Higher orders of perturbation theory and summation of series in quantum mechanics and field theory

V. S. Popov, V. L. Eletskii, and A. V. Turbiner
Institute of Theoretical and Experimental Physics
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The Borel method of summation of a perturbation-theory series with factorially increasing coefficients is considered. The connection between the asymptotic form of the coefficients a_k of this series for k\to\infty and the nature of the singularity of the sum is established. An improved perturbation theory is constructed and the limits of its region of applicability are found. The results obtained are verified for a number of physical problems (the Lagrange function in the nonlinear electrodynamics of the vacuum, the energy levels of an electron in the Coulomb field of a nucleus with Z > 137, the screening of the nuclear charge by the vacuum shell of a supercritical atom, and the Stark effect in the hydrogen atom) for which the coefficients of the perturbation-theory series increase factorially and for which, at the same time, (analytically or numerically) exact solutions are known. Application of the improved perturbation theory to the \( g^4/4! \) scalar field theory makes it possible to establish the behavior of the Gell-Mann-Low function \( \psi(g) \) for \( 0 < g/16\pi < 0.3 \). In this interval \( \psi(g) \) is a monotonically increasing function of the coupling constant g.

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1. INTRODUCTION AND FORMULATION OF THE PROBLEM

In recent years effective ways of calculating higher orders of perturbation theory (PT) in quantum mechanics and statistical physics have been found; the structure of the PT series for the energy levels of the anharmonic oscillator has been investigated in particular detail. In quantum field theory Lipatov has developed a semi-classical method of calculating the functional integral, in which an important role is played by the classical solutions of the field equations, and has applied this technique to the renormalizable scalar field theory with interaction

\[ H_{\text{int}} = \int d^4x \phi^2 \phi, \quad D = 2\pi (n-2). \]

In Ref. 5 the case \( n = m \) was considered and the first term of the 1/n expansion for the Gell-Mann-Low function (GLF) \( \psi(g) \) was found; in Ref. 6 the asymptotic form, for \( k = m \), of the coefficients of the PT series

\[ \psi(z) = \sum_{n=0}^{\infty} a_n (-z)^n, \]

was calculated for arbitrary \( n \). Intensive developments are being made in this direction at the present time. Lipatov's method has been applied to the following problems: the n-dimensional anharmonic oscillator, the scalar field theory with internal-symmetry group \( O(n) \), and the theory of fermions with a Yukawa interaction, scalar electrodynamics, and reggeon field theory in the strong-coupling region. The calculations of the asymptotic forms of PT series in more-realistic field-theory models (quantum electrodynamics, Yang-Mills theory, and so forth) await their turn. In this connection, the following question becomes urgent: what information can be obtained about the behavior of the exact solutions if we know the first few coefficients \( a_k \) of the PT series and their asymptotic form as \( k\to\infty \)?

The present article is devoted to elucidating this question.

Let \( f(z) \) be a function representable by a divergent power series

\[ f(0) = \sum_{n=0}^{\infty} a_n z^n, \]

that is asymptotic for \( z = 0 \) (in field-theory problems the variable \( z \) is usually the coupling constant, but it can also have another physical meaning—see Sec. 5). In most of the theories considered the asymptotic form of \( a_k \) is

\[ a_k = k! \sum_{i=0}^{n} \alpha_i \beta_i \gamma_i \delta_i, \quad k = m, \]

where \( x = \Gamma(z+1) \), and \( \alpha, \beta, \gamma, \) and \( \delta \) are calculable
the distant "tail" of the PT series is also
function f
below.

The first N coefficients a, and of the
asympotic form (3) makes it possible to construct not
only the PT polynomials p,(x) but also the improved
perturbation theory (IPT) functions f (x):}

\[
p(x) = \sum_{k=0}^{\infty} a_k (x-z)^{-k}.
\]

Here,

\[
f(x) = \sum_{k=0}^{\infty} f_k (x-z)^{-k}
\]
is the sum of the series (1) with the asympotic coef-
cients \( f_k \) in view of the factorial increase of \( a_k \) as
k \to \infty this sum must be understood in the generalized
sense, applying one of the methods of summation of
divergent series. The first N coefficients in the
function \( f(x) \) coincide with the exact coefficients, and
the distant "tail" of the PT series is also taken into
account. Therefore, it is natural to expect that as N
increases the functions \( f(x) \) will give a better approxi-
mation to the exact solution than will the PT polyno-
mials \( p_k(x) \), and will enable us to establish it in a
wider range of \( x \). These qualitative arguments can be
given a more exact meaning, and this will be done
below.

We shall describe the content of the paper (a brief
account of the results obtained was published earlier)[15]
in Sec. 2. The series (1) with coefficients of the form (3)
is summed using one of the variants of the Borel
method; the sum \( f(x) \) is obtained in explicit form. In
Sec. 3 the connection is found between the asympotic
form of \( a_k \) as \( k \to \infty \) and the character of the singularity
of the sum at the point \( x = 0 \). In Sec. 4 the limits of the
region of applicability of the IPT are found. These
results are verified in Sec. 5 for several examples for
which the coefficients of the PT series increase factor-
ally and for which, at the same time, exact solutions
are known. In the concluding section (Sec. 6) the IPT
method is applied to the \( g^{\mu / 4} / 4! \) field theory, and
enables us to determine the GLF \( \Phi (x) \) in the interval
\( 0 < x / 10^3 < 0.3 \).

Appendix A contains a summary of the formulas that
make it possible to sum series of the form (1) with
coefficients of the form (2). Appendix B contains a com-
parison of the analytic properties of the exact solution
\( f(x) \) and the function \( f(x) \), which is the Borel sum of the
series (1) with the asympotic coefficients \( f_k \).

2. THE BOREL METHOD OF SUMMATION

This method places the divergent series (1) in cor-
respondence with the generalized sum

\[
f(x) = \sum_{k=0}^{\infty} f_k (x-z)^{-k} = \sum_{k=0}^{\infty} \frac{a_k (x-z)^{-k}}{(ka+k!)}
\]

By choosing the parameter \( \mu > \alpha \) it is possible to quench
the factorial growth of \( a_k \), and make the series for
\( \Phi (x) \) convergent. For the Borel method to be appli-
cable it is sufficient[15] that the function \( \Delta_n (\omega \mu / 1 !) \)
be regular for small \( \omega \) and not have singularities on the
semi-axis \( 0 < \mu < \infty \). Usually one considers the case
\( \mu = 0 \); in accordance with Hardy[15] this method of
summation is designated as the \( (\mu, \mu) \) method. For
\( \mu = 1 \) this method has been treated by Bender and Wu[15]
in an application to the anharmonic oscillator, by
Shirkov[15] for the \( g^{\mu / 4} / 4! \) field theory, and also in a
paper by one of the authors.[17] We shall find it conven-
tient to generalize this definition, introducing the two
parameters \( \mu \) and \( \nu \) in (5). In the case when the series
is summable by the Borel method (i.e., under the con-
dition that the series for \( \Phi (X) \) and the integral in (5) are
convergent), the sum \( f(x) \), naturally, does not depend
on the values of \( \mu \) and \( \nu \).

To calculate the sum (1) it is convenient to use a represen-
tation differing from (2) for the asymptotic form of \( a_k \) as \( k \to \infty \)

\[
a_k = \frac{(ka+k-1)!}{k!} \sum_{m=0}^{\infty} C_m (k+a+k-
\]

The coefficients C and c are related by the linear trans-
formation

\[
c_k = \sum_{m=0}^{\infty} C_m a_k \quad C_m = c_m = \sum_{l=0}^{\infty} \frac{(l+x)}{(l+x)!}
\]
etc. General formulas for the elements of the matrices
\( \Phi (x) \) and \( \Phi (x) \) are given in Appendix A. Substituting (6) into
(5) and assuming that \( b_0 = 0 \), we find (see also Ref. 15)

\[
f(x) = \sum_{k=0}^{\infty} \frac{a_k (x-z)^{-k}}{(ka+k!)} = \sum_{k=0}^{\infty} \frac{a_k (x-z)^{-k}}{(ka+k!)} C_k (a_k x^{-k})
\]

where

\[
l(x; x, p) \sim x^{-1} x^{-1} \sup_{x \to 0} \left( \frac{\Phi (x) \Phi (x)}{\Phi \Phi (x)} \right)
\]

This integral has the following behavior: \( l = a^{-1} a^{-1} \)
for \( x = 0 \), and for \( x \rightarrow \infty \)

\[
l(x; x, p) \sim x^{-1} x^{-1} \sup_{x \to 0} \left( \frac{\Phi (x) \Phi (x)}{\Phi \Phi (x)} \right)
\]

In the frequently encountered cases \( a = 1 \) and \( a = 2 \) it is
expressed in terms of familiar special functions. For
example, for \( a = 1 \),
where $\Gamma(a, x)$ is the incomplete gamma function. It can be shown that the same result (7') is given by summing the series (1) with the coefficients (6) with the aid of the Sommerfeld-Watson integral transform, as suggested by Lipatov.\(^{11}\) The possibility of obtaining an answer for the sum (1) in a closed form containing standard functions is the advantage of the parametrization (6) as compared with (3), except for the case of integer values of $\alpha$.

3. CONNECTION BETWEEN THE ASYMPTOTIC FORM OF $a_k$ FOR $k \to \infty$ AND THE SINGULARITY OF THE SUM

The function $f(x)$ has a branch point $x = 0$ and a cut along $x < 0$. We shall calculate its discontinuity across the cut. Using the known analytic properties of the function $\Gamma(a, x)$, from (7) we obtain

$$
\Delta f(\zeta) = \frac{1}{2} \left[ f(-\zeta + 0) - f(-\zeta - 0) \right]
$$

$$
= -\pi \sin(a \pi) \sum_{n=0}^{\infty} C_n z^n,
$$

where $\zeta = z - \alpha$. If the series $\sum C_n z^n$ has a finite radius of convergence, or even if it is asymptotic for $x = 0$ (which, evidently, occurs in the majority of physically interesting cases), the asymptotic form of $a_k$ for $k \to \infty$ determines the behavior of the discontinuity $\Delta f(\zeta)$ as $\zeta \to -\alpha$.

In the case of arbitrary $\alpha$ this result arises from the following considerations.\(^{11}\) From (2) and (5) we find that the singularity of the function $f(x)$ nearest to zero is located at $x = -\alpha = 1$.

$$
q_0(x) = \frac{1}{\alpha} \left( 1 - e^{-\alpha x} \right),
$$

$$
\Delta f(\zeta) = \frac{1}{\alpha} \left[ f(-\zeta + 0) - f(-\zeta - 0) \right]
$$

The integral

$$
\int_0^\infty e^{-x} q_0(x) dx
$$

is an analytic function of $\zeta$ that acquires a singularity when the point $q_0(x)$ falls on the integration contour $0 < x < \infty$. This happens when $\zeta = -\alpha > 0$. Since $q_0(x) = 1 - e^{-\alpha x}$ at $\zeta = 0$, the discontinuity $\Delta f(\zeta)$ is determined by the behavior of $q_0(x)$ in the vicinity of the singular point $x = 1$, which is known. The elementary calculation of the integral that arises gives

$$
\Delta f(\zeta) = \frac{1}{\alpha} \sum_{n=0}^{\infty} \frac{\pi}{\alpha^{n+1}} \exp(-\alpha^{-n+1}), \quad \zeta > 0
$$

(10)

(for $\alpha = 1$ this formula coincides with the first term of the series (9)). Comparison of the formulas (2) and (10) shows that the faster the increase of the coefficients $a_k$, the weaker is the singularity of the function $f(x)$ at the point $x = 0$. We note that all derivatives of the discontinuity $\Delta f(\zeta)$ vanish at $\zeta = 0$.

The expression (10) contains the only information about the PT sum that can be extracted from the asymptotic form of the coefficients $a_k$ irrespective of the way of parametrizing the corrections in powers of $\alpha^{-1}$ (formula (2) or (6)) and of the summation method applied.\(^{11}\) We note that the asymptotic form of $a_k$ for $k \to \infty$ can be obtained from (10) by means of dispersion relations.\(^{12,14,15}\) The arguments presented above show that it is possible to find the discontinuity of the sum as $x \to 0$ from the form of $a_k$, i.e., the connection between the asymptotic form of $a_k$ and the character of the singularity of $f(x)$ is reciprocal. This will be used below.

4. THE IMPROVED PERTURBATION THEORY

Turning to consider the IPT, we shall investigate the question of the intervals $0 < x < x_0$, in which the functions $f(x)$ give good approximations to the exact solution.

Rewriting (4) in the form

$$
f(x) = f(0) + \sum_{k=0}^{\infty} (b_k - a_k) (-x)^k.
$$

we note that for $N > 1$ we also have $k > 1$ in this sum. Therefore, $a_k = b_k - \frac{1}{k} f(kx) \theta(x)$, whence

$$
f(x) = \frac{1}{1 + (b - a) x},
$$

The value $x = x_0$ at which the last term is comparable in magnitude with $f(x)$ can be regarded as the upper limit of applicability of the IPT. Calculating the integral for $N > 1$ by the method of steepest descents, we arrive at the following equation for $x_0$:

$$
x_0 = \frac{\sqrt{\pi} \Gamma(\frac{1}{2})}{1 + \sqrt{\pi} \Gamma(\frac{1}{2}) \alpha}, \quad \alpha = 2.718.
$$

(11)

We note that the value of $x_0$ does not depend on the parameter $\beta$ in (2) and is insensitive to the form of $f(x)$. The latter is explained by the fact that $|f(x)|^{1/\alpha} \to 1$ if $N \to \infty$, while $f(x)$ is finite or has a power behavior as $x \to 0$.

In certain cases the corrections to the leading term $a_1$ of the asymptotic form may decrease with increase of $k$ by a power law (as in (2)):

$$
a_1 \sim \frac{1}{k} \left( \frac{x}{x_0} \right)^{\alpha}, \quad \alpha = 1.5
$$

(12)

(see the examples I and II in Sec. 5). Then the region of applicability of the IPT is expanded by a factor of $\alpha^{-1}$:

$$
x_0 = \frac{1}{\sqrt{\alpha}} \left( \frac{x}{x_0} \right)^{\alpha}, \quad \alpha > 1.
$$

(13)

The general conclusion is that the region in which the IPT approximates the exact solution contracts with increase of $N$ in accordance with the law $x_0 \propto N^{-\alpha}$, i.e., it contracts more rapidly the greater the parameter $\alpha$. Popov et al. 234 Sov. Phys. JETP 47(2), Feb. 1978
5. SOME EXAMPLES

We shall compare the PT and IPT with the exact solutions for a number of examples for which the PT coefficients have the form (2).

I. The Heisenberg-Euler Lagrangian

The interaction of an electromagnetic field with the vacuum of charged particles (spin \( s \), mass \( m \)) leads to a nonlinear correction \( L' \) to the Maxwell Lagrangian \( L_0 = (E^2 - H^2)/2 \). In the case of constant and uniform fields \( E \) and \( H \) exact expressions \( \delta f_{ik}(z) \) for \( L' \) are known. 

We shall confine ourselves to the case of crossed fields \( E \neq H \neq 0 \), when \( L' \) depends only on the invariant 

\[
\zeta = \frac{e^2 (E^2 - H^2)/m^4}{4}
\]

For \( s = 0 \),

\[
\gamma_{0}(x) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n}
\]

where \( B_n \) are the Bernoulli numbers (the series for \( \gamma_{0}(x) \) converges absolutely for \( |x| < 6 \)). Substituting these expansions into (13) and integrating term by term, we obtain the PT series:

\[
f(z) = \sum_{n=0}^\infty \frac{e^2}{(2n)!} x^{2n}
\]

The asymptotic form of the coefficients \( a_{ik} \) does not depend on the spin \( s \):

\[
a_{ik} = 2e^{2n(2n-3)/m^4}
\]

and, as can be seen from Fig. 1, is established very rapidly. Therefore, the IPT has good accuracy in the present case. Applying the formula (5) with \( \mu = 2 \), \( v = -3 \), we have

\[
f(z) = \sum_{n=0}^\infty \frac{e^2}{(2n)!} x^{2n}
\]

A comparison of \( f(z) \) with \( f_{ik}(z) \) for example of spin is given in Fig. 2, in which the PT polynomials

\[
\psi_{n}(x) = \sum_{n=0}^\infty \frac{e^2}{(2n)!} x^{2n}
\]

and the IPT functions \( f_{ik}(z) \) are also depicted. The change from \( \psi_{n} \) to \( f_{ik} \) extends considerably the region of approximation of the exact solution \( f(z) \). We note that in the given problem the Bore1 sum \( f(z) \) is close to the exact solution. Even for \( x = \infty \),

\[
f(z) = \frac{1}{2} \ln x + \ldots, \quad f_{ik}(z) = \frac{2x+1}{2} \ln x + \ldots
\]

(17)

which coincides with the first term of the exact expression

\[
\delta f_{ik}(z) = \sum_{n=0}^\infty \frac{e^2}{(2n)!} x^{2n}
\]

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\]

FIG. 2. Comparison of perturbation theory with the exact solution. The dashed curves are the PT polynomials and the solid curves are the IPT functions (the numbers on the curves correspond to the values of \( N \)). For \( N \geq 5 \) the functions \( f_{ik}(z) \) coincide with the exact solution \( f(z) \) within the limits of error of the Figure.
sion following from the formula\textsuperscript{206,231} for $\text{Im } L'$ in the case of particles with spin $s$:

$$L'(\xi^2) = \sum s^2 \xi^2 \exp(-\alpha z^2).$$  

(17)

Here, $\beta_{\pm} = (-i)^{s+1}$ for bosons, $\beta_{\pm} = 1$ for fermions, and $E_0 = m^2/e$ is the characteristic field intensity at which nonlinear corrections to $L_0$ become important and pair creation in the electric field ceases to be an exponentially small effect. When $I = 0$ the formula (17) goes over into (17). In this region ($E < E_0$) the discontinuity of the Lagrangian $L'$ depends on the spin $s$ of the particles in a trivial manner:

$$\text{Im } L' = \frac{(2s+1)m^6}{4\pi^2} \exp(-nz).$$

II. Electron energy levels for $Z > 137$

The energy $\epsilon$ of a level near the edge of the lower continuum is determined by the equation\textsuperscript{232}

$$\epsilon = \frac{(2s+1)m^6}{4\pi^2} \left[ \ln(\beta_{\pm}) + (s+1) \right].$$

(18)

which is valid in the limit of small nuclear radius: $|\ln(\beta_{\pm})| \ll 1$. For the lowest level $k = 1$,

$$f(t) = \frac{1}{2s+1} \exp\left(\frac{1}{2s+1} \right).$$

(19)

where $t = z^{1/2} - 1$, $l = z^{1/2} = Z/137$, $Z$ is the nuclear charge, $L_n = Z_n^2$ and $g(s) = \Gamma(s)/\Gamma(s+1)$. Hence,

$$\epsilon_n = \frac{1}{2s+1} \left[ \ln(\beta_{\pm}) + (s+1) \right].$$

(20)

(see curve 1 in Fig. 3). The factorial growth of $a_k$ as $k \to \infty$ and the divergence of the PT series are explained by the fact that Eq. (16) for $Z > 137$ describes a quasi-stationary positron level, the imaginary part of which is determined by the penetrability of the Coulomb barrier and tends to zero exponentially at the positron-creation threshold\textsuperscript{231}.

The Borel sum ($\mu = 2$, $\nu = 0$) is equal to

$$f(t) = \frac{1}{2s+1} \exp\left(\frac{1}{2s+1} \right).$$

(21)

The behavior of $f(t)$ and $f(t)$ is analogous to that in example 1. The principal difference is that the function $f(t)$ and $f(t)$ are far from each other for $z > 0$ (see Eq. 2 in Ref. 16). In particular,

$$f(t) = a_k^{(s)} + \ldots, f(t) = a_k^{(s)} + \ldots$$

(22)

At the same time, the Borel sum accounts well for the discontinuity on the left cut $z = -\xi$:

$$A(t) = a_k^{(s)} \gamma_k^{(s)} + \ldots, A(t) = a_k^{(s)} \gamma_k^{(s)} + \ldots$$

(23)

For the analytic properties of the functions $f(t)$ and $f(t)$ see Appendix B.

Equation (18) determines the dependence of the energy of a level on the nuclear charge. For $Z$ close to $Z$, this dependence can be represented in the form of a PT series:

$$\epsilon = t^{1/2} \left[ \ln(\beta_{\pm}) + (s+1) \right].$$

(22)

The coefficients $a_k$ can be expressed in terms of $a_k$, $1 \leq k \leq b$. As follows from (10) and (20), the asymptotic form of $a_k$ is

$$a_k = \frac{2\pi}{\Gamma(k/2)} e^{-(k-1)/2}. (23)

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III. The relativistic Thomas-Fermi equation

The self-consistent potential $V(r)$ in the vacuum shell of a supercritical ($Z > 1$) atom obeys the equation\textsuperscript{232}

$$\Delta V = -\frac{\mu}{2\pi} \left[ (Z-e_0)/Z \right]^4.$$

(23)

where $\mu(x)$ is the density of protons in the nucleus, and $K = c = m = 1$. We put

$$V(r) = \frac{Z e_0^2}{r} \left[ (Z-e_0)/Z \right]^4, \quad \nu = (Z^2/2).$$

(24)

where $Z = Z - N$ is the charge of the supercritical atom for an external observer. Let $m < 1$ (the case of weak screening\textsuperscript{233}). The solutions of Eq. (23) in the region $r < 1$ possess the property of renormalizability\textsuperscript{234}.

The relativistic Thomas-Fermi equation

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us of the vacuum shell). For \( \Phi(\mu) \) it is possible to obtain a differential equation, \( \Phi^{(2)} + z \Phi = 0 \), which, by means of the substitution \( \mu = e^{2/4} \), \( \Phi(\mu) = 2x^{-1/2} \beta(x) \), is brought to the form

\[
\Phi(x) = F_0 + \sum_{n=1}^{\infty} \frac{a_n}{x^n}. 
\]

We shall study the properties of this equation. The boundary condition for \( q(g) \) follows from PT:

\[
\frac{\partial q}{\partial g} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} a_n, 
\]

for \( g \to 0 \) and \( \beta(\mu) \) can be represented in the form of PT series:

\[ q(g) = 1 - \frac{g^2}{2} + O(g^4). \]

For \( g \to 0 \) the functions \( q(g) \) and \( q(g) \) can be represented in the form of PT series:

\[ q(g) = \frac{1}{1 - \frac{g^2}{2} + O(g^4)}. \]

Substituting (26) into (25) we arrive at the recursion relations

\[ a_n = (2n+1) \sum_{k=0}^{n} a_k b_k, \]

by means of which the coefficients \( a_n \) up to \( n = 200 \) were calculated on a computer. They increase rapidly (see Table I) and have the asymptotic form:

\[ a_n \sim \frac{C_n}{n^{2n}} \exp\left(-\frac{n}{2}\right), \]

with \( C_n = 0.9455/\ldots \). Unlike the preceding examples, this expansion contains terms \( \frac{1}{2}, \frac{1}{3}, \ldots \), and the slow approach of \( a_n \) to the asymptotic form (27) is connected with this. Therefore, other representations were tried:

\[ a_n \sim \frac{C_n}{n^{2n}} \exp\left(-\frac{n}{2}\right). \]

The Stark effect

In an electric field \( F \) the level of an atom (with energy \( E = -\frac{1}{2} \), \( \lambda = \frac{1}{2} \)) is transformed into a quasi-stationary state with complex energy \( E = \frac{1}{2} - i\gamma/2 \). For \( F = F_c \), the Stark shift of the level can be expanded in a TABLE I. The coefficients \( a_n \) for the function \( q(g) \).

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Note: For \( k = 10 \) the exact values of \( a_n \) are given; for larger \( k \) the first five significant figures are given. The figures in brackets indicate the power of ten by which the number given must be multiplied.

The exact solution \( q(g) \), the PT polynomials (the dashed lines) and IPT functions (the solid curves) for the relativistic Thomas-Fermi equation.

PT series:

\[ q(g) = \frac{1}{1 - \frac{g^2}{2} + O(g^4)} \]

(25)

IV. The Stark effect

In an electric field \( F \) the level of an atom (with energy \( E = -\frac{1}{2}, \lambda = \frac{1}{2} \)) is transformed into a quasi-stationary state with complex energy \( E = \frac{1}{2} - i\gamma/2 \). For \( F = F_c \), the Stark shift of the level can be expanded in a

\[ E = E_c - \frac{F}{\Lambda}, \]

where \( \gamma / 2 \) is the Coulomb parameter and \( Z \) is the charge of the atomic core. The asymptotic form of \( a_n \) follows from (29) and (10):

\[ a_n \sim \frac{C_n}{n^{2n}} \exp\left(-\frac{n}{2}\right). \]

A comparison of (30) with numerical calculations \((31) \) for the hydrogen atom (the 1s level, for which \( \lambda = \eta = 1, \lambda = 0 \)) is shown in Fig. 3, curve 4.

The problem of the Stark effect admits an exact solution in the case of a one-dimensional \( \delta \)-potential. The asymptotic form of \( a_n \) is determined by the probability of ionization of the atom in weak \( (F < F_c) \) fields.
V. The zero-dimensional field-theory model

We consider the integral

\[ f(g) = \frac{1}{2} g \frac{d^2 f}{dg^2} \left( \left[ E_0 - \frac{g}{4} + \frac{g^2}{12} \right] \right), \]

(33)

corresponding to the "zero-dimensional model" of field theory. Here, \( n = 4, 6, 8, \ldots \)

\[ a_n = \begin{cases} \frac{1}{n} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} \]

whence

\[ a = \frac{1}{2}, \quad b = -1, \quad c = \frac{1}{2} \left( \frac{n+1}{n-1} \right)^{\frac{n-3}{n}}. \]

(34)

We have investigated the case \( n = 4 \). In this case the integral (33) can be calculated analytically:

\[ f(g) = \frac{1}{2} \hat{a} \left( \frac{1}{n-1} \right)^{\frac{n-3}{n}} E_0 \left( \frac{1}{2} \right). \]

and the discontinuity at \( g = -\xi < 0 \) is equal to

\[ \Delta f(-\xi) = -\frac{1}{2} \left( \frac{3}{2} \right)^{\frac{n-3}{n}} E_0 \left( \frac{1}{2} \right). \]

For \( \xi = 0 \) we obtain \( \Delta f(0) = -2^{-1/2} \pi^{3/4} / \sqrt{g} \), in complete correspondence with formula (10). The convergence of the PT and IPT functions to the exact function \( f(g) \) has also been investigated. The results are analogous to those obtained in the preceding examples.

With increase of \( g \) the function \( f(g) \) decreases monotonically, remaining positive for \( 0 < g < \infty \). When \( g = \infty \), \( f(g) \to 0 \) like \( g^{-1/4} \). On the other hand, the Borel sum

\[ \sum_{n=1}^{\infty} \frac{1}{n^2} (g^{-n} - 1) E_0 \left( \frac{1}{2} \right) \]

vanishes at \( g = 230 \) and then becomes negative: \( \hat{f}(g) = -2^{-1/2} \pi^{1/2} \), for \( g = \infty \). Thus, for this example too, \( f(g) \) and \( \hat{f}(g) \) have different behavior for \( g = \infty \).

The results of the investigation of the asymptotic forms of the PT series are collected in Table II, in which the parameters \( a, \beta, \) and \( \alpha \) for the examples I-V are given. For comparison, the values of these constants for the energy of the ground level of a \( d \)-dimensional anharmonic oscillator \( \phi = \frac{1}{2} x^2 + \frac{1}{4} x^4 \) with nonlinearity \( \phi \), \( \phi \) and also for the GLF in \( \phi^3/4! \) scalar field theory \( \phi^3 \) are also given in the table. We call attention to the analogy between the structure of the series for the GLF in the \( \phi^3/4! \) theory and in the relativistic Thomas-Fermi equation.

On the basis of the examples considered we arrive at the following conclusions.

As a rule, the functions \( f(x) \) approximate the exact solution \( f(x) \) in a wider range of \( x \) than does the PT polynomials of the same degree \( N \). For those \( x \) for which neighboring IPT curves \( f(x) \) and \( f(x) \) are close to each other, they approximate \( f(x) \) well and thereby make it possible to reproduce the exact solution in the interval \( 0 < x < z \). The upper bounds obtained in Sec. 4 for \( z \) are confirmed in all the examples we have considered (this question is analyzed in detail in Ref. 31).

6. CONCLUSION

The application of the IPT to problems in quantum field theory, where exact solutions are absent, is of special interest. Calculation of Feynman graphs gives the first few coefficients \( a_i \) of the PT series, and Lipatov's method determines their asymptotic form for \( k = \infty \). For the \( \phi^3/4! \) scalar field theory the first three coefficients of the GLF\(^{311}\) and the asymptotic form \( \Delta \) \(^{312}\) are known, and this makes it possible to calculate\(^4\) the functions \( \Delta \) and \( \delta \) for \( N = 1, 2, 3, 4 \). The general picture (see Fig. 5) is analogous to the examples considered above, especially the examples II and III. For \( 0 < g \leq 50 \) the curves \( \delta \) are extremely close to each other.\(^5\) From this it is possible to conclude that the exact GLF is also close to them in this range of \( g \). It differs considerably, therefore, from the curve \( L \) obtained by Lipatov\(^{311}\) by expanding in \( 1/\alpha \) in the \( \phi^3/4! \) theory (this difference can be seen clearly from Fig. 6), and this indicates the poor accuracy of the \( 1/\alpha \) expansion when \( n = 4 \). In view of this the conclusion in Ref. 5 that a zero of the GLF exists at \( g = 100 \) appears to us to be doubtful.

Application of the Padé method gives results analogous to those of the IPT. Let \( f(x) \) be the Padé approximate\(^7\) constructed from the PT coefficients \( a_i \). On the other hand, in the IPT functions (cf. (4)) we can replace the polynomial \( \sigma(x) \) by the corresponding Padé approximant; we denote such functions by \( \tilde{f}(x) \). It is obvious that

\[ \tilde{f}(x) = \sigma(f(x)) = \sigma_{a_i}(x). \]

The three known coefficients \( a_i \) and \( \sigma_i \) permit us to construct the approximants \( [3, 0] \) and \( [1, 1] \) for the GLF. Of these \( [3, 0] \) possesses the best accuracy (see Fig. 5). In the given case the Padé method does not lead to the determination of \( \Delta \) in a wider range of \( g \) than does the IPT. Possibly, this is explained by the fact that the number of known coefficients \( a_i \) is too small.

The calculations presented do not enable us to reach...
FIG. 5. The Gell-Mann-Low function in \( g^{\nu/4!} \) scalar field theory. The dashed curves correspond to the PT polynomials \( p_N(g) \) and the solid curves to the IPT functions \( \Phi_N(g) \); \( p(1/2) \) and \( p(1) \) are Padé approximants; the curve \( L \) is taken from Ref. 5.

definite conclusions about the form of the GLF for \( g>50 \) (in particular, one must not assign values to the zeros of the functions \( \Phi_N(g) \) and \( \Phi_J(g) \), as is clear from a comparison of Fig. 5 with Figs. 2 and 4; see also Ref. 14). Where the GLF is reliably determined with the aid of the IPT it is a monotonically increasing function of \( g \). We note that in the case of the relativistic Thomas-Fermi equation (for which the asymptotic form of \( a_1 \) has the same structure as in the \( g^{\nu/4!} \) theory) the GLF increases monotonically and is without zeros for all \( 0<\nu<1 \) (cf. Ref. 26).

The information that can be extracted unambiguously from the asymptotic form of \( a_1 \) concerns the character of the singularity of the GLF. For the \( g^{\nu/4!} \) scalar field theory we have\(^{14} \)

\[
\begin{align*}
q &= \frac{n}{2}, \quad \beta = \frac{n-1}{2(n-2)}, \quad s = \frac{(n-2)(n-3)}{2(n-2)}, \quad \Gamma(0) - \Gamma(n-1),
\end{align*}
\]

whence, with the aid of (10), we find the discontinuity of \( \Phi(g) \) across the cut for \( g=0 \):

\[
\Delta \Phi(g) \sim \left[ g \right]^{(n-2)/2} \exp(-4g^{3/2}),
\]

(35)

where

\[
\rho = \frac{a^{n+2}}{(n-2)!}, \quad \sigma = \frac{3}{n-2}, \quad A = \frac{1}{2(3a^{n+1})(n-2)!}, \quad \Gamma \left( \frac{3a-2}{2(n-2)} \right)
\]

(for \( n=4 \) we have \( \rho=9/2, \sigma=1 \) and \( A=16a^2 \); this case has been considered previously).\(^{13} \)

According to Lipatov\(^{15} \) for \( n=4 \) the zero of the GLF is \( g_0=103 \), i.e., \( g_0 \) is not far from the weak-coupling region: \( g_0=103/16 \pi^2 = 0.65 \). Therefore, the situation with the zero of the GLF in the \( g^{\nu/4!} \) theory can be clarified if the next few coefficients of the IPT series are calculated and a method is found that enables us to establish \( \Phi(g) \) in a wider range of \( g \) than does the IPT (the Padé method is available). However, the behavior of \( \Phi(g) \) in the strong-coupling region (\( \nu \gg 1 \)) is not connected with the asymptotic form of the coefficients \( a_k \) for \( k \to \infty \).

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APPENDIX A

The formulas that make it possible for series with factorially growing coefficients to be summed effectively are collected here. These formulas were used in the calculation of the functions \( f(z) \) and \( f_N(z) \) presented in Figs. 2, 4 and 5.

1. We shall find the sum of the series (1) with \( k=0 \) and coefficients

\[
\begin{align*}
a_{-n-1} &= \frac{(-1)^n n!}{n!},
\end{align*}
\]

(A.1)

where \( n \) is an integer. If \( n=0 \), application of the Borel method (5) with the parameters \( \mu=1 \) and \( \nu=0 \) gives

\[
\begin{align*}
\psi(a) &= \sum_{n=0}^\infty \frac{a^n}{n!} \Gamma(a) = \frac{1}{\Gamma(1-a)} \Gamma^0(-a),
\end{align*}
\]

(A.2)

where

\[
\Gamma^0(x) = \int_0^\infty e^{-x} d\theta
\]

is the incomplete gamma function.\(^{14} \) We note the particular cases

\[
\Gamma(1, x) = e^{-x}, \quad \Gamma(1, -x) = \text{Erf}(x), \quad \Gamma(0, x) = -\text{Erf}(-x),
\]

(A.3)

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where \( E_i(-x) \) is the integral exponential function and \( \text{Erfc}(z) \) is the error integral:

\[
E_i(-x) = \int_x^{\infty} \frac{e^{-t}}{t} \, dt \quad \text{and} \quad \text{Erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^{\infty} e^{-t^2} \, dt.
\]

For \( m = 1, 2, 3, \ldots \), the formulas

\[
\sum_{n=0}^{\infty} (n+m+1)(n+1)!(-1)^{n+1} = \{E_i(-m-1) + \sum_{n=(m+1)}^{\infty} (n-1)! (1 - 1^n)\},
\]

\[
\sum_{n=0}^{\infty} (n+m+1)(n+1)!(-1)^{n+1} = \frac{1}{2} \left[ \frac{e^{-x}}{x} \right] + \frac{1}{2} \int_0^x e^{-t} \, dt.
\]

hold for \( m \geq 0 \). The sum over \( p \) in the right-hand side must be omitted.

2. Turning in (A.1) to the case \( n \geq 1 \), we denote

\[
J(n, b) = \sum_{n+b}^{\infty} \left( \frac{d}{dx} \right)^n \left[ x^n \right] \ln \left[ x^n \right] = f(n, b).
\]

Thus, the sum \( \sum (x+b)^m t^n \) for integer and half-integer values of \( \beta \) is expressed in terms of the standard functions \( E_i(x) \) and \( \text{Erfc}(x^2/2) \), and this is convenient for numerical calculations. In the case of arbitrary \( \beta \) the sum is expressed in terms of the confluent hypergeometric function \( \Phi(a, c; z) \).

3. The representation of \( \phi_h \) in the form (6) is convenient for summing series, while the parametrization (2) is more intuitive. The transformation from (2) to (6) is effected by matrices \( S \) and \( S^{-1} \), the elements of which have the form

\[
S_{\alpha \beta} = \phi_{\alpha \beta}(\beta - 1) \quad (\beta = 0, 1, 2, \ldots).
\]

where \( p_1(x) \) and \( q_1(x) \) are polynomials of degree 2k:

\[
p_1(x) = x^k + \cdots + x + 1, \quad q_1(x) = x^k + \cdots + 1.
\]

APPENDIX B

ANALYTIC PROPERTIES OF THE BOREL SUM \( f(x) \) AND EXACT SOLUTIONS

The analytic properties of functions of the form (1), (2) are of great interest. We shall consider them for the examples of Sec. 5, which has exact solutions.

1. We write formula (13) in the form

\[
\phi(x) = \frac{e^{-x}}{x} \left( 1 + \frac{1}{x} \right) \ln \left( 1 + \frac{1}{x} \right).
\]

Differentiating twice with respect to \( x \) and using formula 3.554 (4) from Ref. 34, we obtain in the case of spin \( \hbar = \frac{1}{2} \):

\[
\phi(x) = \left( \frac{1}{x} \right)^{\frac{1}{4}} \ln \left( \frac{1 + x}{1 + x} \right).
\]

Thus, the sum \( \sum (x+\beta)^m t^n \) for integer and half-integer values of \( \beta \) is expressed in terms of the standard functions \( E_i(x) \) and \( \text{Erfc}(x^2/2) \), and this is convenient for numerical calculations. In the case of arbitrary \( \beta \) the sum is expressed in terms of the confluent hypergeometric function \( \Phi(a, c; z) \).

\[
\phi(x) = \frac{e^{-x}}{x} \left( 1 + \frac{1}{x} \right) \ln \left( 1 + \frac{1}{x} \right).
\]

Integrating this equality and using the expansion of \( \ln(1+x) \) near \( x = 0 \) (see Ref. 34), we obtain for \( x = 0 \) (the region of strong fields \( \left| x^2/2 \right| \gg m^2/\hbar \) )

\[
\phi(x) = \frac{1}{2} \left[ \frac{1}{x} \right]^{\frac{1}{4}} \ln \left( \frac{1 + x}{1 + x} \right).
\]

2. Turning in (A.1) to the case \( n \geq 1 \), we denote

\[
J(n, b) = \sum_{n+b}^{\infty} \left( \frac{d}{dx} \right)^n \left[ x^n \right] \ln \left[ x^n \right] = f(n, b).
\]

Thus, the sum \( \sum (x+b)^m t^n \) for integer and half-integer values of \( \beta \) is expressed in terms of the standard functions \( E_i(x) \) and \( \text{Erfc}(x^2/2) \), and this is convenient for numerical calculations. In the case of arbitrary \( \beta \) the sum is expressed in terms of the confluent hypergeometric function \( \Phi(a, c; z) \).

The values of these polynomials for integer \( s = -1, 0, 1, \ldots \) are equal to

\[
p_1(x) = \frac{1}{2} \left[ \frac{1}{x^2} \right]^{\frac{1}{4}} \ln \left( \frac{1 + x}{1 + x} \right).
\]

In the important particular cases \( \beta = 0, 1, \) these formulas can be transformed to a more convenient form in which all the coefficients in the polynomials are integers:

\[
J(n, b) = \sum_{n+b}^{\infty} \left( \frac{d}{dx} \right)^n \left[ x^n \right] \ln \left[ x^n \right] = f(n, b).
\]

Then, the representation of \( \phi_h \) in the form (6) is convenient for summing series, while the parametrization (2) is more intuitive. The transformation from (2) to (6) is effected by matrices \( S \) and \( S^{-1} \), the elements of which have the form

\[
S_{\alpha \beta} = \phi_{\alpha \beta}(\beta - 1) \quad (\beta = 0, 1, 2, \ldots).
\]

where \( p_1(x) \) and \( q_1(x) \) are polynomials of degree 2k:

\[
p_1(x) = x^k + \cdots + x + 1, \quad q_1(x) = x^k + \cdots + 1.
\]

The values of these polynomials for integer \( s = -1, 0, 1, \ldots \) are equal to

\[
p_1(x) = \frac{1}{2} \left[ \frac{1}{x^2} \right]^{\frac{1}{4}} \ln \left( \frac{1 + x}{1 + x} \right).
\]

In the important particular cases \( \beta = 0, 1, \) these formulas can be transformed to a more convenient form in which all the coefficients in the polynomials are integers:

\[
J(n, b) = \sum_{n+b}^{\infty} \left( \frac{d}{dx} \right)^n \left[ x^n \right] \ln \left[ x^n \right] = f(n, b).
\]
For the relativistic Thomas-Fermi equation it can be shown that, for \( 0 < p < 9/2 \) and \( g = \infty \),
\[
\Phi(g) = \sum_{n=1}^{\infty} (-g^n)^{p} = \left( \frac{d^n}{dg^n} \Phi(g) \right)_{g=0}, \quad n \neq 0 \text{ is not an integer}
\]
\[
\Phi(g) = \sum_{n=0}^{\infty} \frac{1}{n!} \left( -g \right)^n = \left( \frac{d^n}{dg^n} \Phi(g) \right)_{g=0}, \quad n = 0 \text{ is an integer}
\]

where
\[
A^{p} - z^{1-p}(1 - p) - 1^{1-p}/z^{1-p}
\]

It can be shown that, for \( 0 < p < 9/2 \) and \( g = \infty \),
\[
\Phi(g) = \sum_{n=1}^{\infty} (-g^n)^{p} = \left( \frac{d^n}{dg^n} \Phi(g) \right)_{g=0}, \quad n \neq 0 \text{ is not an integer}
\]
\[
\Phi(g) = \sum_{n=0}^{\infty} \frac{1}{n!} \left( -g \right)^n = \left( \frac{d^n}{dg^n} \Phi(g) \right)_{g=0}, \quad n = 0 \text{ is an integer}
\]

where
\[
A^{p} - z^{1-p}(1 - p) - 1^{1-p}/z^{1-p}
\]

The behavior of the functions \( \Phi(g) \) in the region of large \( g \) depends essentially on \( p \), although the coefficients \( \Phi_N \) for \( h = 1 \) differ only by terms of order \( 1/d \).

Note added in proof (December 15, 1977). We have now considered a further example: the perturbation-theory series for the energy \( E_n(g) \) of the ground level of an anharmonic oscillator with nonlinearity \( g \) in a space of \( d \) dimensions. For \( d = 1 \) the coefficients of this series were calculated by Bender and Wu,\( ^{27} \) and this permits us to construct the PT polynomials and the IPT functions. A comparison of these with the exact values of \( E_n(g) \) obtained by numerical solution of the Schrödinger equation leads to the same conclusions as in Sec. 5. For \( N = 2 \) the region of applicability of the \( N \)-th approximation of perturbation theory contracts: \( E_2 \) \( g \to 3k \). On the other hand, the calculation by the Padé-approximant method of the energy \( E_2(g,\beta) \) leads to the result that the region in which the exact solution \( E_2(g) \) is approximated increases with increase of \( N \). This is an important advantage of the Padé method in the summation of series with factorially increasing coefficients.

However, this advantage is manifested only in cases when a sufficient number of the first coefficients \( \Phi_n \) of the series are known.

1) See Ref. 18, in which an analogous method was used to establish the relationship between the nearest singularity of the scattering amplitude \( f(\theta) \) and the asymptotic form of the partial amplitudes \( f_1(\theta) \) for \( l = \infty \).

2) This statement assumes the absence of an essential singularity in the exact solution of terms of the type \( \Phi(g) = \alpha(g) \exp(-z^m) \), where \( m > 0 \) and \( \Phi(g) \) is a function that does not have an essential singularity at \( z = 0 \). It is obvious that the PT series does not give any information about the presence of such terms in the exact solution, since all derivatives of the function \( \Phi(g) \) vanish at \( z = 0 \). The presence of such terms should be established from independent considerations (see, e.g., Ref. 7). It is known that the existence of such terms, which are not present in the problems we are considering, lead to the appearance of such terms.

3) The quantities \( \alpha, \beta, \gamma, \) and \( \delta/\gamma \) were obtained from the recursion relations for \( \Phi_n \) given above. The coefficient \( \gamma \) was found by comparing the asymptotic form (27) with the exact values of \( \Phi_N \) for \( h = 150 - 200 \).

4) It should not be thought that the good approximation of \( f(\theta) \) by the ITP functions is due to the fact that the Borel sum \( f(\theta) \) is close to the exact solution. This occurs only in exceptional cases (example 3); usually, these functions are very far from each other.

5) In the calculation of \( E_n(g) \) the asymptotic form of the coefficients was written in the form \( \Phi_n = c_1 \Phi_1 + \ldots \Phi_n \) and formula (B.8) from Appendix A was used.

On the other hand, the polynomial \( p_n(g) \) with \( N = 2, 3 \) and 3 are close to each other only in the interval \( 0 < g < 20 \). For
larger values of $g$ they differ considerably, both from the IPT functions and amongst themselves (e.g., for $g=10$ we have $p_1=0.27$ and $p_2=0.22$, while $p_1=0.01$). The advantage of the IPT over the PT is clear from this.

11., the fraction $P_n(g)/P_M(g)$, where $n$ and $M$ are the degree of the numerator and denominator (33).

12. For $(a + 2b + 1)^{n}$ the series in (2.3) is convergent, but the first term $a^n$ has a branch point at infinity. The possibility of obtaining a convergent expansion for $L^n$ in inverse powers of the external field was noted earlier (35).


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