c. \)

As is well known, at values of the magnetic field cor-
responding to the inequality \( q < 1 \), a helicon wave exists
in the "minimum" circular polarization. In the region \( q^2 > 1 \), the logarithm in (2) has a finite imaginary part due
to the collisionless absorption that makes propagation of the helicon impossible and leads to the anomalous
skin effect.

In the region of weak magnetic fields, where \( q^2 \to 1 \), the
approximate expression for the function \( F \) is of the form
\[
F(q) = - \frac{2\pi}{q} + \frac{q}{q^2}. \tag{4}
\]
The first term on the right side corresponds to col-
losionless absorption of the wave by electrons with small longitudinal velocities (Landas damping), while
the second describes the dispersion of the nondissipa-
tive part of \( \sigma_n \) due to electrons with large velocities.

Substituting (4) in (1), we have
\[
\begin{align*}
\sigma_n = & \frac{3\pi}{2} \left[ \frac{a}{|q|} \left( \frac{1}{q^2} \right) \right]. \tag{5}
\end{align*}
\]
The first term in (5) does not depend on \( \nu \), while the
numerical coefficient in the second term is of the same
order as in the first. Therefore, in the region \( q \approx 1 \)
the second component is much smaller than the first, whence
it follows that the magnetic field has a weak effect on the
anomalous skin effect.

The situation is different for anisotropic metals if the
magnetic field \( H \) is oriented relative to the Fermi sur-
faced in such a way that the plot of \( \delta S / \delta \nu \) for the basic
group of carriers has a minimum, i.e., there are no
electrons among them with small displacements along
the field. In particular, this takes place in noble metals
at \( k|H| [110] \) and in aluminum [112] at \( k|H| [100]. \) In the
study of dopplerons in copper in Ref. 7, a model of the
Fermi surface was considered with minimum \( \delta S / \delta \nu \),
for which the function \( F \) which describes the electron
contribution to the conductivity, is of the form
\[
F(q) = - \frac{q^2}{2} \left[ \frac{1}{|q^2-1|} \right]. \tag{6}
\]
where
\[
q = \frac{k c}{2 \pi n c} \left[ \frac{S S}{\delta \nu} \right]. \tag{7}
\]

At large \( q \), this function is real, \( F = -q^2 \), i.e., the
electrons make no contribution to the collisionless ab-
sorption. Moreover, it must be noted that the sign of
\( F \) is opposite the sign of the second term in (4). This
is also a consequence of the minimum of \( \delta S / \delta \nu \).

Besides the electrons, copper has hole orbitals of the
"dog bone" type, located near the central cross section
of the Fermi surface. Among these carriers there
are holes with small displacements, causing collision-
less absorption. However, the hole concentration in
copper is small, therefore the dissipative conductivity
term associated with them is numerically small. Thus,
the asymptotic expression for the conductivity at \( q > 1 \)
is of the form
\[
\sigma_n = \frac{\alpha^2}{F^2} \left( \frac{q^2}{|q^2-1|} \right), \tag{8}
\]
where \( \alpha \) is a small numerical coefficient.

We emphasize that the asymptotic form (8) is not the
consequence of the model and occurs in all cases in
which the basic group of carriers has a minimum of \( \delta S / \delta \nu \).

2. We want to describe the behavior of the impedance
not only in fields that are small in comparison with the
helicon threshold field, but also in the region of the
helicon and the doppleron that exists in the opposite
circularization.11-14 Therefore, we consider a simple
Fermi-liquid model for which the nonlocal conductivity
has the correct asymptotic forms at both large and
small \( q \):
\[
\begin{align*}
\sigma_n &= \frac{\sigma_0}{F^2} \left[ \frac{q^2}{|q^2-1|} \right], \tag{9}
\end{align*}
\]
The second term represents the contribution of the elec-
tron Fermi surface, which has the shape of a paraboloid
of revolution.\textsuperscript{10} We have written unity in the deno-
imator of the first term so that the collisionless absorp-
tion vanishes as \( q \to 0 \), as is the case in real metals.

The dispersion equation for circular polarizations of the
electromagnetic field \( k E^2 = 4 \pi i \omega \sigma \), can be written
in this model in the form
\[
D_\nu(\nu) = 0, \tag{10}
\]
\[
D_\nu(q) = q^2 - \frac{\sigma_0}{F^2} \left[ \frac{q^2}{|q^2-1|} \right], \tag{11}
\]
where
\[
1 = \text{const.} \tag{12}
\]
for the paraboloid \( \delta S / \delta \nu = 2np \), const. In the second
term in square brackets, we have neglected the small
imaginary terms of order \( \gamma \).

We consider first the region of fields satisfying the
inequality
\[
0 < |q < 1. \tag{13}
\]
Under the condition (13), the first component in the
square brackets in (11) is small and can be neglected in
comparison with the second. Here the dispersion equa-
tion has two solutions: \( q_+ \) and \( q_- \), which describe the
distribution of the radiofrequency field in the metal and
the value of the impedance. In the minus polarization,
both roots are real at \( |q| < 1 \). One of them refers to the

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helicon, and the other to the doppleron. At \( \lambda > 1 \), both roots are essentially complex. In the plus polarization, the root \( q_0 \) is imaginary; it is connected with the "damped helicon." The root \( q_1 \) which is real, characterizes the doppleron.

We assume the reflection of the electrons from the surface to be diffuse. Then the surface impedance \( Z_s \) of the semi-infinite metal is determined by the expression of Ref. 11, which can be written in the form

\[ Z_s = \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{R_s(q)}{q^2} dq \right)^{-1}. \tag{14} \]

\[ s = \frac{4\pi\epsilon v_H}{e}. \tag{15} \]

In the considered region of fields, the integral in (14) is simply expressed in terms of the solution of the dispersion equation:

\[ Z_s = -\frac{\alpha v_H}{q} \tag{16} \]

If we substitute the explicit expressions for \( q_0 \) and \( q_1 \) in (16) (the roots of the dispersion equation, which have positive imaginary parts), we then obtain

\[ Z_s = \frac{\alpha v_H}{q} \left( 1 + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{R_s(q)}{q^2} dq \right)^{-1}, \tag{17} \]

\[ Z_s = \frac{\alpha v_H}{q} \left( 1 - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{R_s(q)}{q^2} dq \right)^{-1}, \tag{18} \]

These are in fact interpolation formulas. They describe correctly the behavior of the impedance both in the region of existence of the helicon in strong fields (\( \lambda < 1 \)) and in fields that are much smaller than the threshold value (\( \lambda > 1 \)). The dependence of the impedance \( Z_s \) on the field in the vicinity of the helicon threshold is sensitive to the model of the Fermi surface and one should not attribute a special significance to it.

We consider in more detail the skin effect in the region of comparatively weak fields, where

\[ \lambda > 1 > \lambda^2. \tag{19} \]

Here the expressions (17)–(18) for the impedance are simplified and take the form

\[ Z_s = \frac{\alpha v_H}{q^2} \left( 1 - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{R_s(q)}{q^2} dq \right)^{-1}. \tag{20} \]

It follows from (20) with account of (12) and (15) that \( R_s \) and \( X_s \) are proportional to \( (\omega_H/\gamma)^{3/4} \), while \( R_L = (\omega_H/\gamma)^{1/4} \). In the minus polarization, the electromagnetic field decays exponentially over a distance \( \delta = 1/\beta^2 \), where \( \beta^2 \) is the imaginary part of \( \beta \) and is a solution of the dispersion equation. The distance is

\[ \delta = \frac{\beta^2}{\lambda_H} \left( \frac{\alpha v_H}{q^2} \right) \tag{21} \]

In the plus polarization, the field is the sum of the doppleron and skin terms. The doppleron term has small damping, while the skin term decays over a distance of the order of \( \delta \).

3. We now study the region of weak magnetic fields \( \omega_H \gg \gamma \). In this case, the dispersion equation (10) can be written approximately in the form

\[ \gamma^2 \left( 1 + \frac{\alpha^2}{2\pi^2} \right) \gamma^2 + \frac{\alpha^2}{2\pi^2} = 0. \tag{22} \]

The expression in square brackets, which is connected with the nonlocal conductivity, is defined on the real \( q \) axis. We continue it analytically into the complex \( q \) plane, defining the first sheet by the cuts shown in Fig. 1. Then the dispersion equation for the plus polarization has in the upper halfplane of the first sheet the solution

\[ q^2 = (\alpha^2)^{1/2} \pm i \omega. \tag{23} \]

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\[ q^2 = (\alpha^2)^{1/2} \pm i \omega. \tag{23} \]

The locations of these roots are shown in Fig. 1. The remaining roots of the dispersion equations lie in the lower halfplane and we shall not be interested in them in what follows.

The surface impedance is determined by the integral (14). We carry out integration by parts and deform the contour of integration to the cut in the upper halfplane, isolating the pole components. Then, using the smallness of \( \lambda_H \), we displace the cuts to the real semi-axis. Finally, replacing \( q^2 \) by \( s \), we obtain

\[ s = \frac{2\pi \alpha v_H}{q} \left( 1 - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{R_s(q)}{q^2} dq \right)^{-1}, \tag{24} \]

where \( q_s \) are the dispersion-equation roots lying in the upper halfplane on the first sheet.

Further calculation of the integrals for different polarization is carried out separately. As a result of uncomplicated but tedious calculations with the use of the inequality (22), we find

\[ Z_s = Z_L^0 \left( 1 + \frac{\alpha^2}{2\pi^2} \right), \tag{27} \]

\[ Z_s = \frac{\alpha v_H}{q} \left( 1 - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{R_s(q)}{q^2} dq \right)^{-1}, \tag{28} \]

\[ \lambda_H^2 = \frac{2\pi \alpha v_H}{q} \left( 1 - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{R_s(q)}{q^2} dq \right)^{-1}. \tag{29} \]

The quantity \( Z_s \) does not depend on \( \gamma \). It represents the impedance of a metal with carrier concentration \( n_c \).
under the conditions of the anomalous skin effect at \( N = 0 \). The functions \( \Delta \), describe the change in the impedance with the magnetic field. Thanks to the fact that \( f \) enters into (29) only in combination with the small factor \( a^4 \), the functions \( \Delta \), (in contrast with the case of the alkali metal),\(^{14}\) change rapidly with \( H \). It should be emphasized that the impedance for linear polarization, \( Z_{xx} = Z_{xy} = (Z_+ + Z_-)/2 \), does not contain a logarithmic term and therefore is a more slowly varying function of the field. Moreover, we note that in specular reflection of the carriers from the surface, the impedance has likewise no logarithmic singularities.

4. Plots of the surface resistance and the reactance for both circular polarizations are shown in Fig. 2. These are obtained by numerical integration of (14) with the conductivity (9). The calculation was performed for the following values of the parameters: \( n = 5 \times 10^{27} \text{ cm}^{-3} \), \( n = 0.2 \); \( p_L = 0.47 \times 10^{-8} \text{ cm} \), the free path length of the electrons \( l = 0.05 \text{ cm} \), frequency \( \omega = 2 \times 10^{12} \text{ Hz} \). Since the chosen value of \( r \) is not very small, it is impossible to expect numerical agreement of these curves with the asymptotic formulas calculated earlier. However, the character of the change of \( R \) and \( X \), with the field \( H \), is in complete agreement with the expressions (17)-(18) and (27)-(29). It is seen from the drawing that in weak fields the functions \( Z_+(H) \) undergo a sharp change; hence the signs of the derivatives of \( dR_+/dH \) and \( dX_+/dH \) are opposite to the signs of these functions in the alkali metal.\(^{12}\) The significant difference from the alkali metal lies in the fact that the functions \( R_+, (H) \) and \( X_+, (H) \) have extrema on going to the region of moderate fields. It should be noted in this case that, in correspondence with Eqs. (17)-(18) and (27)-(29), the extrema of \( R_+ \) and \( X_+ \) are located at smaller fields than the extrema of \( R_- \) and \( X_- \).

5. We noted above that the character of the reflection of the electrons affects the change in the impedance of the semi-infinite metal in weak fields. It is also known that the behavior of the impedance in the vicinity of the helicon and doppleron thresholds\(^{15,16}\) is very sensitive to the character of the reflection. It was shown in Refs. 13, 14 that in diffuse reflection, the amplitude of the doppleron oscillations in compensated metals in strong fields is much greater than that in specular reflection. Is it of interest to study the effect of diffuse reflection on the doppleron oscillations of the uncompensated metal. We do this with the help of the model (9) described above. We determine the amplitude of the doppleron oscillations of the impedance of the plate for the plus polarization in the region of moderate fields (13). Since the holes play a small role in this region, we can use the results of Ref. 13 for the amplitude of the doppleron oscillations in the paraboloidal model. In order to obtain the formula for the impedance of the plate in the plus polarization we must replace the quantity \( 1 - i \) in Eq. (29) of Ref. 13 by \(-1 + i\) and \( a_+ \) by \( a_- \). Assuming that the thickness of the plate \( d \) is large in comparison with the damping length of the doppleron, we can represent the oscillatory part of the impedance \( \Delta Z \), in the form

\[
\Delta Z = \int \frac{1-\exp(-\delta H)}{2i(\omega H)^2} \exp(i \alpha H) \, dH.
\]

In the case of specular reflection, and

\[
\Delta Z = \int \frac{\exp(-i \alpha H)}{2i(\omega H)^2} \exp(i \alpha H) \, dH.
\]

The solutions of the dispersion equations (10)-(11) for the plus polarization in the region (13) are of the form

\[
q_\alpha = (\omega L/2) \left((\omega L/2)^2 + 1/\omega^2\right)^{1/2} + \omega 2 \left((\omega L/2)^2 + 1/\omega^2\right)^{1/2}.
\]

In (33) we have taken into account a small imaginary term (proportional to \( a \)) that is due to the dissipative hole conductivity. This term determines the damping of the doppleron and, consequently, the value of the exponential in (30)-(31), although it does not affect the pre-exponential factors. Here and below, we shall not write for the damping of the doppleron the obvious component \( i \) due to collisions.

Substituting (33)-(35) in (30)-(31), we get for the amplitude of the doppleron oscillations

\[
|\Delta Z| = 22(1-\exp(-\delta H)^2)^{1/2}(\omega L/2)^2 + \omega \exp(-\delta H)^2.
\]

In the case of specular reflection, and...
in the case of diffuse reflection; $K_2 = 2m_g$.

In the range of fields in which $l > 1$, the amplitudes of the oscillations in both cases are the same:

$$|\Delta E| = \frac{4l}{\pi} \frac{K_2 m_g}{\sin^2 \theta} \exp \left( -\frac{2\pi n}{\lambda} \right) \frac{\sin \theta}{\sin \theta - \sin \theta_0}.$$  \hspace{1cm} (26)

At small $l$, the amplitude of the oscillations in the diffuse case falls off as $1/l$, while in the specular case it contains the extra factor $4l$ and falls off as $l^{-1}$.

**EXPERIMENT AND DISCUSSION**

The $R(H)$ and $X(H)$ dependences of silver, copper, and aluminum were studied experimentally in the case of circular polarizations of the exciting field. The samples were plane-parallel plates, cut by the electric spark method from single crystal blanks. After the cutting, the plates of silver and copper were treated by the method described in Refs. 15 and 16. The normals to their surfaces coincide with the direction of the [110] axis. The dimensions of the samples of silver and copper after treatment amounted to $14 \times 7 \times 0.81$ and $10 \times 8 \times 0.25$ mm, respectively, while the resistance ratios $\rho_{0s}/\rho_{0c}$ were $1.6 \times 10^4$ and $1.4 \times 10^4$, respectively. The normal to the aluminum plate coincided with the [100] axis. The sample was etched in concentrated nitric acid before the measurements. Its dimensions were $12 \times 6 \times 0.47$ mm and the resistance ratio $\rho_{0s}/\rho_{0a} = 7 \times 10^3$.

The measurements of the impedance were carried out in a magnetic field perpendicular to the surface of the plate, at a temperature of 4.2 K and in the radiofrequency range. The real and imaginary parts of the impedance were studied by means of an amplitude bridge and an autodyne, in the coil of the tank circuit of which the sample was placed. The measurement apparatus assured a reliable discrimination of the signals proportional to $R(H)$ and $X(H)$. The method of measurements and the method of calibration of the impedance changes are described in detail in Ref. 17. The coil of the tank circuit, and also the auxiliary coil which permits production of a circularly polarized field, were wound of brass wire of diameter 0.08 mm. The use of copper coils led to a significant distortion of the experimental plots as a consequence of the dependence of the impedance of the wire on the magnetic field.

Figures 3 and 4 show the $R_s(H)$ and $X_s(H)$ curves for silver and aluminum. We subtracted from the plotted values a correction connected with the weak dependence of the impedance of the brass coils on the magnetic field. Above all, note should be made of the presence of a strong dependence of the impedance of silver on the magnetic field intensity over the entire interval of its change.

The $R_s$ and $X_s$ curves in Fig. 3 are nonmonotonic; in fields $H < 3$ kOe they have extrema. In stronger fields, the resistance $R_s$ decreases with increase in $H$ (if we disregard small oscillations) and $X_s$ increases. The
signs of the change in the reactivities $X \gamma$ in the region $\gamma > 3$ IOE are proportional to the signs of the change of the corresponding resistances $R$. Since the signs of the change of $R_1$ and $R_2$, and also of $X_1$ and $X_2$, are different, the dependence of the impedance on the field in the linear polarization turns out to be much weaker (the impedance in the linear polarization is equal to the half-sum of $Z_1$ and $Z_2$). It should be noted that in most experimental studies of the impedance of metals in a magnetic field, the derivative $dR/d\gamma$ or $dX/d\gamma$ was measured for linear polarizations with copper coils. This apparently makes difficult the observation of the singularities of the impedance.

The oscillations of the impedance on the curves of Fig. 3 are connected with the penetration of the electromagnetic field through the plate. The oscillations of $R_1$ and $X_1$ in strong fields are due to the excitation of the helicon. In the plus polarization, oscillations are seen that are connected with the doppleron wave. The properties of dopplerons in silver were studied in the region of very weak fields ($H < 1$), where our theory is inapplicable.

A comparison of the curves in Figs. 3 and 4 with the corresponding curves on Fig. 2 shows that the extremal and theoretical results are in qualitative agreement.

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