Particle-like solitons in superfluid $^3$He phases

G. E. Volovik and V. P. Mineev

1. INTRODUCTION

The solitons in superfluid $^3$He that have thus far been investigated are either moving domain walls, i.e., plane solitons (see, for example, Maki and Kumar’s paper(12)), or coreless vortices, i.e., linear solitons (see Refs. 2–5). The possibility of the existence of particle-like solitons was pointed out in Ref. 6 by the present authors. These states are characterized by an integral topological invariant and a finite size, energy, and momentum. A well-known example of such states in superfluid $^3$He is the ring vortex. However, in contrast to ring vortices, the particle-like solitons considered here do not possess singularities anywhere in the order-parameter field. The purpose of the present paper is to study in detail the structure of these states in both phases of superfluid $^3$He with allowance for the spin-orbit interaction.

From the topological point of view there can exist in superfluid $^3$He only two types of particle-like solitons, to each of which corresponds its own integral topological invariant. For solitons of the first type this number is the degree of the mapping of a three-dimensional sphere into a three-dimensional sphere ($S^3 \to S^3$). Solitons of this type are first considered, using as an example the A phase (see Ref. 6). The components $R_{ik}$ of the characteristic spin-orbit force acting on a solid immersed in the A phase is also considered in this section. Its appearance is connected with the fact that the body is stationary.

To solitons of the second type (see Sec. 4) corresponds the so-called Hopf invariant, which arises in the mapping of a three-dimensional sphere onto a two-dimensional sphere ($S^3 \to S^2$). Like solitons of the first type in the B phase, solitons of the second kind are unstable against a decrease of their dimensions, and can appear only in dynamical processes.

2. SOLITONS OF THE FIRST KIND. THE B PHASE

Let us consider solitons in the B phase with dimensions $R < R_0$, where $R_0$ is the characteristic spin-orbit (dipole-dipole) interaction range (see the review article by Leggett(17)). In this case the order parameter $A_4$ is given by an arbitrary rotation matrix $R_4$:

$$A_4 = R_4 A_0 R_0^{-1}$$

The components $R_{ij}$ are functions of three parameters. These are, for example, the direction, $\omega$, of the rotation axis and the angle, $\theta$, of rotation. We choose as these parameters the components of the unit four-dimensional vector

$$n_a = (a, \omega, \rho), \quad n_a n_a = 1 = a$$

so that

$$R_{ab} = n_a n_b$$

(Here $a = 1, 2, 3, 4$; the Latin indices run through the values $1, 2, 3$.)

Thus, the vector $n_4 \vec{n}$ which defines the mapping of the coordinate space of the vector $\vec{n}$ onto the three-dimensional sphere $n_4 n_4 = 1$, is specified at each point of the vessel with the $^3$He-B. At large distances from the soliton the order-parameter field is unperturbed: $R_{44} = R_0$. Let us, for definiteness, choose $R_{44} = 0$. This implies that to all infinitely remote points of the coordinate space corresponds the vector $n_4 = (0, 0, 0, 1)$. A three-dimensional space whose points at infinity are all equivalent is, from the standpoint of its topological structure, a three-dimensional sphere $S^3$ in a four-dimensional space, in the same way as a plane with identical points at infinity is equivalent to a two-dimen-
the integral invariant

\[
\int f^2 d^3 \mathbf{r} = \int f^2 \rho \, d \mathbf{r}.
\]

The characteristic dimension of the integration domain is of the order of \(R_s\), and, consequently,

\[
E = \frac{\rho^2}{c^2} \int \left( \frac{d\mathbf{v}}{dt} \right)^2 + \left( -\omega \mathbf{v} \right)^2 d^3 \mathbf{r}
\]

(2.9)

Since the energy is proportional to the soliton dimension, such solitons are unstable, since they can continuously reduce their radius, conserving the invariant (2.4) in the process. As soon as \(R_s\) becomes \(\ll \xi\), the coherence length, the order parameter ceases to be described by the rotation matrix \(R_{\mathbf{n}}\), and the topological invariant (2.4) ceases to have meaning. Therefore, a soliton at these distances can vanish, although the possibility of its being stable at distances \(R_s^*\) is not to be excluded. This question, as well as the question of the possibility of the stabilization of solitons with dimensions \(R_s^* \approx \xi\), e.g., by a spin current, remains open.

Let us write out the expression for the spin current \(j_s\) (a current of spin \(S_3\) in the direction \(\mathbf{n}\)) at large distances from the soliton:

\[
\mathbf{j} = \rho^2 \frac{\mathbf{R}}{c^2} \left( \mathbf{e}_3 - \frac{e_3}{|e_3|} \right).
\]

(2.10)

3. SOLITONS OF THE FIRST KIND. THE A PHASE

The order parameter of the \(A\) phase has the form

\[
\mathbf{A} = \mathbf{A}(\mathbf{r}) V(\mathbf{A}(\mathbf{r})),
\]

(3.1)

where \(V\) is a unit vector characterizing the spin motion, while \(\mathbf{A}^*\), \(\mathbf{A}^\dagger\), and \(1 + \mathbf{A}^\dagger \mathbf{A}^*\) are unit vectors characterizing the orbital motion. We shall consider the solitons in orbital motion. With that end in view, let us write the orbital part of the order parameter in the form

\[
\Delta^\dagger \mathbf{r} \mathbf{A}(\mathbf{r}) \mathbf{A}(\mathbf{r})^\dagger \mathbf{r} = R_s(\mathbf{r}) \mathbf{r},
\]

(3.2)

where \(R_s\) is a three-dimensional rotation matrix; \(\mathbf{e}_1\), \(\mathbf{e}_2\), \(\mathbf{e}_3\) are the base vectors of the Cartesian coordinate system. Thus, there exists at each point of the coordinate space, as in the \(B\) phase, a rotation matrix that can be parametrized with the aid of the formula (2.3) in terms of the four-dimensional unit vector \(n_0\). Consequently, as in the \(B\) phase, there exist particle-like solitons characterized by the invariant (2.4).

An investigation of the phase trajectories of Eq. (2.7) shows that this equation does not possess a solution with a continuous derivative and the asymptotic forms (2.8). It is, however, easy to construct a continuous solution whose derivative is discontinuous at \(r = R_s\). This discontinuity can actually be neglected, since we must take into consideration at distances \(r \approx \xi\) from the discontinuity surface the terms of fourth order in the gradients in the Hamiltonian (2.3), which allows us to construct a solution with a continuous derivative in the entire region \(0 < r < \xi\).

Because of the rapid convergence of the energy integral

\[
E = \frac{\rho^2}{c^2} \int \left( \frac{d\mathbf{v}}{dt} \right)^2 + \left( -\omega \mathbf{v} \right)^2 d^3 \mathbf{r}
\]

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Let us write out the expression for the spin current \(j_s\) (a current of spin \(S_3\) in the direction \(\mathbf{n}\)) at large distances from the soliton:

\[
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\]

(2.10)
Let us construct the corresponding solution for $N=1$. In contrast to the $B$ phase, the solution for the vector $n$, will not be spherically symmetric, since there is a preferred direction of the vector $l$ for $r = \infty$, namely, $l = l_0$. Therefore, generally speaking, the solution to the Ginzburg-Landau equation should be sought in the form

$$n_0 \sim (\cos \theta / 0)^+ \sin \theta / 0, x(r, \theta), \cos \phi(r, \theta).$$  \tag{3.6}$$

However, the nonlinear equations for $\theta$ and $\chi(r, \theta)$ cannot be solved analytically. Therefore, we shall consider the asymptotic behavior of the solution for $r = \infty$ and $r = 0$.

Since $\sin \theta \rightarrow 0$ for $r = \infty$ and $r = 0$, the expression (3.6) can be rewritten in these limiting cases in the form

$$n_0 \rightarrow (\sin \theta, 0, 0)$$  \tag{3.7}

Retaining in the Hamiltonians (2.5) only the terms quadratic in the gradients $n$, we obtain for $R < R_D$ ($V = \text{const}$)

$$E = \frac{1}{2} n^2 / 0, 1 / 2 \{1 + \frac{1}{2} (2n_0)^+ \sin^2 \theta - \sin \theta + \cos \theta / 0\}^2.$$  \tag{3.8}

Varying this functional, we obtain the equation

$$\Delta \chi + \frac{1}{2} \chi + 2 \chi_x / 0 = 0.$$  \tag{3.9}

The solution has the form

$$n_0 \rightarrow (\sin \theta, 0, 0), r > R_D, n_0 \sim 0, r < R_D$$  \tag{3.10}

i.e., the asymptotic behavior of the solution is exactly the same as in the $B$ phase.

The superfluid velocity $v^s$ for $r = \infty$ and $r = \infty$ is determined from the formula

$$n^\mu = -\nabla, -\nabla \times n, \nabla \cdot n = 0, \nabla \cdot n = 0,$$  \tag{3.11}

The current density for $v^s = 0, j_1 = \partial \nabla \times n + C_1 (\text{curl} J_1)$ has the following form:

$$j = -\nabla \times n + C_1 (\text{curl} J_1), r > R_D.$$  \tag{3.12}

The same velocity ($v^s$) and current ($j$) field distributions arise in the motion of a solid in an ideal liquid. This motion is characterized by a momentum $P$ and a soliton-drift velocity $u$:

$$P = v^s \cdot n, u = \frac{P}{m}.$$  \tag{3.13}

These expressions resemble the corresponding expressions for quantized vortex rings in a superfluid liquid, except for the absence of the factor $\ln(R/\xi)$ in the expression for $u$, since solitons do not possess singular cores.

As in the $B$ phase, the soliton energy

$$E = v^s \cdot (n \times \partial / \partial t).$$  \tag{3.14}

Solitons in the $A$ phase are stable owing to the coupling of the dimension $R$ to the momentum, which, in the absence of a normal component, is conserved during the motion.

Notice in this connection that, in solving the present problem, it would have been more rigorous to have considered the functional $E = P - u$, i.e., the functional $E$ for a given momentum $P$. Such a treatment alters the asymptotic behavior of $v^s$, but not the dipole character of $v^s$ and $j$. In this case there appears a characteristic soliton dimension $R/D$ as a result of which the scaling invariance of the Ginzburg-Landau equation is destroyed. Consequently, the solution to this equation with a continuous derivative can exist in the entire space, and, in contrast to the $B$ phase, it is not necessary to take terms of higher order in the gradients into account.

Such solitons can arise in the presence of a flow of the superfluid component relative to the normal component, effecting, just as vortex rings do in He II, the transfer of momentum from the superfluid component to the normal component.

The influence of the soliton states on the superfluid properties of $^4\text{He}$ has been discussed by Anderson and Toulouse.\(^{44}\) Although these authors considered vortices without singularities, i.e., linear solitons, the arguments they adduce in their paper are applicable to the case of particle-like solitons, especially as the above-considered solitons with $N=1$ is a vortex without a singularity bent into a ring.

Thus far we have considered solitons with $R < R_D$ in the $A$ phase. It can be shown that the asymptotic forms of the solutions for solitons with dimension $R > R_D$ differ from those obtained for $R < R_D$ only by a scale transformation along the $z$ axis. To wit, for $R > R_D$ it is necessary to set $Y = 1$ in the order parameter (3.1) (see Ref. 7). In the process the coefficients of the free-energy expansion (3.8) change and new terms appear, with the result that Eq. (3.9) will have a more complex form. Nevertheless, the $v^s$ and $j$ fields at large distances retain their dipole character, with the only difference that the vector $x$ in the formulas (3.11) and (3.12) should be replaced by $x - \xi x - \xi^2$, where the constants $\xi$ and $\xi^2$ depend on the coefficients in the energy expansion.

The asymptotic expression (3.11) does not allow a judgment to be made about the magnitude of the angular momentum, $I_\phi$, of the particle-like soliton:

$$L = \int \partial \partial k.$$

To estimate $L$, let us consider the following term of the expansion of $v^s$ in powers of $n$ for $r = \infty$:

$$v^s \cdot n = \frac{P}{m} v^s \phi / m, \nabla \phi, \nabla \phi / r.$$  \tag{3.15}

It can be seen from (3.15) that the angular momentum has the order of magnitude...
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and is equal to a unit three-dimensional vector:

\[
L - \rho R \overline{a} \frac{dS}{m}
\]  

(3.10)

This should lead to the appearance of the Magnus force when a superfluid current perpendicular to \(L\) flows around the soliton. And in this respect a particle-like soliton is similar to a normal solid with dimension \(R\) immersed in \(\text{He}-A\). Even if the body is at rest relative to the liquid, there arise near it nonzero angular momentum \(L - \rho R \overline{\omega} \frac{dS}{m}\). This is connected with the fact that we should have fulfilled on the surface of the solid the boundary condition \(1\), where \(\nu\) is a vector normal to the surface (see Ref. 8). This boundary condition can be satisfied with the aid of a single singular point (a simple hedgehog) in the 1-vector field inside the solid. Since the 1 field at infinity is uniform, this hedgehog should be compensated by an anti-hedgehog on the surface of the body: an "island" of zero dimension, in Mermin's terminology (see Fig. 11). In Ref. 10 the present authors showed that a hedgehog and an anti-hedgehog in the field of the vector 1 are connected by a vortex with two circulation quanta. This vortex is located inside the solid. Therefore, around it exists a nonzero circulation of the velocity \(v\). Consequently, if a superfluid current \(I\) flows around a solid in the \(A\) phase, then the solid will experience a Magnus force:

\[
F = \frac{A}{m} R \nu
\]  

(3.17)

4. SOLITONS OF THE SECOND KIND

It remains for us to consider the spin solitons with dimensions \(R < R_{\text{p}}\) in the field of the vector \(V\) in the \(A\) phase and the solitons with dimensions \(R > R_{\text{p}}\) in the \(B\) phase. In both cases the order parameter is determined by a unit three-dimensional vector: \(V\) in the \(A\) phase and \(w\) in the \(B\) phase, respectively. The angle, \(\theta\), of rotation about the \(w\) axis for \(R > R_{\text{p}}\) is fixed by the dipole-dipole interaction, and is equal to 104°.

We are required to construct a mapping of a coordinate space with identical points at infinity, i.e., of a three-dimensional sphere \(S^3\), onto the space of variation of the vector \(V\) (or \(w\)), i.e., into a two-dimensional sphere \(S^2\). Let us construct this mapping in the following manner. Let the field of the four-dimensional unit vector \(n_{\mu}\) define an \(S^3 - S^2\) mapping of degree \(N\). Let us define the rotation matrix \(R_{\mu
u}(n_{\mu}\overline{a})\) according to the formula (2.3), and let this matrix act on a constant vector, e.g., \(\overline{a}\). The resulting vector will give the \(S^3 - S^2\) mapping:

\[
V(t) = R_{\mu
u}(n_{\mu}\overline{a}) \overline{a}
\]  

which is characteristic by the index \(N\), called the Hopf invariant.

A similar mapping, called the Hopf stratification in topology, arose in the case of solitons of the first kind in the 1-vector field (see (3.2)). Precisely because of this, the invariant (2.4) could be rewritten in the form (3.5), which depends only on \(i\), a fact which can easily be verified with the aid of the relation (3.4) (see also Faddeev's lectures).

The asymptotic forms of the solutions for the vectors \(n_{\mu}\) in a soliton corresponding to the Hopf invariant \(N = 1\) retain the form of (3.11) up to a scale transformation along the \(z\) axis, and, therefore, the spin current falls off at large distances from the soliton in the dipole fashion. Like solitons of the first kind in the \(B\) phase, solitons of the second kind are unstable, since the field of the spin variables \(V\) and \(w\) possesses no momentum. The possibility of the production of similar states by a spin current requires a separate investigation.

In conclusion, we consider it our pleasant duty to thank S. P. Novikov for valuable consultations, as well as N. D. Mermin for sending us his preprint. 

This circumstance, which is a consequence of the scaling invariance of Eq. (2.7), was pointed out to us by A. M. Polyakov, E. B. Bogomol'nyi, and V. A. Fateev.

11This preprint, as an example of the scaling invariance of Eq. (2.7), was pointed out to us by A. M. Polyakov, E. B. Bogomol'nyi, and V. A. Fateev.


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Translated by A. K. Agrej.