# Singularity of the coefficient of absorption of long-wavelength sound in a phase transition of order 2 1/2

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The anomalous dependence of the absorption coefficient $\Gamma_e$ for absorption of long-wavelength sound by electrons in metals on $z=\epsilon_F-\epsilon_c$ near $z=0$ is considered ($\epsilon_F$ is the Fermi energy and $\epsilon_c$ is the critical energy at which the topology of the electron constant-energy surfaces changes). It is shown that the character of the anomaly depends in an essential way on the topology of the transition: when a new cavity is formed $\Gamma_e$ has a discontinuity $\Delta\Gamma_e$, and when a neck is broken $\Gamma_e$ increases logarithmically and, when the point $z=0$ is crossed, has at the same time a finite discontinuity $\Delta\Gamma_e$ of the same order of magnitude as in the case of formation of a new cavity in the Fermi surface ($\Delta\Gamma_e$ coincides in order of magnitude with $\Delta_e$, the total electronic absorption coefficient).

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It is well known[1] that when the topology of the Fermi surface changes as a result of an external perturbation (e.g., pressure) the anomalies that have become known as a phase transition of order $2\frac{1}{2}$ (PT $-2\frac{1}{2}$) or Lifshitz transition[2] are observed in the metal.

Attention has been drawn repeatedly to the fact that a PT $-2\frac{1}{2}$ should be accompanied by anomalies in the kinetic characteristics, and the coefficient of absorption of short-wavelength sound has been found to be particularly sensitive to a PT $-2\frac{1}{2}$.[3] The task of this communication is to show that at a PT $-2\frac{1}{2}$ the electronic part $\Gamma_e$ of the coefficient of absorption of long-wavelength sound ($ql\ll 1$, $q=\omega/s$, where $\omega$ is the frequency and $s$ the velocity of sound, and $l$ is the electron mean free path) experiences substantial changes, the character of the anomaly being dependent on the topology of the transition: when a new cavity is formed $\Gamma_e$ changes discontinuously, and as a "neck" is broken $\Gamma_e$ increases logarithmically. Since the absorption coefficient for long-wavelength sound is expressed in terms of components of the electron-viscosity tensor $\eta_{iklm}$,[4] the statement about the singularities of $\Gamma_e$ for $ql\ll 1$ implies that $\eta_{iklm}$ possesses corresponding singularities.

In other words, a PT $-2\frac{1}{2}$ should be manifested in all those properties (internal friction, etc.) in which the viscosity of the metal is important and the electronic component plays an appreciable role in the viscosity tensor of the metal.

The electronic subsystem plays a no less important role than the ionic subsystem in the propagation of sound through a metal.[5] This can be seen particularly clearly from the derivation of the expression for the sound velocity based on the simple model in which the role of the electrons reduces to screening of the Coulomb field (see, e.g., Ref. 6, Chap. 6, Sec. 12). According to this model the square of the velocity of longitudinal sound is inversely proportional to $\nu(\epsilon_F)$—the density of states at the Fermi surface:

$$s^2=NZ^2/M\nu(\varepsilon_F), \tag{1}$$

where $N$ is the number of ions of charge Ze in unit volume and $M$ is the ion mass.

It can be seen from (1) that at a PT-$2\frac{1}{2}$ the longitudinal-sound velocity should have a root singularity along with $\nu(\epsilon_F)$.[2] Below we shall show that the electronic sound-attenuation coefficient $\Gamma_e$ possesses a sharper singularity.

If we confine ourselves to the $\tau$-approximation, which is absolutely sufficient to elucidate the character of the anomaly, the electronic part $\Gamma_e$ of the absorption coefficient can be given the following form,[7] valid for arbitrary relative values of the wavelength of the sound and the electron mean free path:

$$\Gamma_e=\frac{2}{\rho(2\pi\hbar)^3}\oint_{\varepsilon(\mathbf{p})=\varepsilon_F}\frac{dS}{lv^2}\frac{|\Lambda|^2}{(s/v-\mathbf{nv})^2+(1/ql)^2}, \tag{2}$$

where $\rho$ is the metal density, $\mathbf{n}=\mathbf{q}/q$, $\boldsymbol{\nu}=\mathbf{v}/v$, $\mathbf{v}=\partial\epsilon(\mathbf{p})/\partial\mathbf{p}$ is the velocity of an electron with quasi-momentum $\mathbf{p}$ and energy $\epsilon(\mathbf{p})$, the integration is performed over the Fermi surface $\epsilon(p)=\epsilon_F$, $dS$ is an element of area on this surface, and $\Lambda$ is the corresponding component of the deformation potential and depends on the polarization of the sound and on its direction of propagation. Naturally, the quantities $\Lambda$ are different in the cases of longitudinal and transverse sound. However, there are no reasons to expect that $\Lambda$ vanishes at those points in p-space at which the topology of the Fermi surface changes.

We shall assume that the principal mechanism of electronic dissipation is scattering by crystal defects; therefore,

$$1/l=N_{\text{imp}}\,\sigma, \tag{3}$$

where $N_{\text{imp}}$ is the number of scattering centers per unit volume and $\sigma$ is the effective scattering cross-section. The latter expressions gives us the formal right to assume that $l$ does not depend on the change of topology of the Fermi surface.

The temperature-dependent part $l(T)$ of the mean free path can, of course, have an anomaly at a PT $-2\frac{1}{2}$;

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in the first place, however, this will be weaker than the anomalies discovered here (see below), since the calculation of $l(T)$ reduces, in essence, to integration of the coefficients $\Gamma_e(q)$ over the phonon momenta, and, secondly, for $l \ll l(T)$ (which is always the case at low temperatures), the singularity due to $l(T)$ will contain the small parameter $l/l(T) \ll 1$. Thirdly and finally, the frequency dependence permits us to distinquish the contribution of a particular scattering mechanism, and, therefore, to investigate the singularities of the coefficient of absorption due to this mechanism.

Strictly speaking, the formula (2) is valid at absolute-zero temperature, since the relation

$$-\partial f_F/\partial\varepsilon=\delta(\varepsilon-\varepsilon_F),$$

was used in its derivation ($f_F(\epsilon)$ is the Fermi function). For temperature effects, see below.

We shall describe the PT $-2\frac{1}{2}$ by the variation of the parameter $z=\epsilon_F-\epsilon_c$, where $\epsilon_c$ is the value of the energy at which the topology of the constant-energy surface changes. For small $z$ that part of the Fermi surface whose topology is changing can be described by an expression quadratic in the components of the quasi-momentum: in the formation of a new cavity,

$$\varepsilon_p=\frac{p_1^2}{2m_1}+\frac{p_2^2}{2m_2}+\frac{p_3^2}{2m_3}=z,\quad \varepsilon_p=\varepsilon(\mathbf{p})-\varepsilon_c \tag{4}$$

and in the breaking of a neck,[1)]

$$\varepsilon_p=\frac{p_1^2+p_2^2}{2m_\perp}-\frac{p_3^2}{2m_\parallel}=z. \tag{5}$$

The anomalous part $\Gamma_e(z)$ of the attenuation coefficient is expressed by formula (2) with the assumption that the integration is performed either over the surface (4) or over the surface (5). Since, in both cases, the anomalous parts of the Fermi surface are located near a point in p-space at which the velocity vanishes ($p_1=p_2=p_3=0$; $\mathbf{v}=0$), it can be seen from formula (2) that $\Gamma_e(z)$ is sensitive to a PT $-2\frac{1}{2}$.

For $ql \ll 1$ the expression (2) can be simplified substantially:

$$\Gamma_e=\frac{2q^2}{\rho(2\pi\hbar)^3}\oint_{\varepsilon(\mathbf{p})=\varepsilon_F} l|\Lambda|^2\frac{dS}{v^2}, \tag{6}$$

which is valid for $z \gg \tilde{m}s^2(ql)^2$ (in order of magnitude, $\tilde{m}$ coincides with the effective mass of the electron). Since $\tilde{m}s^2 \simeq 10^{-17}$ erg $\simeq 0.1$ K, it is evidently of academic interest to go beyond the limits of this approximation. Nevertheless, we shall give the corresponding formulas for $z \ll ms^2(ql)^2$ to prove the consistency of the entire treatment.

Thus,

$$\Gamma_e(z)=\frac{2q^2}{\rho(2\pi\hbar)^3}\oint_z l|\Lambda|^2\frac{dS}{v^2}=\frac{2q^2l_0|\Lambda_0|^2}{\rho(2\pi\hbar)^3}\iiint\frac{\delta(\varepsilon_p-z)}{v(\mathbf{p})}d^3p, \qquad |z|\ll\tilde{m}s^2(ql)^2. \tag{7}$$

Since the anomalous part $\Gamma_e(z)$ is determined by the integration over a small region about the coordinate origin ($p_1=p_2=p_3=0$), we have taken the quantities not possessing singularities ($\Lambda$ and $l$) outside the integral. The change to the three-dimensional integral makes the calculations easier. In the case of breaking of a neck the shape of the surface (5) does not restrict the region of integration along the $p_3$ axis. Therefore, we shall assume that the integration over $p_3$ is taken from $-p_{max}$ to $p_{max}$; as we shall see, the character of the anomaly does not depend on $p_{max}$.[2)]

On the formation (or disappearance) of a cavity in the Fermi surface, we have, according to (7) and (4),

$$\Gamma_e(z)=\begin{cases}0 & z<0,\\ \dfrac{4\pi q^2l_0|\Lambda_0|^2}{\rho(2\pi\hbar)^3}(m_1m_2m_3)^{1/2}\beta, & z>0,\end{cases} \tag{8}$$

where

$$\beta=\frac{1}{4\pi}\iint(\gamma_1n_1^2+\gamma_2n_2^2+\gamma_3n_3^2)^{-1/2}\,dO, \tag{9}$$

here, $n_i$ are the direction cosines ($n_i^2=1$), $\gamma_1=(m_2m_3/m_1^2)^{1/3}$, $\gamma_2=(m_1m_3/m_2^2)^{1/3}$ and $\gamma_3=(m_1m_2/m_3^2)^{1/3}$. For $m_1=m_2=m_3$ the parameter $\beta=1$. We emphasize that $\Gamma_e(z)$ for $z>0$ is of the same order of magnitude as the sound-absorption coefficient due to a large cavity in the Fermi surface.

Now let $z \ll \tilde{m}s^2(ql)^2$. Then, according to (2),

$$\Gamma_e(z)=\frac{2|\Lambda_0|^2}{\rho(2\pi\hbar)^3l_0s^2}\oint_z dS=\frac{8\pi(m_1m_2m_3)^{1/2}|\Lambda_0|^2z}{\rho(2\pi\hbar)^3l_0s^2}. \tag{10}$$

Figure 1 shows the dependence $\Gamma_e(z)$. We note that $\Gamma_e(z)$ for $z \ll \tilde{m}s^2(ql)^2$ does not depend on the frequency. However, it is necessary to remember that it is not possible, apparently, to observe the decrease of $\Gamma_e(z)$ corresponding to formula (10), since the PT $-2\frac{1}{2}$ is smeared out by the temperature (the width of the smearing with respect to $z$ is of the order of $T$).

When a neck is broken, from (7) and (5) we have

$$\delta\Gamma_e(z)=\frac{4\pi l_0|\Lambda_0|^2q^2}{\rho(2\pi\hbar)^3}m_\perp m_\parallel\left(\frac{m_\perp}{m_\perp+m_\parallel}\right)^{1/2}\int_{x_{min}}^{x_{max}}\frac{dx}{(x^2+\operatorname{sign}z)^{1/2}}$$

where

$$x_{min}=\left(\frac{m_\perp+m_\parallel}{m_\parallel}\right)^{1/2},\quad z<0;$$
$$x_{min}=0,\quad z>0;$$
$$x_{max}=p_{max}\left(2m_\parallel|z|\frac{m_\parallel}{m_\perp+m_\parallel}\right)^{-1/2}.$$

The absorption coefficient is nonzero both for $z<0$ and for $z>0$. As $z\to 0$ the absorption coefficient $\Gamma_e(z)$

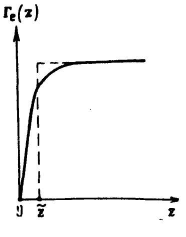


FIG. 1. The dependence $\Gamma_e(z)$ when a new cavity appears in the Fermi surface; $\tilde{z}=\tilde{m}s^2(ql)^2 \ll \tilde{m}s^2$; $\Gamma_e$ is given by formula (8).

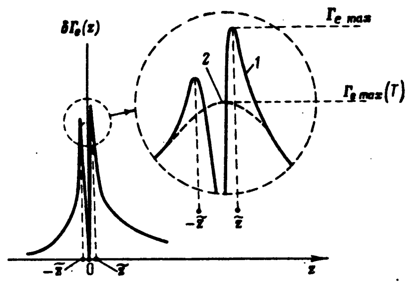


FIG. 2. Dependence of $\delta\Gamma_e$ on $z$ when a neck is broken: (1) at absolute-zero temperature, (2) with allowance for temperature smearing; $\tilde{z}=\tilde{m}s^2(ql)^2 \ll \tilde{m}s^2$, $\Gamma_{e\,\max}=\Gamma_e\ln(\epsilon_0/\tilde{z})$, $\Gamma_{e\,\max}(T)=\Gamma_e\ln(\epsilon_0/T)$.

has a logarithmic singularity:

$$\delta\Gamma_e(z)=2\pi m_\perp m_\parallel\left(\frac{m_\perp}{m_\perp+m_\parallel}\right)^{1/2}\frac{q^2 l_0|\Lambda_0|^2}{\rho(2\pi\hbar)^3}\ln\frac{\varepsilon_0}{|z|}. \quad (11)$$

The value of

$$\varepsilon_0=\frac{p_{max}^2}{2m_\parallel}\frac{m_\perp+m_\parallel}{m_\parallel}$$

does not play any special role (in a more exact calculation, the value of $\epsilon_0$ should be expressed in terms of the parameters of the Fermi surface). We call attention to the fact that, in addition to the logarithmic singularity, $\Gamma_e$ has a finite discontinuity when we pass through $z=0$:

$$\delta\Gamma_e(z)=\frac{4\pi l_0|\Lambda_0|^2 q^2}{\rho(2\pi\hbar)^3}m_\perp m_\parallel\left(\frac{m_\perp}{m_\perp+m_\parallel}\right)^{1/2}\ln\left[\left(1+\frac{m_\perp}{m_\parallel}\right)^{1/2}+\left(\frac{m_\perp}{m_\parallel}\right)^{1/2}\right]. \quad (12)$$

The anisotropy of the masses enhances the discontinuity $\Delta\Gamma_e$, which is of the same order of magnitude as in the case of formation of a new cavity.

For $z \ll \tilde{m}s^2(ql)^2$ the calculation of the absorption coefficient $\Gamma_e(z)$ reduces to calculation of the area of the part of the Fermi surface near the conical point. It is easy to show that

$$\delta\Gamma_e(z)=\frac{2\pi m_\parallel|\Lambda_0|^2}{\rho(2\pi\hbar)^3 s^2 l_0}\left(\frac{m_\perp}{m_\perp+m_\parallel}\right)^{1/2}|z|\ln\frac{\varepsilon_0}{|z|}. \quad (13)$$

Thus, in order of magnitude, the maximum value of $\Gamma_e(z)$ when a neck is broken is equal to

$$\Gamma_{e\,\max}=\Gamma_e\ln\frac{\varepsilon_0}{\tilde{m}s^2(ql)^2}, \quad T=0$$

(see Fig. 2).

However, since the temperature smears out the singularity, for $T \gg \tilde{m}s^2(ql)^2$ the maximum value of the absorption coefficient is

$$\Gamma_{e\,\max}(T)=\Gamma_e\ln(\varepsilon_0/T), \quad T\gg\tilde{m}s^2(ql)^2. \quad (14)$$

Since the finite discontinuity (12) is of the same scale as $\Gamma_e$, even after the temperature smoothing one can observe, in addition to the logarithmic increase of $\Gamma_e$ (cf. (14)), an increase (or decrease) of the absorption coefficient at the PT $-2\frac{1}{2}$.

Kontorovich and Sapogova[8] have shown that the existence of conical points in the electron spectrum of metals leads to singularities in the dispersion of the short-wavelength phonons. The formulas (2) and (6) show that conical points at which $v=0$ should also be manifested in the attenuation of long-wavelength sound. In particular, their disappearance (at the phase transition) should lead to a substantial change (as a rule, a decrease) in the absorption coefficient.

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1) In order not to clutter up the calculations we have assumed the presence of axial symmetry.

2) An analogous situation occurs in the calculation of anomalies in the thermodynamic characteristics,[2] and also of $\Gamma_e$ for $ql \gg 1$ (see Ref. 3).


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