Calculation of structure functions of deep-inelastic scattering and e^+e^- annihilation by perturbation theory in quantum chromodynamics

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A model of fermions connected with Yang-Mills fields of the nonabelian gauge group $SU(N)$ is considered. A method based on an analysis of the Feynman diagrams makes it possible to write down in the principal logarithmic approximation a closed expression for the inclusive cross sections of electron scattering by a quark (gluon) in e^+e^- annihilation into a quark (gluon). A specially chosen gauge makes it possible to describe the structure of the deep-inelastic processes in the language of the parton model with virtual quarks and gluons in the role of the partons. The asymptotic properties of the parton distributions are analyzed. An indication of certain duality between the quasi-elastic and Regge limits is discussed. The Gribov-Lipatov relation and the analytic connection (in a certain sense) between the scattering and annihilation channels (the Drell relation) hold in the model under consideration. In addition, a "sum rule in O^2," which unique to the Yang-Mills theory, has been established for the distributions of the number of the partons; this rule singles out a model in which the number of "flavors" is equal to the number of colors. The results are directly applicable to an analysis of the experimental situation in lepton-hadron reactions.

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Nonabelian gauge fields are being comprehensively studied ever since the discovery, in 1973, of asymptotic freedom.[1] The model of colored quarks connected with the Yang-Mills gluon field makes claim to a prominent role as a dynamic theory of strong interactions.

Quantum chromodynamics (QCD) has attracted attention because of the enticing prospect of combining asymptotic freedom, which is responsible for the pointlike structure of hadrons at short distances, with the "infrared slavery" for the color states. The enthusiasts of QCE are usually cool to the fact that progress in the explanation of the "confinement" mechanism is restricted so far to the premise that a *possible* connection exists between the infrared pole and the catastrophic enhancement, at large distances, of the colored interaction that confines the quarks. Gribov[2] has shown that an analysis of the region $\bar{g}^2 \sim 1$ by modern field-theoretical methods meets with serious difficulties that are peculiar to the case of a nonabelian gauge field. The problem of constructing a quantum theory on the basis of the Yang-Mills Lagrangian is at present in the stage of correct formulation.

Nonetheless, it might be assumed that the quark-gluon field model is applicable at short distances, where the interaction is weak, $\bar{g}^2(r) \sim |\ln r|^{-1}$, and the customary methods of expansion in the coupling constants are applicable. A classical example of processes in whose development short distances play a role are the lepton-hadron reactions $e^-p \to e^- + \ldots$, $e^+e^- \to$ hadrons, where the hadron structure is prepared by a strongly virtual photon.

As applied to processes of the first type, the renormalization approach served as a basis for the development of an angular-momentum technique[3] that makes it possible to explain the picture of the phenomena in deep-inelastic scattering of electrons and neutrinos (see, e.g.,[4]). Semiphenomenological attempts are made to reconstruct the structure function from the known asymptotic form of the angular momenta (e.g.,[5]). The chromodynamic interaction of quarks has been a successful concept in the physics of new particles.[6]

In the present paper are calculated the structure functions of deep-inelastic scattering and e^+e^- annihilation in a model wherein the fermions interact with vector Yang-Mills fields of the nonabelian gauge group $SU(N)$. The procedure is the following: We separate from the photon forward-scattering amplitude, which determines the structure function, the diagrams that yield the highest degree of $\ln(|q^2|/\mu^2)$ (q is the momentum transfer and μ is the normalization mass):

$$W^{(n)} = \left(g_\mu^2 \ln \frac{|q^2|}{\mu^2} \right)^n f_n(\omega).$$

The sum of these contributions yields W in the principal logarithmic approximation with

$$g_\mu^2 \ll 1, \quad g_\mu^2 \ln(|q^2|/\mu^2) \sim 1.$$

This program was realized in 1972 by Gribov and Lipatov in models wherein fermions are coupled with pseudoscalar ($L_{int} = \bar{\psi}\gamma_5\psi\varphi$) and massive vector ($L_{int} = \bar{\psi}\gamma_\mu\psi A^\mu$) mesons.[7]

The zero-charge problem does not appear in the present model, so that we can go over to infinitely large q^2. The behavior of the structure function in the Bjorken limit $|q^2| \to \infty$, $\omega = \text{const}$ is determined by a conflict between two tendencies: the form-factor induced drop of each individual process and the growth of the cross section on account of newly produced channels. As a result, the asymptotic form in the region of academically large q^2, $\ln\ln|q^2| \gg 1$, is of the form

$$W \sim (\ln|q^2|)^{-(4C_2/\beta_2)\ln\ln\ln|q^2|},$$

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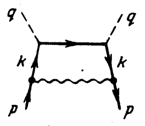


FIG. 1. Diagram of first order in g^2 for the fermion structure function.

where C_2 and β_2 are certain numbers defined below (Eq. (45)). The weak dependence of the structure function on q^2 in a wide range of ω allows us to speak of an approximate gauge invariance.

From among the known infrared singularities of gauge theories, our analysis pertains, strictly speaking, to the case of broken gauge symmetry, when the vector mesons acquire a mass, e.g., via the Higgs mechanism. In the situation of actual interest (chromodynamics), where the strong coupling of the color fields at large distances plays the principal role, the infrared integrals should be cut off at characteristic virtualities $\sim\mu^2$ connected with the average transverse momentum of the quark in the hadron wave function, i.e., in fact with the radius of the infrared prism. Calculating the structure function of the quark by perturbation theory, we shall formally saturate the optical theorem with quark and gluon states, and ignore the stage of transition of the colored fields into colorless hadrons. The simplest assumption whereby the hadron is made up of valent quarks consists in the fact that the gluons that bind the quarks together do not carry any noticeable fraction of the momentum whatever in the system where the proton is fast. This hypothesis, together with the quark structure functions obtained below, makes it possible to describe both the quantity W and the character of scaling violation at $1-x\ll 1$ and $x\ll 1$ for the most unpleasant choice of the probability of finding the quark in the proton and the normalization point μ. In this paper we shall not dwell on the description of the experimental data.

The structure of the paper is the following: In Secs. 1 and 2 is described the procedure of selecting the diagrams that are essential in the principal logarithmic approximation (PLA or the Gribov–Lipatov approximation), and write down the Bethe-Salpeter equation for deep inelastic scattering of a photon by a quark. Sections 3–5 are devoted to the interpretation of the PLA results in the language of the parton model with a cutoff parameter $\Lambda^2\sim q^2$,[8] an interpretation made possible by a special choice of the "physical" gauge; the asymptotic properties of W and the indication of duality between the quasi group and the Regge regions are discussed. In Secs. 6–8 are studied the relations between the structure functions of the scattering and of the e^+e^- annihilation, and it is indicated that a unique "sum rule in q^2" exists in chromodynamics for the parton distribution in the quark and the gluon.

1. DEEP-INELASTIC SCATTERING BY QUARK. SELECTION OF DIAGRAMS THAT ARE ESSENTIAL IN THE PRINCIPLE LOGARITHMIC APPROXIMATION. AXIAL GAUGE

Consider the forward-scattering amplitude of a photon by a quark in the PLA

$$\frac{g_\mu^2}{4\pi^2}\ll 1,\quad \frac{g_\mu^2}{4\pi^2}\ln\frac{|q^2|}{\mu^2}\sim 1, \tag{1a}$$

$$\omega\sim 1. \tag{1b}$$

The assumption of the limiting value (1b)

$$\frac{g_\mu^2}{4\pi^2}\ln\frac{1}{\omega-1}\sim 1 \quad \text{(quasielastic region)},$$

$$\frac{g_\mu^2}{4\pi^2}\ln\omega\sim 1 \quad \text{(Regge region)}$$

will be discussed below.

We designate the quark momentum by p_μ. Corrections of order are not controlled by the approximation, so that we can put $p^2=0$. We introduce a light-like vector q'_μ that defines together with p the scattering hyperplane in momentum space:

$$q_\mu=q_\mu'-\frac{1}{\omega}p_\mu,\quad (q_\mu')^2=0,\quad \omega=-\frac{2pq}{q^2}\approx-\frac{2pq'}{q^2}\equiv-\frac{s}{q^2} \tag{2}$$

and represent all the momenta, following Sudakov, in the form

$$k_i=\alpha_i q'+\beta_i p+k_{i\perp}. \tag{3}$$

In the Born approximation, only a γ photon with transverse polarization has a large cross section for scattering by a fermion, since the forward-scattering-amplitude discontinuity, which determines the cross section, is given by

$$\operatorname{Im}A_{\mu\nu}=\operatorname{Im}\left[e^2\frac{1}{2}\operatorname{Sp}\left\{\hat p\gamma_\mu\frac{1}{-(\hat p+\hat q)-i\varepsilon}\gamma_\nu\right\}\right]\approx-g_{\mu\nu}{}^{\perp}e^2\pi\delta\left(1-\frac{1}{\omega}\right). \tag{4}$$

The relation that follows from (4) between the structure functions W_1 and νW_2 is preserved in all orders of the PLA, so that it is necessary to calculate only one scalar function

$$W(\omega,q^2)=2mW_1=\omega\nu W_2,\quad -g_{\mu\nu}{}^{\perp}W=\pi^{-1}\operatorname{Im}A_{\mu\nu}. \tag{5}$$

Which processes make the main contribution to W? We examine qualitatively the simplest diagram shown in Fig. 1. It differs from the case of an abelian gauge field only by an additional factor

$$F^aF^b\delta_{ab}=\frac{N^2-1}{2N}=C_2$$

which is the Casimir operator of the fundamental representation of the $SU(N)$ group, in accord with which, we assume, the fermion fields are transformed under colored rotations. The large logarithm $\ln q^2$ is accumulated here upon integration with respect to $k_\perp$ in a wide range $\mu^2\ll-k_\perp^2\ll|q^2|$. Closing the integral with respect to α around the pole of the gluon propagator we obtain (without writing out the integral with respect to β)

$$A\sim g_\mu^2\int\frac{d^2k_\perp}{(k_\perp^2)^2}\operatorname{Sp}\{\hat p\gamma_\sigma\hat k\ldots\hat k\gamma_\tau\}g_{\sigma\tau}, \tag{6}$$

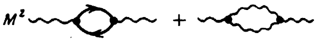


FIG. 2. Decay in final state of a produced particle with virtuality.

$$\alpha s \sim k_\perp^2, \quad \beta \sim 1 \quad (0<\beta\leqslant 1). \tag{7}$$

In (6) we have used for concreteness the Feynman gauge for the Green's function of the vector field.

It is easily seen that in the diagram of Fig. 1 only the transverse polarizations of the gluon are operative. Indeed, from among the two terms of the expansion

$$g_{\sigma\tau} = g_{\sigma\tau}^{\perp} + (p_\sigma q_\tau' + p_\tau q_\sigma')/(pq') \tag{8}$$

a substantial contribution to (6) is made only by the first, for otherwise the trace contains two neighboring matrices $\hat p$, $\hat p \hat p = p^2 \approx 0$.

We note a circumstance of importance for the analysis of more complicated diagrams. The denominator of the integrand in (6) contains a high degree of $k_\perp$ on account of two propagators. To separate the principal logarithmic contribution in (6) it suffices now to retain in the spinor numerator the term of order $k_\perp^2$. Using the expansion (3) and (7) we obtain

$$A \sim g_\mu^2 \int \frac{d^2 k_\perp}{k_\perp^4} \mathrm{Sp}\{\gamma_\sigma^{\perp}\gamma^{\sigma\perp}\hat p \hat k_\perp \ldots \hat k_\perp\} \sim g_\mu^2 \int_{\mu^2}^{|q^2|} \frac{d(-k_\perp^2)}{-k_\perp^2}. \tag{9}$$

In all the subsequent analysis, just as in the derivation of (9), we are dealing with large virtualities

$$|k^2| \sim |k_\perp^2| \gg \mu^2. \tag{10}$$

This means that we must take into account the effects of small distances, which reduce in the PLA (1) to replacement of the virtual propagators and vertices by the renormalized quantities in the single-loop approximation. Using the cell considered by us as an example, this leads to the appearance of factors

$$d_F^2(k_\perp^2)\Gamma^2(k_\perp^2) \tag{11}$$

under the integral sign in (9).

Since we are interested in the total interaction cross section, we must consider besides the pole diagram also diagrams with decay of the produced gluon. The PLA admits of an invariant pair mass (Fig. 2) $M^2 \lesssim -k_\perp^2$. Replacement of the pole expression by the complete imaginary part of the propagator with virtuality $-k_\perp^2 \gg \mu^2$ reduces effectively, as can be easily verified with the aid of the Kallen–Lehmann representation, to additional multiplication of the integrand in (9) by $d_G(k_\perp^2)$. We obtain ultimately

$$A \sim g_\mu^2 \int \frac{d(-k_\perp^2)}{-k_\perp^2} d_G(k_\perp^2) d_F^2(k_\perp^2)\Gamma^2(k_\perp^2) = \int_{\mu^2}^{|q^2|} \frac{d(-k_\perp^2)}{-k_\perp^2} \bar g^2(k_\perp^2). \tag{12}$$

The natural variable that determines the q^2-dependence of the structure function, is the quantity[7]

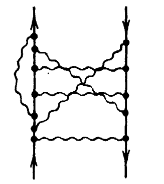


FIG. 3. Example of nonplanar diagram that appears in the vector theory.

$$\xi = \frac{1}{16\pi^2}\int_{\mu^2}^{|q^2|} \frac{dk^2}{k^2}\bar g^2(k^2) = \frac{1}{\beta_2}\ln\frac{g_\mu^2}{\bar g^2(q^2)} = \frac{1}{\beta_2}\ln\left(1+\beta_2\frac{g_\mu^2}{16\pi^2}\ln\frac{|q^2|}{\mu^2}\right), \tag{13}$$

where β_2 is the highest-order coefficient in the Gell-Mann–Low function

$$\frac{d}{d\ln k^2}\ln \bar g^2(k^2) = -\frac{1}{16\pi^2}\beta_2 \bar g^2(k^2) + \ldots; \tag{14}$$

$\beta_2 = \frac{11}{3}N - \frac{2}{3}n_f$ for the $SU(N)$ group with n_f flavors of the N-colored quarks.

The logic of the analysis of the diagram in Fig. 1 can be easily extended to ladder diagrams (Figs. 6a and 6b below), and a diagram with n cells makes a contribution

$$W^{(n)} \sim \int_0^{\xi} d\xi_n \int_0^{\xi_n} d\xi_{n-1} \ldots \int_0^{\xi_2} d\xi_1 \sim \xi^n,$$

which is connected with the ordered momentum region

$$\begin{gathered}\mu^2 \ll -k_{1\perp}^2 \ll -k_{2\perp}^2 \ll \ldots \ll -k_{n\perp}^2 \ll |q^2| \sim s,\\ \alpha_i s \sim k_{i\perp}^2, \quad \beta_i \sim 1 \quad (1 \geqslant \beta_1 \geqslant \beta_2 \geqslant \ldots \geqslant \beta_n > 0).\end{gathered} \tag{15}$$

We see therefore that the quantity $\xi = \xi(q^2)$ is the true expansion parameter of the amplitude

$$W(\omega, q^2) = \sum_{n=0}^{\infty}\left(g_\mu^2 \ln\frac{|q^2|}{\mu^2}\right)^n f_n(\omega) \to \sum_{n=0}^{\infty} \xi^n(\bar g^2(q^2)) f_n'(\omega). \tag{16}$$

We proceed to the analysis of more complex diagrams. We have seen with the aid of a simple example that substantial contributions to diagrams of the ladder type are made only by vector mesons with polarizations orthogonal to the reaction plane. In the vector theory, however, a comparable contribution to W is made also by nonplanar diagrams in which the gluons subtend over an arbitrary number of cells. Let us verify that in this case the gluons turn out to be longitudinally polarized—"bremsstrahlung" gluons (Fig. 3).

Let us recall the logic underlying the analysis of nonplanar diagrams.[7] In order not to disturb the ladder diagrams and simultaneously see to it that the integration with respect to k' lead to the appearance of a large logarithm, it is necessary to subject the momentum of the encompassing meson to the inequality (Fig. 4)

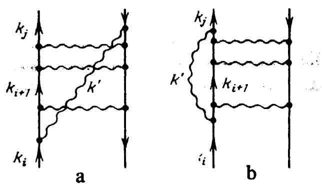


FIG. 4. Emission of a real (a) and virtual (b) longitudinally polarized (bremsstrahlung) vector meson.

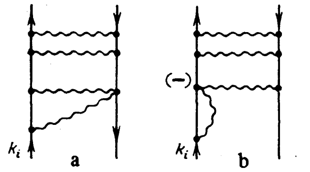


FIG. 5. Reduced diagrams that describe the joint contribution due to emission of real (a) and virtual (b) bremsstrahlung vectors from cell i into cells having jumbers $j > i$.

$$\mu^2 \ll \ldots \ll -k_{i\perp}^2 \ll -k_\perp'^2 \ll -k_{i+1\perp}^2 \ll \ldots \ll -k_{j\perp}^2 \ll \ldots \ll |q^2|. \tag{17}$$

The only critical circumstance in this case is the following: by virtue of the condition (17) we have $-k_\perp'^2 \ll -k_{j\perp}^2$ and the integral with respect to $k_\perp'$ (after integrating with respect to α') contains not two virtual fermion propagators, as before, but only one:

$$A \sim \int \frac{d^2k_\perp'}{k_\perp'^2} \mathrm{Sp}\{\hat{p}\gamma_\sigma(\hat{k}_i - \hat{k}')\ldots\} d_{\sigma\tau}(k') \tag{18}$$

(cf. (6)). We are therefore forced to retain from the spinor numerator the structure

$$\mathrm{Sp}\{p\gamma_\sigma p(\beta_i-\beta')\ldots\} d_{\sigma\tau}(k') \to p_\sigma d_{\sigma\tau}(k') \mathrm{Sp}\{p\ldots\}. \tag{19}$$

In a standard gauge (e.g., $d_{\sigma\tau}(k') = g_{\sigma\tau}$), we can make with logarithmic accuracy the substitution

$$p_\sigma \to k_\sigma'/\beta' \tag{20}$$

and, using Ward's identity, represent the total contribution from the emission of longitudinally polarized particles from the i-th cell to all cells numbered $j > i$ in local form (see Fig. 5). The reduced diagrams of Fig. 5 can be represented in the following manner. The charged particle, radically altering its virtuality ($|k_i|^2 \ll |k_{i+1}|^2$), "jars" the bremsstrahlung field. The diagrams in Figs 5a and 5b describe respectively the emission of real and virtual bremsstrahlung quanta. (The minus sign in front of diagram b emphasizes the fact that the virtual emissions decrease the cross sections.)

In chromodynamics the gluons are charged, so that besides the new ladder-type diagrams of the 6c type it is necessary to take into account also the bremsstrahlung from the gluon. The procedure described above for the summation of the contributions of the longitudinal vectons becomes obscure in the Yang–Mills model, owing to the appearance of ghosts and of an additional four-gluon vertex. It turns out that instead of using current conservation in each individual diagram in which bremsstrahlung particles appear, it is possible to choose initially a gauge such that the gluon propagator does not propagate at all in the PLA of states polarized in the (p, q) plane. All that need be required of the sought gauge is

$$p_\mu d_{\mu\nu}(k) = O(k_\perp). \tag{21}$$

In this case the integral (18) is determined by the region near the upper limit, which leads in final analysis to a loss of $\ln|q^2|$ in the cross section and makes it necessary to leave out the diagrams of Figs. 3 and 4 in the PLA.

Condition (21) is satisfied by the so-called axial gauge

$$q'^\mu A_\mu(k) = 0, \quad d_{\mu\nu}(k) = g_{\mu\nu} - \frac{k_\mu q_\nu' + k_\nu q_\mu'}{(kq')}; \quad q'^2 = 0. \tag{22}$$

In the gauge (22), which was applied by Lipatove[6] to the calculation of the structure functions, the set of the PLA diagrams is accounted for fully in electrodynamics by planar ladder diagrams of the type 6a and 6b. In chromodynamics, the axial gauge is convenient in two ways: first, like any gauge of the type

$$a^\mu A_\mu^a(k) = 0, \quad a^\mu = \mathrm{const} \tag{22a}$$

it is free of ghosts[1)]; second, it preserves the ladder character of the principal diagrams in $\ln q^2$ when account is taken of the gluon self-action (Fig. 6c).

To prove the last statement we consider a nonplanar diagram with emission from a gluon (Fig. 7a or 7b). Attempting to separate the principal logarithmic region (17) we obtain, as before, for the "skew" cell the expression

$$\int \frac{d^2k_\perp'}{k_\perp'^2} \{\Gamma_{\pi\rho\sigma}(k_i, k', k_i - k') d_{\sigma\sigma'}(k') d_{\rho\rho'}(k_i - k')\}. \tag{23}$$

The numerator now contains in place of the trace the product of the Yang–Mills vertex by the axial propagators, which by virtue of the property

$$d_{\mu\nu}(k') = g_{\mu\nu}^\perp + O(k_\perp') \tag{24}$$

(we recall that $\alpha' \sim k_\perp'^2$) must be replaced by[2)]

$$\int \frac{d^2k_\perp'}{k_\perp'^2} \{\Gamma_{\pi\rho\sigma}(k_i, k') g_{\sigma\sigma'}^\perp g_{\sigma\sigma'}^\perp\}. \tag{25}$$

It is easy to verify, by examining the i-th cell, that

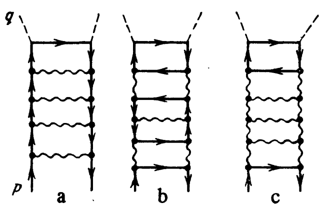


FIG. 6. Ladder diagrams that determine the structure function W in the principal logarithmic approximation.

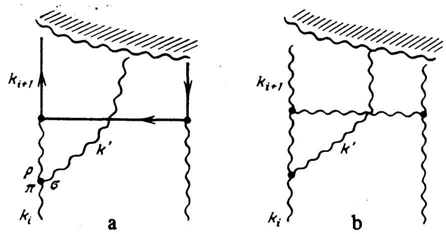


FIG. 7. Emission of a bremsstrahlung gluon by a gluon.

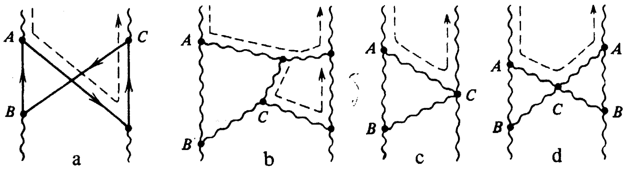


FIG. 8. Examples of diagrams that make not contribution to the PLA in an axial gauge. The dashed lines mark the "paths" of large momenta (see the text).

the third index of the gluon vertex should also be transverse. Consequently the expression in the curly brackets cannot be prevented from growing: $\Gamma_{\pi\rho\sigma}$ is a linear combination of the π, ρ, and σ components of the momenta that enter in the vertex, and is inevitably proportional to $k'_\perp$. Thus, there is no logarithmic contribution, and any diagram such as 7a or 7b makes no contribution to the PLA in the axial gauge.

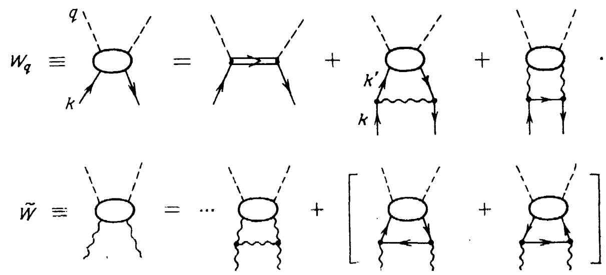
The remarks made concerning the axial gauge in the two considered cases (Figs. 4 and 7) are sufficient for the derivation of the selection rule.

In the gauge (22), *no contributions are made* in the principal logarithmic approximation by diagrams with: 1) more than a two-particle state in the t channel 2) four-gluon vertices.

Consider for example certain diagrams that contribute in a standard (say, Feynman) gauge but need not be taken into account in an axial gauge (Fig. 8). The logic of the analysis is the following: in the logarithmic region all the momenta are strictly ordered (15). We consider the cell corresponding to a momentum k with the minimal virtuality (contour ABC in Fig. 8). In the propagators marked by the dashed lines, large momenta predominate, so that the integral with respect to $k_\perp$ (after integration with respect to α, which reduces to a residue at a pole of the propagator BC) is determined only by one virtual propagator AB. Therefore, just as in the examples analyzed above, the integral with respect to $k_\perp$ turns out to be nonlogarithmic.

2. THE BETHE-SALPETER EQUATION

It follows from the discussion in the preceding section that the deep-inelastic scattering by a quark is determined by processes of the type shown in Fig. 9, and to find the structure function W in the PLA it is necessary to gather ladder diagrams of Fig. 6, calculated in the gauge (22). This is easily done with the aid of the system of Bethe–Salpeter equations, which has the graphic form:



(26)

The blocks denote here the amplitudes, averaged over the target polarizations, of the γ-quantum scattering by a quark and a transverse gluon.

Leaving out the unit matrices that fix the colorless state in the t channel, we express (26) in analytic form:

$$W_q(\omega_k, \xi_k; \xi) = \frac{e_q^2}{d_F(\xi)}\delta\left(1 - \frac{1}{\omega_k}\right) + \int_{\xi_k}^{\xi} d\xi' \int_0^1 \frac{dx}{x}\Phi_F{}^F(x)\cdot$$
$$W_q(x\omega_k, \xi'; \xi) + \int_{\xi_k}^{\xi} d\xi' \frac{d_G(\xi')}{d_F(\xi')}\int_0^1 \frac{dx}{x}\Phi_F{}^G(x)\tilde{W}(x\omega_k, \xi'; \xi), \tag{26a}$$

$$\tilde{W}(\omega_k, \xi_k; \xi) = \int_{\xi_k}^{\xi} d\xi' \int_0^1 \frac{dx}{x}\Phi_G{}^G(x)\tilde{W}(x\omega_k, \xi'; \xi)$$
$$+ \int_{\xi_k}^{\xi} d\xi' \frac{d_F(\xi')}{d_G(\xi')}\int_0^1 \frac{dx}{x}\Phi_G{}^F(k)\sum_{q,\bar{q}=1}^{n_f}[W_q(x\omega_k, \xi'; \xi) + W_{\bar{q}}]. \tag{26b}$$

The antiquark structure function $W_{\bar{q}}$ satisfies Eq. (26a) with the substitutions $W_q \to W_{\bar{q}}$, $e_q \to e_{\bar{q}}$; in the case of electromagnetic (but not weak!) interaction, $W_{\bar{q}}$ coincides with W_q.

We present some explanations of Eqs. (26).

1) These equations determine the amplitudes off the "mass shell" as functions of $\xi_k \equiv \xi(k^2)$, $\omega_k = -2kq/q^2 = \beta_k\omega$. It is necessary to put in the answer $\xi_k = 0$ and $\omega_k = \omega$. The parameter $\xi = \xi(q^2)$ determines the initial condition: at $\xi_k = \xi$ there are no ladder diagrams and the process is determined by a Born diagram.

2) In the Born cross section (the first term in (26a) account is taken of the form factors $\gamma_{em}(q^2)$ and of the disintegration of the leading quark into states with invariant mass all the way to $M^2 \sim |q^2|$. Then, using the Ward identity for the electromagnetic vertex, we have

$$\gamma_{em}^2(q^2)d_F(q^2) = d_F^{-1}(\xi). \tag{27}$$

3) In the integrals that connect W with W and $\tilde{W}$ with $\tilde{W}$, the entire dependence on the virtuality is concentrated in the virtual charge $d_F^2 d_G \Gamma_{FFG}^2 = \bar{g}^2 = d_G^3\Gamma_{GGG}$. In the two other cases there is no cancellation of the ratio of the propagators

$$d_F(\xi') = \exp(-\gamma_F\xi'), \quad d_G(\xi') = \exp(-\gamma_G\xi'), \tag{28}$$

where γ_F and γ_G are the anomalous dimensionalities of the quark and gluon fields in the axial gauge.

Let us write out, in concluding this section, the functions $\Phi_B^A(x)$ that determine the dependence of the kernels of the Bethe–Salpeter equations on the energy fraction x carried away by the particle B in the decay of the parti-

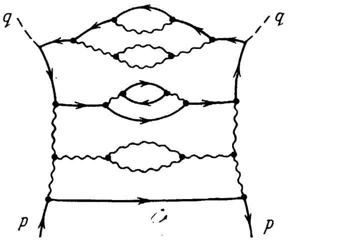


FIG. 9. Parton structure of the deep-inelastic scattering of a photon by a quark (p) in the PLA in the axial gauge.

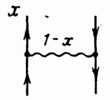
cle $A(A, B=F, G$; the subscript F will henceforth label any of the $2n_f$ fermion states—an F quark or antiquark of n_f flavors). It is convenient here to separate the group factors from the summation over the color indices:

$$\Phi_F{}^F(x)=C_2V_F{}^F(x),\ \Phi_G{}^G(x)=NV_G{}^G(x),$$
$$\Phi_F{}^G(x)=C_2V_F{}^G(x),\quad \Phi_G{}^F(x)={}^1/_2V_G{}^F(x); \tag{29}$$

here

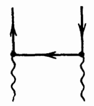
$$C_2\delta_{ij}=F_{ik}{}^aF_{kj}{}^a,\quad N\delta_{ab}=f_{acd}f_{bcd},\quad {}^1/_2\delta_{ab}=F_{ik}{}^aF_{ki}{}^b.$$

After separating the color factors, the first three kernels of those listed below coincide with the corresponding expressions in the abelian case; the last kernel is calculated in accord with the same logic: from the product of the Yang–Mills vertices by the axial propagator we separate the terms that are logarithmic in $k_\perp^i$; the corresponding x-dependences ($x\equiv\beta'/\beta_k$) make up the kernel $V_G^G(x)$. We have

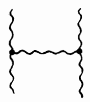


$$\equiv V_F^F(x)=2\frac{1+x^2}{1-x}, \tag{30a}$$

$$\equiv V_F^G(x)=2\frac{1+(1-x)^2}{x}, \tag{30b}$$



$$\equiv V_G^F(x)=2[x^2+(1-x)^2], \tag{30c}$$



$$\equiv V_G^G(x)=4x(1-x)\left[1+\frac{1}{x^2}+\frac{1}{(1-x)^2}\right]. \tag{30d}$$

3. VIRTUAL QUARKS AND GLUONS AS PARTONS. SYMMETRIES OF DECAY PROBABILITIES

A gauge condition of the type (22a) has enabled us to get along in the system (26) without a third paraistic equation for the "ghost structure function." Let us show now that the axial gauge not only simplifies the calculations thanks to the absence of longitudinal polarizations and ghosts, but also makes it possible to interpret the PLA result in the language of the parton model with a variable cutoff parameter $\Lambda^2\sim|q^2|$ in the transverse momenta, by choosing a propagator that propagates only the physical states of the vector field.[8] Indeed, in the PLA the total photon scattering cross section, as we have seen, is determined by tree diagrams, and the interference of the amplitudes drops out (see, e.g., Fig. 8b). We can therefore describe the development of the process with the aid of classical probabilities.

An initial quark with $k^2\sim\mu^2$ ("dressed" quark) decays successively into quarks and gluons that play the role of partons (the bare field-theory particles) with increasing $k_{i\perp}$ (and virtualities); the γ quantum becomes fragmented on the most virtual parton with an initial-momentum fraction $\beta_n\approx1/\omega\equiv x$. The remaining partons with positive virtuality carry a total longitudinal momentum $1-x$ and form the target-fragmentation region, decaying independently into quarks and gluons with finite virtuality $k^2\sim\mu^2$ (see Fig. 9).

In this interpretation, the functions $V_A^B(x)$ determine the probability of the decay of a dressed particle A into two partons in the Born approximation. They satisfy the following symmetry relations[8,9]:

1. Permutation of the decay products ($A=F, G$):

$$V_F{}^F(x)=V_F{}^G(1-x),\quad V_G{}^A(x)=V_G{}^A(1-x). \tag{31a}$$

2. The crossing relation:

$$V_A{}^B(x)=(-1)^{2s_B-2s_A+1}x^{-1}V_B{}^A(x^{-1}). \tag{31b}$$

By virtue of the properties (31a) and (31b), only one of the three nuclei with fermion participation is independent. The quantity $V_G^G(x)$, which is absent in the abelian case, goes over into itself under the transformations (31a) and (31b) and seems to be unconnected with the remaining nuclei.

3. The following equality, however, does hold:

$$V_F{}^F(x)+V_F{}^G(x)=V_G{}^F(x)+V_G{}^G(x). \tag{31c}$$

This equality could be interpreted as equality of the probability of finding a parton of any sort with momentum fraction x in a quark and gluon (in the Born approximation). This interpretation of (31c), however, is somewhat arbitrary, since the V_A^B do not include the color multipliers. The number of fermion and gluon degrees of freedom are different, so that such a statement is generally speaking incorrect for the total probabilities $D_A^B(x)$ of finding a parton B in a particle A.

The relation for the parton densities in the quark and gluon, which follows from (31c) in the special case $N=n_f$, was discussed in the last section.

4. "PARTONOMETRY." SOLUTION OF THE BETHE-SALPETER EQUATIONS FOR THE PARTON-NUMBER DISTRIBUTION

We have written down the system of Bethe-Salpeter equations for the scattering of a virtual photon. The structure of Eqs. (26), however, does not depend on the sort of the scattering particle r ($r=\gamma, \nu, \bar{\nu}\ldots$): the type of the process influences only the form of the Born terms. The cross section of any deep-inelastic interaction can therefore be expressed in terms of the universal distributions D_A^B of the parton number[3)]:

$$\sigma_A{}^r(x,q^2)=\sum_B\int_0^1\frac{dx'}{x'}\sigma_B{}^r(x',x)\eta^BD_A{}^B(x',\xi(q^2)). \tag{32}$$

Here $\sigma_B^r(x', x)$ are the cross sections, averaged over the polarizations and the color of the field B, for scattering by the virtual particle B.

The quantities σ_B^r are determined by the simplest diagrams and coincide as a rule with the corresponding cross sections of the parton model. Thus, for the scattering of a transversely polarized photon, e. g.,

$$\sigma_q^{1,T}=\sigma_{\bar q}^{1,T}\backsim e_q^2\delta\left(1-\frac{x}{x'}\right),\quad \sigma_G^{1,T}=0;$$

the weak interaction, in contrast to the electromagnetic one, distinguishes between the quark and the antiquark.

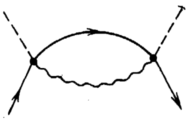
The joint description of various experiments on lepton–hadron interactions makes it possible in principle to reconstruct all the parton distributions D^B from formula (32). From this point of view it can be stated that each type of deep-inelastic process is a definite "measuring instrument"—*partonometer*—that responds with different degrees of sensitivity to the presence of a parton of sort B in the nucleon.

In contrast to the orthodox parton model, where the transverse momenta are bounded, in the logarithmic field theory the probability of finding the "parton" depends on q^2, i. e., on how deep the virtual photon or the W-boson penetrates into the parton cloud of the hadron, or so to speak on the "resolving power of the partonometer." This dependence leads, as is well known, to a definite character of scaling violations that can serve as an argument in favor of field theory. However, to make a decisive choice between the parton model and chromodynamics we need direct "measurements" of the gluon distribution in the hadron.

The gluons do not take direct part in either the electromagnetic or the weak interaction, so that to find the distributions D^G we must "construct" more refined partonometers. In principle, the role of such a partonometer could be played by the graviton. In real experiments the gluon-number density determines, for example, the cross section for the photoproduction (or electroproduction if $|q^2|\lesssim m_c^2$) of a heavy quark.[6]

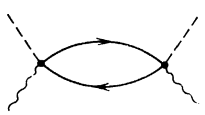
By way of one more example we point to the scattering of a longitudinal photon. In this case the logarithmic integration is lost in the upper cells of the diagrams of Fig. 6, and the corresponding $2\to 2$ amplitude, which are concentrated at short distances, determine the partonometric cross sections $\sigma_F^{\gamma,L}$ (Fig. 6a) or $\sigma_G^{\gamma,L}$ (Figs. 6b and c). The calculation leads to the following results for $\sigma_B^{\gamma,L}$ ($x_n\equiv x/x'$):

$\sigma_F^{\gamma,L}\backsim$



$$=x_n\frac{\bar g^2(q^2)}{4\pi^2}C_2e_F^2\theta(1-x_n),\tag{33a}$$

$\sigma_G^{\gamma,L}\backsim$



$$=x_n(1-x_n)\frac{\bar g^2(q^2)}{4\pi^2}\,\frac{8}{3}\sum_{F=1}^{2n_f}e_F^2\theta(1-x_n).\tag{33b}$$

This example is of interest also because the cross sections (33) discriminate strongly between the field model and the standard parton picture, which predicts a power-law smallness of σ^L in the Bjorken limit:

$$\sigma^L/\sigma^T\sim 1/q^2.$$

The σ^L partonometer is particularly sensitive to the gluon distribution at small x. In the region of large $\omega\equiv x^{-1}$, where the small contribution of the sea predominates in the structure function W (see Sec. 5B below), we can obtain the following simple formula for the ratio of the cross sections for the scattering of longitudinally polarized photons by quarks:

$$\frac{\sigma^L}{\sigma^T}\approx\frac{\bar g^2(q^2)}{4\pi^2}\left[\frac{C_2}{2}+\frac{8}{3}N\left(\frac{\ln\omega}{4N\xi}\right)^{1/2}\right]\quad\text{for}\quad\left(\frac{\ln\omega}{4N\xi}\right)^{1/2}\gg 1\tag{34}$$

(see, however, the restriction (62) below).

The terms in (34) pertain respectively to the partonometric cross sections $\sigma_F^{\gamma,L}$ and $\sigma_G^{\gamma,L}$. Thus σ^L is determined at small x mainly by the scattering of the photon by the parton–gluon (33b).

Equations (26) are solved with the aid of the Laplace transformation:

$$W(\omega,\xi_k;\xi)=\exp(\gamma_F\xi_k)\int\frac{d\nu}{2\pi i}\exp[\nu(\xi-\xi_k)]\int\frac{dj}{2\pi i}\omega^jW_\nu(j),$$
$$\tilde W(\omega,\xi_k;\xi)=\exp(\gamma_G\xi_k)\int\frac{d\nu}{2\pi i}\exp[\nu(\xi-\xi_k)]\int\frac{dj}{2\pi i}\omega^j\tilde W_\nu(j).\tag{35}$$

To find all the distributions D_A^B we must supplement (26b) with the Born term

$$e_G^2\delta(1-1/\omega_k)\exp(\gamma_G\xi),$$

by considering a fictitious partonometer $r(B)$ that interacts only with one of the $2n_f+1$ fields (quark, antiquark, or gluon) with a single renormalized charge.

The substitution (35) transforms (26) into an algebraic system. We introduce the j, ν and the mixed j, ξ representation for the functions D:

$$D_A{}^B(x;\xi)=\int\frac{dj}{2\pi i}x^{-j}D_A{}^B(j;\xi)=\int\int\frac{d\nu\,dj}{(2\pi i)^2}x^{-j}e^{\nu\xi}D_A{}^B(j,\nu).$$

The parton number distributions take the following forms[4]:

1. Distribution of the quark (antiquark) in the sea, $D^{sea}=D_F^{F'}$ $(F\neq F')$:

$$\eta^FD^{sea}(j,\nu)=\frac{\Phi_F{}^G\Phi_G{}^F}{(\nu-\nu_0)\Xi_\nu}=\frac{1}{2n_f}\left[\frac{\nu-\Phi_G}{\Xi_\nu}-\frac{1}{\nu-\nu_0}\right],$$
$$\eta^FD^{sea}(j,\xi)=\frac{1}{2n_f}\left[\frac{\nu_0-\nu_-}{\nu_+-\nu_-}\exp(\nu_+\xi)+\frac{\nu_+-\nu_0}{\nu_+-\nu_-}\exp(\nu_-\xi)-\exp(\nu_0\xi)\right].\tag{36a}$$

2. Distribution of the quark-parton in a quark of the same flavor, $D_q^q=D_{\bar q}^{\bar q}$:

$$\eta^FD_q{}^q(j,\nu)=\eta^FD^{sea}(j,\nu)+\frac{1}{\nu-\nu_0},$$
$$\eta^FD_q{}^q(j,\xi)=\eta^FD^{sea}(j,\xi)+\exp(\nu_0\xi).\tag{36b}$$

3. Gluon distribution in a quark (antiquark):

$$\eta^G D_F{}^G(j,\nu)=\frac{\Phi_F{}^G}{\Xi_\nu},$$

$$\eta^G D_F{}^G(j,\xi)=\frac{\Phi_F{}^G}{\nu_+-\nu_-}\left[\exp(\nu_+\xi)-\exp(\nu_-\xi)\right]. \tag{36c}$$

4. Quark (antiquark) distribution in gluon:

$$\eta^F D_G{}^F(j,\nu)=\frac{\Phi_G{}^F}{\Xi_\nu},$$

$$\eta^F D_G{}^F(j,\xi)=\frac{\Phi_G{}^F}{\nu_+-\nu_-}\left[\exp(\nu_+\xi)-\exp(\nu_-\xi)\right]. \tag{36d}$$

5. Gluon-parton in gluon:

$$\eta^G D_G{}^G(j,\nu)=\frac{\nu-\Phi_F}{\Xi_\nu},$$

$$\eta^G D_G{}^G(j,\xi)=\frac{\nu_+-\nu_0}{\nu_+-\nu_-}\exp(\nu_+\xi)+\frac{\nu_0-\nu_-}{\nu_+-\nu_-}\exp(\nu_-\xi). \tag{36e}$$

The following notation is used in formulas (36):

$$\Xi_\nu\equiv(\nu-\Phi_F)(\nu-\Phi_G)-2n_f\Phi_F{}^G\Phi_G{}^F=(\nu-\nu_+)(\nu-\nu_-)$$

is a quadratic form that determines the "frequencies" of the coupled system of "oscillators" with natural frequencies Φ_F and Φ_G;

$$\nu_\pm=\nu_\pm(j)\equiv{}^1/_2\{\Phi_F+\Phi_G\pm[(\Phi_F-\Phi_G)^2+8n_f\Phi_F{}^G\Phi_G{}^F]^{1/2}\}. \tag{37}$$

The functions Φ denote here the kernels of Eqs. (26) in the j-representation.

$$\nu_0\equiv\Phi_F=\Phi_F(j)\equiv\int_0^1\frac{dx}{x}\Phi_F{}^F(x)x^j+\gamma_F=C_2\cdot 2\int_0^1\frac{dx}{x}\frac{1+x^2}{1-x}x^j$$

$$+\gamma_F=-4C_2\psi(j+1)+C_2\left[3-4\gamma+\frac{2}{j(j+1)}\right], \tag{38a}$$

$$\Phi_G=\Phi_G(j)\equiv\int_0^1\frac{dx}{x}\Phi_G{}^G(x)x^j+\gamma_G$$

$$=N\cdot 4\int_0^1 dx(1-x)\left[1+\frac{1}{x^2}+\frac{1}{(1-x)^2}\right]x^j+\gamma_G$$

$$=-4N\psi(j+1)+\beta_2-4N\gamma+N\cdot\frac{8(j^2+j+1)}{j(j^2-1)(j+2)}, \tag{38b}$$

$$\Phi_F{}^G=\Phi_F{}^G(j)\equiv\int_0^1\frac{dx}{x}\Phi_F{}^G(x)x^j=C_2\cdot 2\int_0^1\frac{dx}{x}\frac{1+(1-x)^2}{x}x^j$$

$$=2C_2\frac{j^2+j+2}{j(j^2-1)}, \tag{38c}$$

$$\Phi_G{}^F=\Phi_G{}^F(j)\equiv\int_0^1\frac{dx}{x}\Phi_G{}^F(x)x^j=\frac{1}{2}\cdot 2\int_0^1\frac{dx}{x}[x^2+(1-x)^2]x^j$$

$$=\frac{j^2+j+2}{j(j+1)(j+2)}. \tag{38d}$$

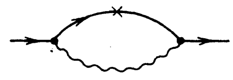
Here

$$\psi(z)\equiv\frac{d}{dz}\ln\Gamma(z),$$

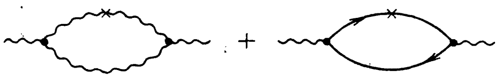
γ is the Euler constant ($\gamma=0.5772\ldots$).

In formulas (38) we used expressions for the anomalous dimensionalities γ_F and γ_G, which are connected with the renormalization of the wave functions of the quark and gluon fields in the axial gauge.

Differentiating the self-energy of the particle with respect to momentum, we can relate γ_A with the integral (first moment) of the functions Φ_A^B that determine the decay of the particle A into two partons in the Born approximation:

$-\gamma_F =$ 

$$=\int_0^1 dx\Phi_F{}^F(x)=\int_0^1 dx\Phi_F{}^G(x), \tag{39a}$$

$-\gamma_G =$ 

$$=\int dx\left(\frac{1}{2}\Phi_G{}^G(x)+n_f\Phi_G{}^F(x)\right). \tag{39b}$$

Substituting in (39) the expressions for the kernels Φ_A^B (see (29), (30)), we get:

$$\gamma_F=-C_2\left[1+4\int_0^1 dx\frac{x}{1-x}\right],$$

$$\gamma_G=-\frac{2}{3}n_f-N\left[\frac{1}{3}+4\int_0 dx\frac{x}{1-x}\right]. \tag{40}$$

We note that since the integral of Φ_A^A with respect to x in (38a) and (38b) as well as γ_A (see (40)) diverge, whereas the resultant quantities $\sim\Phi_A(j)$ are finite. This is a reflection of the fact that the standard cancellations of the infrared infinities due to emission by particle A of real (the integral of Φ_A^A) and virtual (γ_A) bremsstrahlung gluons takes place in the total cross section.

5. ASYMPTOTIC PROPERTIES OF PARTON DISTRIBUTIONS

The distributions D_A^B (36) can be represented by a single formula[9]:

$$\eta^B D_A{}^B(x;\xi)=\int\frac{dj}{2\pi i}x^{-j}\left\{\delta_{A,F}\delta_{B,F}\left(\delta_{A,B}-\frac{1}{2n_f}\right)\exp[\nu_0(j)\xi]\right.$$

$$\left.+\sum_{\sigma=+,-}C_A(\sigma,j)C^B(\sigma,j)\exp[\nu_\sigma(j)\xi]\right\}, \tag{41}$$

$$A,B=q,\bar q,G\quad(q,\bar q=1,2,\ldots,n_f),$$

where

$$C_G(\sigma,j)=(\nu_\sigma(j)-\nu_0(j))G_\sigma^{-1/2}(j),\qquad C^G=C_G/\eta^G,$$

$$C_F(\sigma,j)=\Phi_F{}^G(j)G_\sigma^{-1/2}(j),\qquad C^F(\sigma,j)=\Phi_G{}^F G_\sigma^{-1/2}/\eta^F,$$

$$G_\sigma(j)=(\nu_+(j)-\nu_-(j))\,|\nu_\sigma(j)-\nu_0(j)|.$$

The first term in (41), which appears only when both the target and the sought parton are fermions, can be expressed, just as the contributions of the "trajectories" $\nu_\pm(j)$, in factorized form. To this end it is necessary to regard $\nu_0(j)$ as a trajectory that is $2n_f$-fold degenerate in C-parity and in isospin of the group of flavors:

$$\nu^{T,P_C}(j)=\nu_0(j),$$
$$T=1,2,\ldots,n_f,\quad P_C=\pm 1.$$

The aggregate of the thus-introduced "residues" satisfies the conditions of orthogonality and completeness[9] (see footnote 3). It follows directly from the factorized form (41) that, for example, the fraction of the momentum carried by a given parton does not depend on the sort of the particle A at large q^2 (when we can confine ourselves to the leading term $\exp(\nu_+\xi)$). We shall not discuss in detail the sum rules that are connected with the individual angular momenta j and are successfully used.

Let us dwell on the asymptotic properties of the parton distributions. For the sake of argument we consider the structure function of the deep inelastic scattering of an electron by a quark

$$W_q(\omega,\xi)=\sum_F e_F^2 D_q^F(1/\omega;\xi(q^2)),$$

$$W_q(j,\nu)=\frac{e_q^2-\overline{e^2}}{\nu-\nu_0(j)}+\overline{e^2}\frac{\nu-\Phi_G(j)}{(\nu-\nu_+(j))(\nu-\nu_-(j))}. \qquad (42)$$

Here $\overline{e^2}$ is the rms charge of the sea,

$$\overline{e^2}=\frac{1}{2n_f}\sum_{F=1}^{2n_f} e_F^2.$$

A. Quasielastic region

We consider the kinematic region

$$1-x\ll 1,\quad \xi\sim 1\quad \text{or}\quad x\sim 1,\quad \xi\gg 1. \qquad (43)$$

In these cases the phase volume open to the particles produced in the target fragmentation region is small:

$$\sum_{k=1}^{n}\beta_k=1-x,\qquad \langle\beta\rangle=\frac{1-x}{\langle n\rangle}\ll 1,$$

therefore only the bremsstrahlung-field contributions that are logarithmic in β_k survive. In the integral (35) the important role is played by large j, where

$$\nu_+(j)\approx\Phi_F(j)=-4C_2\ln(j+{}^1/_2)+C_2(3-4\gamma)+O(1/j^2),$$
$$\nu_-(j)\approx\Phi_G(j)=-4N\ln(j+{}^1/_2)+\beta_2-4N\gamma+O(1/j^2). \qquad (44)$$

Substitution of (44) in (42) makes a leading contribution to W in the form

$$W_q\approx e_q^2x^{1/2}\exp[C_2(3-4\gamma)\xi](\ln x^{-1})^{4C_2\xi-1}/\Gamma(4C_2\xi). \qquad (45)$$

At finite ξ and $1-x\ll 1$ we get

$$W\approx e_q^2\exp[C_2(3-4\gamma)\xi](1-x)^{4C_2\xi-1}/\Gamma(4C_2\xi). \qquad (46)$$

The form-factor-governed amplitude decrease, which is connected with virtual bremsstrahlung fields, remains uncompensated by emission of real particles, and furthermore to a greater degree the smaller the phase volume of the final state $(1-x)$.

How close can x be brought close to unity without going outside the framework of the considered approximation? In the elastic scattering region,

$$1-x\sim\mu^2/|q^2|, \qquad (47)$$

formulas (45) and (46) are certainly incorrect, for in this case a new parameter appears in the problem. The formal contradiction consists in the fact that when (47) is substituted Eq. (46) in the doubly logarithmic region

$$g_\mu^2\ln\frac{|q^2|}{\mu^2}\ll 1,\qquad g_\mu^2\ln^2\frac{|q^2|}{\mu^2}\sim 1$$

corresponds to a form factor that differs from that of Sudakov

$$\Gamma(q^2)\approx\exp\left(-C_2\frac{g_\mu^2}{16\pi^2}\ln^2\frac{|q^2|}{\mu^2}\right) \qquad (48)$$

only by the factor of 2 in the argument of the exponential.

Gribov and Lipatov[7] have shown how, without going outside the framework of the PLA, to write down an interpolation formula valid up to

$$\frac{g_\mu^2}{4\pi^2}\ln\frac{1}{1-x}\lesssim 1 \qquad (49)$$

and is joined together with the doubly logarithmic calculation (48). To this end it is necessary to take into account the limitation imposed on the integration with respect to $k_\perp'$ by the parametric smallness of the phase volume

$$-k_\perp'^2\ll\beta'|q^2|, \qquad (50)$$

where β' and $k_\perp'$ are the Sudakov components of the momentum of the bremsstrahlung gluon.

In view of the limitation (50), the integrals with respect to $k_\perp'$ and β' become coupled in the argument of the exponential for $W(j)$

$$\nu_+(j)\xi\approx-4C_2\xi\int_0^1 dx\,x\frac{1-x^{j-1}}{1-x}\approx-4C_2\int_0^1\frac{d\beta'}{\beta'}(1-e^{-j\beta'})\int_0^{\xi(q^2)}d\xi' \qquad (51)$$

and (51) goes over into

$$-4C_2\int_0^1\frac{d\beta'}{\beta'}(1-e^{-j\beta'})\int_0^{\xi(\beta' q^2)}d\xi'. \qquad (51a)$$

An estimate of the integral by the saddle-point method transforms (51a) into

$$W_q\approx e_q^2\int\frac{dj}{2\pi i}e^{j(1-x)}\exp\left\{\int_0^1\frac{dy}{y}\Phi_F^F(y)[y^i-y]\xi((1-y)q^2)\right\}$$
$$\approx e_q^2\exp\{3C_2\xi-4C_2\gamma\xi'\}(1-x)^{-4C_2/\beta_2}\frac{(1-x)^{4C_2\xi'-1}}{\Gamma(4C_2\xi')}$$
$$\times\exp\left\{-\frac{4C_2}{\beta_2}(\xi-\xi')\frac{e^{\beta_2\xi}}{g_\mu^2/16\pi^2}\right\}. \qquad (52)$$

Here $\xi'\equiv\xi(q^2/j_0)$, and the saddle-point value of the angular momentum j_0 is determined by the equation

$$j_0(1-x)=4C_2\xi',\quad j_0\gg 1. \tag{53}$$

Expression (52) goes over into (46) at

$$(\xi-\xi')\ll 1/\beta_2,\quad (\xi-\xi')\ln j_0\lesssim 1/2C_2,$$

and can be regarded in the doubly logarithmic limit as an interpolation formula that takes into account the singly logarithmic corrections to the Sudakov form factor (48).

B. Regge region

In the reverse limiting case $x\ll 1$, $\xi\sim 1$ the integral (35) is determined by the extreme right singularity in the j plane. The pole of the kernel $\Phi_G(j)$ (as well as of $\Phi_F^G(j)$) at the point $j=1$, which corresponds to the state of two vector particles in the t channel, causes now the leading trajectory to be

$$\nu_+(j)\approx\Phi_G(j)=\frac{4N}{j-1}+\ldots-4N\int_0^1 dx\left[\frac{x}{1-x}-\frac{x^j}{1-x}\right]. \tag{54}$$

Thus in the region of partons with $x\ll 1$ (wee-partons) the dominant contribution is that of the sea, which is made up in its interior of pure gluon cells.

We have obtained in the principal logarithmic approximation the essential singularity of the partial wave

$$W(j)\sim\exp\left[4N\xi/(j-1)\right]. \tag{55}$$

This singularity leads to a rapid growth of the structure function in the Regge limit $\omega=x^{-1}\to\infty$. An estimate by the saddle-point method yields the formula

$$\nu W_2=\frac{1}{\omega}W\approx\overline{e^2}n_f\frac{64C_2}{3(2\pi)^{1/2}}\frac{e^{-a\xi^2}}{(16N\xi\ln\omega)^{3/4}}\exp[(16N\xi\ln\omega)^{1/2}],$$
$$j_{\text{saddle}}-1\approx\left(\frac{4N\xi}{\ln\omega}\right)^{1/2}\ll 1,\quad a=\frac{11}{3}N+\frac{2}{3}n_f\left(1-\frac{2C_2}{N^2}\right). \tag{56}$$

The statement concerning the growth of the sea (56) is valid in a limited region of not too large ω

$$(g_\mu^2/4\pi^2)\ln\omega\ll 1. \tag{57}$$

What happens at *parametrically high energies*?

Let us verify whether the character of the singularity (55) is a direct continuation of the result of Kuraev, Lipatov, and Fadin,[10] who studied gluon scattering in the Regge region $g^2\ln\omega\sim 1$ at arbitrary virtualities of the external lines. The following equation was obtained in[10] for the partial amplitude in the vacuum channel (forward gluon-gluon scattering):

$$[(j-1)-2\alpha(k_\perp^2)]F_j(k_\perp)=A_0^{-1}(j-1)$$
$$+\frac{g_\mu^2}{(2\pi)^3}\frac{N}{2}\int d^2k_\perp'\frac{K^{(0)}(k_\perp,k_\perp')}{(k_\perp'^2-m^2)^2}F_j(k_\perp'). \tag{58}$$

The leading singularity of F_j, as follows from an analysis of (58), is a branch point at

$$j_{\text{vac}}-1=Ng_\mu^2\pi^{-2}\ln 2. \tag{59}$$

To approach the kinematic region of interest to us we must consider the case of strongly vertical ends $|k_\perp^2|\gg m^2$ and retain in (58) only the principal logarithms $\ln|k_\perp^2|$:

$$[(j-1)-2\alpha(k_\perp^2)]F_j(k_\perp)=\text{const}\cdot(j-1)$$
$$-\frac{g^2N}{16\pi^3}k_\perp^2\int d^2k_\perp'\frac{4}{k_\perp'^2(k-k')_\perp^2}F_j(k_\perp'), \tag{60}$$

$$F_j(k_\perp)\approx\text{const}+\frac{1}{j-1}\frac{g_\mu^2N}{4\pi^2}\int_{m^2}^{-k_\perp^2}\frac{d(-k_\perp'^2)}{-k_\perp'^2}F_j(k_\perp'). \tag{61}$$

After this simplification of (58), the square-root branch point (59) on the right of unity will look like an essential singularity at $j=1$ (61). Expanding $\xi^{5)}$:

$$\xi(k^2)\approx\frac{g_\mu^2}{16\pi^2}\ln\frac{|k^2|}{\mu^2},$$

we verify that (55) and (61) coincide.

Thus, whereas in the quasielastic region it was necessary to adapt the PLA to a correct description of double logarithms, in the Regge limit the fit is effected directly. The reason is that the new phenomenon that arises when the energy is increased—the reggeization of the gluon exchanges—does not appear in the vacuum channel: the contribution of the virtual bremsstrahlung fields responsible for the reggeization is cancelled by the emission of real quanta (the integral in the expansion (54) for $\Phi_G(j)$ vanishes near $j=1$; in the language of Eq. (58) this means that in the PLA in $k_\perp$ the gluon trajectory $\alpha(k_\perp^2)$ is cancelled in the right and left sides of (60)).

The restriction (57) can therefore be relaxed:

$$\frac{g_\mu^2}{4\pi^2}\ln\omega\ll\frac{g_\mu^2}{4\pi^2}\ln\frac{|q^2|}{\mu^2} \tag{62}$$

and thus the conclusion that the sea grows is correct up to energies such that there is no need to give preference to the logarithm of ω over $\ln|q^2|$.

To explain the true character of the vacuum singularity it is necessary to go beyond the framework of the PLA and take s-channel unitarity into account.

6. e^+e^- ANNIHILATION. THE GRIBOV-LIPATOV RELATION

We have dealt so far with deep-inelastic scattering. It is possible to investigate similarly in the principal logarithmic approximation the structure function of e^+e^- annihilation

$$W^{(a)}(\omega,q^2),\quad\omega=\frac{2p_aq}{q^2}\quad(p_a=-p);\quad\omega\lesssim 1,$$

where p_a is the momentum of the quark or gluon released in the final state. In full analogy with the result of Gribov and Lipatov,[7] the set of essential diagrams turns out to be the same as in the case of deep-inelastic scattering, the only difference being that the Sudakov mo-

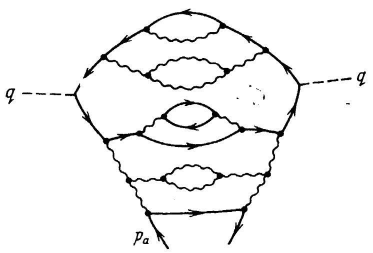


FIG. 10. Structure of the e^+e^- annihilation into an antiquark (p_a).

menta β_i do not increase but increase upward along the ladder (see Fig. 10). Now the parton no longer carries part of the target momentum, but conversely, p_a is the fraction x of the momentum $q^{1/2}/2$ with which the partons from the disintegration of the γ quantum move apart.

A reflection of this fact is the agreement between the distribution of the partons B in the particle A, on the one hand, and the density of the number of dressed particles B in the fragmentation of the parton A:

$$D_A{}^B(x)=\bar{D}_A{}^B(x). \tag{63}$$

This equality leads to the Gribov-Lipatov relation[7]

$$W^{(a)}(\omega,\xi)=-\omega^{-1}W(1/\omega,\xi), \tag{64}$$

which connects the structure functions of the deep-inelastic scattering and the e^+e^- annihilation in the physical regions. In the heretofore considered model examples[7-9] the role of the partons was played by virtual nucleons and mesons, while (64) was formulated directly for the observable cross sections with participation of hadrons.

In chromodynamics, where the hadrons are regarded as composite objects, the rule (64) is satisfied in the PLA for the fundamental fields—quarks and gluons:

$$e^-q\to e^-+\ldots\leftrightarrow e^+e^-\to\bar{q}+\ldots,$$
$$e^-G\to e^-+\ldots\leftrightarrow e^+e^-\to G+\ldots$$

7. DRELL'S RELATION

Drell *et al.*[11] have advanced the hypothesis that the structure functions $W[\omega\geq 1, q^2<0]$ and $W^{(a)}[\omega\leq 1, q^2>0]$ are connected by analytic continuation

$$W(\omega,q^2)=W^{(a)}(\omega,q^2). \tag{65}$$

In vector theory this is literally incorrect—the cross section becomes complex on going through the point $\omega=1$ (see (46)). Lipatov[8] has shown that Drell's relation (65) can be explained if the non-analytic term encountered in the continuation, of the form $(\ln\omega)^a$, are understood in their arithmetic sense: $\ln\omega\to|\ln\omega|$ at $\omega<1$.

Bukhvostov, Lipatov and Popov[9] proved for a model of the electrodynamics type the following representation:

$$D_A{}^B(x)=\int\frac{d\lambda}{2\pi i}|\ln x|^{-2\lambda}f_A{}^B(\lambda,x;\xi), \tag{66}$$

where $f_A^B(x)$ is analytic at $x=1$ (with the exception of a simple pole $1/(1-x)$), and satisfies the crossing relation

$$f_A{}^B(x)=(-1)^{2s_B-2s_A+1}x^{-1}f_B{}^A(x^{-1}). \tag{66a}$$

The ensuing functional equation for D

$$D_A{}^B(x)=(-1)^{2s_B-2s_A+1}x^{-1}D_B{}^A(x^{-1}) \tag{67}$$

together with new identity (63) yields a new relation

$$\bar{D}_A{}^B(x)=(-1)^{2s_B-2s_A+1}x^{-1}D_B{}^A(x^{-1}), \tag{67a}$$

which is equivalent to Drell's relation (65).

Before we prove the representation (66) in chromodynamics, we call attention to a curious structure of the complex λ plane. The leading singularities, which determine the asymptotic limit of the effective form factor of the quark and the gluon

$$\Gamma^2_{\text{эфф}(A)}(\varepsilon)\equiv\int_1^{1+\varepsilon}d\omega D_A{}^A(\omega),\quad A=F,G \tag{68}$$

as $\varepsilon\to 0$, are (respectively) *poles*

$$\lambda_F=-2C_2\xi,\quad \lambda_G=-2N\xi, \tag{69}$$
$$D_A{}^A\sim(\omega-1)^{-2\lambda_A-1},\quad \Gamma_{\text{eff}(A)}-(\varepsilon^{-1})^{\lambda_A}. \tag{70}$$

The corrections to (70), as well as distributions of the type $D_A^B(A\neq B)$ have the structure of Mandelstam branch cuts formed by the trajectory λ_F or λ_G and the immobile poles $\lambda_0=0$.

Notice must be taken of a certain duality between the quasielastic region $\omega\to 1$ and the Regge limit $s\to\infty$. The "trajectories" λ_F and λ_G introduced by us coincide with the Regge trajectories of the fermion and gluon! In fact, if we expand ξ (see footnote 5) in formulas (69), we obtain the equalities ($t\equiv q^2$)

$$\lambda_F=-C_2\frac{g^2}{8\pi^2}\ln(-q^2)=\alpha_F(t),$$
$$\lambda_G=-N\frac{g^2}{8\pi^2}\ln(-q^2)=\alpha_G(t),$$

where $\alpha(t)$ are the Regge trajectories of the fermion and gluon (after subtracting the spin) at large t.[12,10b] From the general philosophical point of view this agreement is not accidental: both phenomena, the rapid decrease of the particle form factor and the particle reggeization, are due to emission of infrared bremsstrahlung fields and indicate that the particle in question is not elementary in the sense of field theory.

We proceed to prove the representation (66). We separate in the distributions D_F^F and D_G^G the "valent" part of the contribution of the sea:

$$\eta^F D_{q_i}{}^{q_k}(j,\nu)=\frac{\delta_{ik}}{\nu-\Phi_F}+\frac{1}{2n_f}\frac{\chi}{(\nu-\Phi_F)[(\nu-\Phi_F)(\nu-\Phi_G)-\chi]},$$
$$\eta^G D_G{}^G(j,\nu)=\frac{1}{\nu-\Phi_G}+\frac{\chi}{(\nu-\Phi_G)[(\nu-\Phi_F)(\nu-\Phi_G)-\chi]}. \tag{71}$$

Here $\chi=\chi(j)\equiv 2n_f\Phi_F^G(j)$. Using the explicit expressions for the kernels of $\Phi(j)$ and the method developed in[9] we readily obtain

1) for the valent contributions (the first terms in (71))

$$f_A^{(val)}(\lambda,x;\xi)=\frac{\sqrt{x}}{\ln x^{-1}}\int\frac{dz}{2\pi i}e^z(\Omega_A(x,z))^{\lambda_A}\frac{1}{\lambda-\lambda_A},$$

$$\Omega_F=z^2\exp\left\{2K(z,x)-\frac{3}{2}-4\frac{\ln^2 x}{4z^2-\ln^2 x}\right\},$$

$$\Omega_G=z^2\exp\left\{2K(z,x)-\frac{\beta_2}{2N}-16\frac{(4z^2+3\ln^2 x)\ln^2 x}{(4z^2-\ln^2 x)(4z^2-9\ln^2 x)}\right\}, \quad (72)$$

$$K(z,x)=\gamma+\int_0^\infty d\tau\, e^{-z\tau}\left(\frac{1}{\tau}-\ln x/2\,\mathrm{sh}\frac{\tau\ln x}{2}\right);$$

2) for the contribution of the "sea" in (71), and also for the partial waves of the distributions D_F^G and D_G^F, which we shall express in unified fashion for convenience $(A\neq B)$

$$f_A^B(\lambda,x,\xi)=\frac{\sqrt{x}}{\ln x^{-1}}\int\frac{dz}{2\pi i}e^z\int_0^1 d\theta(\Omega_F)^{\theta\lambda_F}(\Omega_G)^{(1-\theta)\lambda_G}\frac{\xi}{\lambda-[\theta\lambda_F+(1-\theta)\lambda_G]}P_A^B. \quad (73)$$

The functions $P_A^B=P_A^B(\theta,x,z,\xi)$ for different distributions take the form

$$P_F^G=\frac{1}{\eta^G}\Phi_F^G(j)I_0(2\xi[\theta(1-\theta)\chi(j)]^{1/2}),$$

$$P_G^F=\frac{1}{\eta^F}\Phi_G^F(j)I_0(2\xi[\theta(1-\theta)\chi(j)]^{1/2}),$$

$$P_{F(F\neq F')}^{F'}=\frac{1}{\eta^F\cdot 2n_f}\left[\frac{\theta}{1-\theta}\chi(j)\right]^{1/2}I_1(2\xi[\theta(1-\theta)\chi(j)]^{1/2}), \quad (74)$$

$$P_G^G=\frac{1}{\eta^G}\left[\frac{1-\theta}{\theta}\chi(j)\right]^{1/2}I_1(2\xi[\theta(1-\theta)\chi(j)]^{1/2}).$$

In formulas (74) it is necessary to substitute $j=z/\ln x^{-1}-\frac{1}{2}$; I_0 and I_1 are modified Bessel functions. The functions $f_A^B(x)$ are analytic for $x=1$. The functional equation (66a) follows from

$$\Omega_A(x)=\Omega_A(x^{-1}),\quad P_A^B(x)=(-1)^{2s_B-2s_A}P_B^A(x^{-1}). \quad (75)$$

The properties of (75) are directly connected with the crossing relation (31b) for the kernels of the Bethe-Salpeter equation.

8. PROBABILITY OF "PARTONIZATION" OF THE QUARK AND THE GLUON

From among the symmetry properties (31) of the kernels of the Bethe-Salpeter equation, the first two are valid in any theory,[7-9] while the third (31c) appears in the Yang-Mills model. The first relation (31a) is connected with sum rule for the parton momentum, the crossing transformation ensures satisfaction of the Drell relation, and from the third identity we can obtain a specific "sum rule in q^2" for the distributions of the parton number in the quark and in the gluon.

By varying q^2—the "resolving power" of the partonometer—we penetrate to different "depths" of the parton cloud of the dressed particle and examine how frequently a parton fermion (quark or antiquark of any flavor) or gluon or a gluon with fixed momentum fraction x is encountered. It is clear that as $x\to 1$, for example we shall encounter for the most part a parton gluon if the target is chosen to be a gluon, and a fermion otherwise; at $x\ll 1$, on the contrary, gluons will predominate regardless of the spin of the target. However, if we add the fermion and gluon distributions (normalized to one degree of freedom), then such a quantity, which characterizes the total probability of the partonization of the given target particle, turns out, after integration with respect to q^2 (over all "depths") the same for the quark and the gluon:

$$\int_0^{g_\mu^2}\frac{d\bar g^2}{\bar g^2}[D_F^{(F)}(x)+D_F^G(x)]=\int_0^{g_\mu^2}\frac{d\bar g^2}{\bar g^2}[D_G^F(x)+D_G^G(x)]. \quad (76)$$

It remains to be added that relation (76) is valid only if $N=n_f$.

To prove (76) it is necessary to consider four distributions:

$$D_F^b(j)=\frac{1}{2N\cdot 2n_f}\frac{\nu-\Phi_G(j)}{\Xi_\nu},\quad D_F^G(j)=\frac{1}{2(N^2-1)}\frac{\Phi_F^G(j)}{\Xi_\nu},$$

$$D_G^F(j)=\frac{1}{2N}\frac{\Phi_G^F(j)}{\Xi_\nu},\quad D_G^G=\frac{1}{2(N^2-1)}\frac{\nu-\Phi_F(j)}{\Xi_\nu}, \quad (77)$$

separate the color factors in Φ, use the property (31c) in the j-representation, and to verify that at $N\to n_f$ we have complete cancellation of the kernels $V(j)$ as well as of the constant terms connected with γ_F and γ_G (see (38)). The remaining expression, which is proportional to ν, vanishes upon integration with respect to ξ.[7]

The fact that a model having the same number of quark multiplets as colors, such as relation (76), is singled out may indicate that the Yang-Mills Lagrangian contains an implicit symmetry that connects the character of the gauge group with the number of fermion fields that are included in the theory.

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[1]In the Faddeev–Popov quantization formalism, the determinant of an infinitesimally small transformation of the gauge condition does not depend on the potential A:

$$\delta(a^\mu A_\mu)\to\delta(a^\mu A_\mu')=\delta\left(a^\mu\left[S^{-1}A_\mu S+S^{-1}\frac{\partial S}{\partial x^\mu}\right]\right)=\delta\left(a^\mu S^{-1}\frac{\partial S}{\partial x^\mu}\right)=\text{const}.$$

[2]The terms $\sim k_{i\perp}$ are discarded here since they disturb the logarithmic character of the integration in the i-th cell.

[3]We recall that by parton, in the sense of field theory, we mean a virtual particle (quark, antiquark, or gluon), which carries a definite fraction x' of the momentum of target A and has an arbitrary transverse momentum $k_\perp^2$ all the way to q^2. The distributions D_A^B are normalized to one degree of freedom of the fields B and A and differ from those used in[8,9] by a factor η^B equal to the number of states of the parton B: $\eta^F=2N$, $\eta^G=2(N^2-1)$.

4)Equations (26) are in essence the integral form of the renormalization-group equations. We have recently received a preprint by Buras,[13] in which distributions analogous to (36) were obtained by diagonalization of the differential equations of the renormalization group.[3]

5)In papers on reggeization no account was taken of the effects due to the distance dependence of the charge $\bar{g}^2$.

6)In the abelian case one of the poles of (69) is a standing one, $\lambda_G = 0$. This agrees with the fact that the photon is not reggeized in electrodynamics.

7)In the general case we can construct the following "non-interpretable" relation

$$\int_0^{g_\mu^2} \frac{d\bar{g}^2}{\bar{g}^2} (\bar{g}^2)^{2(N-n_f)(N^2-1)/3(N^2+1)} \left\{ \frac{n_f}{N} D_F^{\langle F\rangle}(x) + D_F{}^G(x) - D_G{}^F(x) - D_G{}^G(x) \right\} = 0.$$

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Critical charge for anomalous nuclei and the effect of screening

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The dependence of the critical charge of nuclei on photon density is found. Values of Z_{cr} for anomalous nuclei are obtained. The calculations of Z_{cr} have been carried out taking into account the diffuse nature of the nuclear boundary and also the effect of screening of the Coulomb nuclear potential by the electron shell. The Thomas–Fermi statistical method is employed for describing the electron density in the shell. Two cases are considered: 1) screening by the usual electron shell formed by electrons occupying levels of the discrete spectrum ($-m < \epsilon < m$); 2) screening by a vacuum shell which is formed by electrons situated in levels of energy $\epsilon < -m$. In the first case the dependence of Z_{cr} on the degree of ionization of the atom $q = (Z-N)/Z$ is also obtained. The properties of electron states at the critical point are considered in detail. Asymptotic formulas for the solutions of the Thomas–Fermi equation in two limiting cases have been obtained in the Appendix: $q \to 0$ (weakly ionized atom) and $q \to 1$.

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1. INTRODUCTION

The critical charge of a nucleus[1] and the spontaneous production of positrons for $Z > Z_{cr}$ have been investigated in many papers (a discussion of the different aspects of this problem, and of its significance for the verification of quantum electrodynamics in strong external fields and references to the literature of the subject can be found in Refs. 2–5). The usually quoted[6–9] values of Z_{cr} refer to the normal density of nuclear matter $n_0 \approx 0.17$ nucleon $\cdot$ F^{-3}. At the same time there are theoretical indications[10–15] of the possibility of existence of anomalous nuclei with a density which differs significantly from n_0. Such a possibility was investigated for the first time by Migdal[10] who showed that nuclear matter beginning with a certain density $n = n_c$ becomes unstable with respect to the production of π mesons, and this leads to a phase transition of the nucleus into a superdense state with the formation of a pion condensate. Subsequently this problem was considered in greater detail[11–13]; also the possibility of the existence of neutron ($N \gg Z$) and supercharged ($Z \sim 137^{3/2}$) nuclei was discussed.[12] Lee and Wick also gave arguments in favor of the existence of stable superdense nuclei.[14,15] At present a large number of papers is devoted to the problem of the π condensate, to its effect on different properties of nuclei and

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