Relaxation of high-current electron beams and the modulational instability

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The modulational instability of the plasma noise in resonance with high-current high-energy beams in a plasma becomes important when these beams relax. The short-wavelength transfer of the oscillations caused by such an instability appreciably protracts the process of the collective relaxation of the beam. We derive and solve in this paper equations describing the dynamics of the relaxation under such conditions. We study the cases of relativistic and non-relativistic beam energies.

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1. The collective relaxation of an electron beam penetrating a "cold" plasma is caused by the resonance interaction between the beam particles and the plasma oscillations excited through the beam instability. The theory of such a relaxation based on the weak turbulence equations (see, e.g.,[1–3]) is well known; it is inapplicable for high-current high-energy beams when in the spectral region of the plasma noise in resonance with the beam the effect of the modulational instability becomes important.[4,5]

As a result of the instability cavities—bunches of Langmuir energy with plasma flowing out of it—are formed on the initially uniform background of the plasma oscillations. The cavities collapse to a size where resonance absorption by the plasma electrons of the plasmons trapped in them (collapse of Langmuir waves, first studied by Zakharov[6]) occurs. The short-wavelength transfer of oscillations caused by the modulational instability and the collapse can appreciably lower the level of the plasma noise in resonance with the beam and, hence, protract the collective relaxation process.

Sudan[7] was the first to draw attention to the important role of the non-linear effects connected with the modulational instability for the relaxation of high-current beams. Subsequently attempts were made to study qualitatively the relaxation process under those conditions.[8,9] However, the construction of a quantitative theory of the quasi-stationary Langmuir turbulence excited when high-current beams relax in a plasma is made difficult by the invalidity of the approximation of a weak coupling between the modes (strong turbulence).

In the present paper we expound a phenomenological approach to the problem, based upon the results of a non-linear theory of the modulational instability.[10] According to that theory, the development of the modulational instability creates a mechanism for the dissipation of long-wavelength plasmons in resonance with the beam, the velocity of which is determined by the "effective" frequency of the scattering of plasmons by density fluctuations:

$$\nu_{eff}\approx\alpha\omega_p W/n_0T_e, \qquad (1)$$

W is the energy of the oscillations; α is a numerical coefficient; according to the results of a computer simulation[11,12] $\alpha\approx\frac{1}{3}$.

In the problem considered by us the noise excited by the beam plays the role of the long-wavelength pumping for the Langmuir turbulence which arises as a result of the modulational instability. In final reckoning the competition between the buildup of the oscillations by the beam and the dissipation produced by the collapse leads to the establishment of a quasi-stationary turbulence. The energy going into plasma turbulence due to the modulational instability is transferred to short-scale lengths in the collapsing cavities and absorbed by the particles. The corresponding balance condition has the form

$$\nu_{eff}\frac{E_0^2}{16\pi}=\gamma_M W,\qquad \gamma_M=\omega_p\left(\frac{m}{M}\frac{W}{n_0T_e}\right)^{1/2}, \qquad (2)$$

γ_M is the growth rate of the modulational instability which determines the speed of the collapse.

The connection between the energy W of the plasma turbulence and the pumping amplitude E_0 can be found from Eqs. (1) and (2):

$$\frac{W}{n_0T_e}=\frac{M\alpha^3}{m}\left(\frac{E_0^2}{16\pi n_0T_e}\right)^2. \qquad (3)$$

The plan of the paper is the following. In Sec. 2 we give a short summary of the results of the relaxation theory based upon the weak turbulence equations and we establish the limits of the applicability of such a theory. Here we give a qualitative study of the relaxation process, taking the modulational instability into account. A quantitative theory of the relaxation of a nonrelativistic high-current beam is constructed in Sec. 3. We restricted ourselves to a study of the kinetic beam instability which arises when "smeared out" beams relax, $\Delta v/v_0>(n_1/n_0)^{1/3}$ (Δv is the spread in the velocities in the beam, n_1 and n_0 are, respectively, the beam and the plasma densities, $n_1\ll n_0$). The spectrum of the oscillations excited by the beam is assumed to be one-dimensional which, strictly speaking, corresponds to the presence of a strong magnetic field parallel to the beam motion, but one can use the formulae obtained for estimates also when there is no field present. Finally, in Sec. 4 we consider the influence of the modulational instability on the relaxation of relativistic beams. We study the case of the relaxation of smeared-out beams when there is no magnetic field, which is most important for many applications (heating of a plasma target by a beam, and so on).

2. The process of the relaxation of electron beams injected into a plasma has recently been studied in detail both in the quasi-linear approximation[1,2] and also taking into account non-linear effects in the weak turbulence framework.[3] In the quasi-linear relaxation theory the diffusion of beam particles in the field of the oscillations excited by them leads to the establishment of a "plateau" on the velocity distribution function of the beam at characteristic distances:

$$l_{QL}=\frac{v_0}{\omega_p}\frac{T_e}{mv_0^2}\frac{n_0}{n_1}\Lambda, \quad (4)$$

Λ is the logarithm of the ratio of the final to the thermal noise which is of the order of magnitude of the Coulomb logarithm. In the final state about half of the power is transferred to the plasma oscillations; because the group velocity of the oscillations is small, $v_g/v_0\sim T_e/mv_0^2$, their energy density exceeds the energy density in the beam considerably:

$$\left[\frac{E_0^2}{16\pi}\right]_{QL}\approx\frac{n_1mv_0^2}{15}\frac{mv_0^2}{T_e} \quad (5)$$

The important role of the non-linear effects in the problem considered is, in fact, connected with this. In weak turbulence the main non-linearity is caused by the induced scattering of the oscillations excited by the beam by the ions in the plasma which leads to a transfer of these oscillations from the spectral range in resonance with the beam to the region of large phase velocities (small k). In a separate scattering process the plasmon wavenumber changes by an amount

$$\delta k\sim kv_{Ti}\Big/\frac{d\omega}{dk}\sim\lambda_D^{-1}\left(\frac{m}{M}\frac{T_i}{T_e}\right)^{1/2}$$

which is usually much smaller than the width of the spectrum of the oscillations excited by the beam $\Delta k\sim\omega_p\Delta v/v^2$ so that the transfer by ions is differential in character. The characteristic growth rate of this process is

$$\gamma_i\approx\omega_p\frac{E_0^2}{16\pi n_0T_e}\frac{m}{M}\frac{1}{k^2\lambda_D^2}. \quad (6)$$

The transfer appreciably affects the relaxation process of the electron beam if for the maximum energy of the oscillations, given by (5), the growth rate of the induced scattering exceeds the growth rate of the beam instability for a strongly smeared out ($\Delta v\sim v$) beam: $\gamma_b\approx\omega_pn_1/n_0$.

The corresponding condition has the form

$$\varepsilon\approx10\frac{M}{m}\left(\frac{T_e}{mv_0^2}\right)^3<1. \quad (7)$$

If $\varepsilon\ll1$ the spectral transfer of the oscillations by the ions leads to a stabilization of the beam instability for the following level of plasma noise in resonance with the beam:

$$\left[\frac{E_0^2}{16\pi}\right]_{WT}=\left[\frac{E_0^2}{16\pi}\right]_{QL}\varepsilon\Lambda. \quad (8)$$

As a result of this the relaxation length of the beam increases by a factor $\varepsilon^{-1}\Lambda^{-1}$ as compared to the quasi-linear length (4).

We have already noted in Sec. 1 that the applicability of the relaxation theory expounded here is limited due to the effect of the modulational instability which cannot be described in the framework of weak turbulence. The condition for the occurrence of such an instability $W/n_0T_e>k^2\lambda_D^2$ corresponds to the possibility of trapping of plasmons in the density wells caused by the high-frequency pressure force. For plasmons in resonance with the beam this condition has the form

$$\frac{E_0^2}{16\pi}>W^*=n_0T_e\frac{T_e}{mv_0^2}. \quad (9)$$

Using (5) and (8) we can then write down the following relations to determine the limits of applicability of the quasi-linear theory and the weak turbulence in the problem of the relaxation of an electron beam:

$$\frac{n_1}{n_0}=10\left(\frac{T_e}{mv_0^2}\right)^3,\quad \frac{n_1}{n_0}=\Lambda^{-1}\frac{m}{M}. \quad (10)$$

The limits of applicability of the various theories in the plane of the parameters mv_0^2/T_e, n_1/n_0 are shown in Fig. 1. The lines in this figure correspond to conditions (10) and the condition $\varepsilon\approx\Lambda^{-1}$ so that I is the region of applicability of the quasi-linear theory, II is the weak turbulence region, taking into account induced scattering by ions, and III is the strong turbulence region where the modulational instability of the oscillations excited by the beam is important.

The rate of dissipation caused by the modulational instability is so large that normally above the limit given by condition (10) there lie beam density ranges in which the threshold value of the dissipation speed ν_{eff}^{thr} is appreciably larger than the growth rate γ_b of the beam instability. The dissipation produced by the modulational instability then "freezes" the noise in resonance with the beam at a level W^*. The length over which the electron beam relaxes then turns out to be equal

$$l_{NL}^{(1)}\approx\frac{v_0}{\omega_p}\frac{n_0}{n_1}\frac{T_e}{mv_0^2}\Lambda^*+\frac{1}{2\pi}\frac{v_0}{\omega_p}\left(\frac{mv_0^2}{T_e}\right)^2. \quad (11)$$

The first term in this formula is the length over which

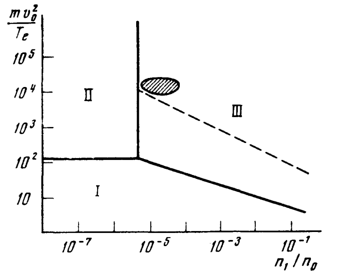

FIG. 1. Regions of applicability of various theories of beam relaxation. I, quasi-linear theory, II: weak turbulence, III: strong turbulence. We have used in the calculations the values $M/m=2\times10^4$, $\Lambda=10$, $\alpha=\frac{1}{3}$, and the shaded region corresponds to high-energy electrons in the ionosphere.

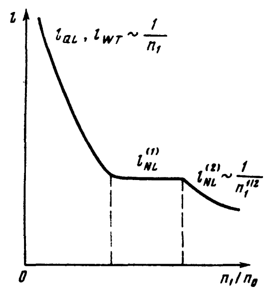

FIG. 2. Relaxation length of an electron beam as a function of its density.

the plasma noise grows to the level W^*, Λ^* is the logarithm of the ratio of the "threshold" noise W^* to the thermal fluctuation energy. The second term is the relaxation length for a "threshold" level of oscillations. Sufficiently far from the limits (10) of this regime the relaxation length given by Eq. (11) enters a plateau as function of the parameter n_1/n_0 (see Fig. 2). The induced scattering of the oscillations by the ions does not play a role and only two processes are important: the pumping of the oscillations by the beam and the dissipation due to the modulational instability. The upper limit of the plateau on the relaxation length is determined by the condition $\gamma_b = \nu_{eff}^{thr}$:

$$\frac{n_1}{n_0} = \frac{M\alpha^3}{m}\left(\frac{T_e}{mv_0^2}\right)^2. \tag{12}$$

(the region below the dashed line in Fig. 1 corresponds to the plateau of the relaxation length).

At large beam densities a "free" development of the modulational instability is possible and the connection between the energy of the plasma oscillations W and the pumping amplitude E_0 follows from Eq. (3). The level of the noise in resonance with the beam which performs the role of a constantly acting long-wavelength pump can be found from the condition that the energy transfer to the resonance region compensates the dissipation in that region as a result of the development of the modulational instability (see[13]):

$$\gamma_b E_0^2 = \nu_{eff}(W) E_0^2. \tag{13}$$

The energy of the resonance oscillations determined from this condition corresponds to a relaxation length of the beam which slowly ($\propto n_1^{-1/2}$) decreases with increasing density and which is appreciably larger than the relaxation length found in the weak turbulence framework:

$$l_{NL}^{(2)} \approx \frac{1}{2\pi}\frac{v_0}{\omega_p}\frac{mv_0^2}{T_e}\left(\frac{n_0}{n_1}\frac{M\alpha^3}{m}\right)^{1/2}. \tag{14}$$

We show in Fig. 2 how the relaxation length depends on the beam density.

3. We consider in more detail the relaxation process of an electron beam under the conditions of a free development of the modulational instability, i.e., for beam densities larger than those given by (12). We can use for the description of such a process the quasi-linear equations modified by taking into account two non-linear effects: 1) dissipation produced by the modulational instability; 2) macroscopic deformations of the plasma density as a result of the spatial inhomogeneity of the oscillations excited by the beam[1)] $E_0^2(z)$:

$$v\frac{\partial f}{\partial z} = \frac{e^2}{m^2}\frac{\partial}{\partial v}\left[\frac{|E_k|^2}{v}\frac{\partial f}{\partial v}\right], \tag{15}$$

$$v_g\frac{\partial |E_k|^2}{\partial z} = \left(\frac{4\pi^2 e^2}{mk^2}\omega_p\frac{\partial f}{\partial v} - \nu_{eff}\right)|E_k|^2, \quad kv = \omega_p, \tag{16}$$

$$\frac{1}{2\pi}\int dk|E_k|^2 = E_0^2.$$

In these equations $\omega = [\omega_p^2(z) + 3k^2v_{Te}^2]^{1/2}$ is the frequency of the plasma oscillations, $v_g = 3kT_e/m\omega_p$ is their group velocity, $\omega_p = (4\pi e^2 n(z)/m)^{1/2}$, $n(z)$ is the plasma density profile established as the result of the balance between the high-frequency and the gas-kinetic pressures:

$$n(z)T_e + W(z) = \text{const.}$$

The spatial distribution of the energy of the oscillations excited by the beam, obtained from the set of Eqs. (15), (16), is shown in Fig. 3. For small z the energy of the oscillations increases exponentially, as in the linear theory, $E_0^2 \propto \exp(2\gamma_b z/v_g)$. When the energy of the oscillations grows the first non-linear mechanism to appear is the breaking of the resonance between the beam and the oscillations due to the change in the wavenumbers of the oscillations in a plasma with a non-uniform density. The inhomogeneity is produced as the result of the plasma flowing away from the region where the oscillations are excited by the high-frequency pressure force. The shift of the wavenumbers by the width of the resonance with the beam Δk takes place when the energy of the oscillations changes by an amount

$$\Delta\frac{E_0^2}{16\pi} = U \equiv n_0 T_e k\Delta k\lambda_D^2.$$

This mechanism can not fully stabilize the beam instability (it is in general switched off when $E_0^2 = $ const) but if we take it into account the regime indicated by the number 2 on Fig. 3 is established. In this regime the spectrum of the oscillations is broadened in the direction of larger k as the result of spectral transfer caused by the macroscopic density inhomogeneity; the energy of the noise in resonance with the beam is maintained at a constant level $[E_0^2/16\pi]_{res} = U$ and the total energy of the oscillations increases linearly:

$$\frac{d}{dz}\frac{E_0^2}{16\pi} = 2U\frac{\gamma_b}{v_g}. \tag{17}$$

When the energy of the plasma oscillations increases

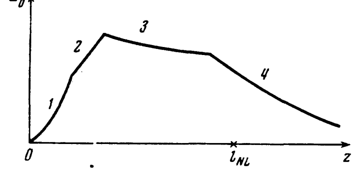

FIG. 3. Spatial distribution of the plasma noise in resonance with the beam.

further, caused by the collapse dissipation sets in, and in final reckoning the beam instability is eliminated as a result of the cancellation of the pumping by the damping $\gamma_b = \nu_{eff}$. The total energy of the oscillations, and with it also the plasma density, change rather slowly (on the scale of the quasi-linear growth length v_g/γ_b) so that the resonance pumping of the oscillations proceeds as in a uniform plasma.

The formula for the distribution in the beam in the regime considered is obtained from the condition that the right-hand side of Eq. (16) vanishes. When integrating that equation we find that in the wavenumber region in which pumping occurs ν_{eff} = const. The result has the form

$$f(z,\ v)=A(z)-B(z)/v, \tag{18}$$

where we have written

$$B=\frac{2}{\pi}n_0\frac{\nu_{eff}}{\omega_p},$$

$$A=2\frac{n_1 v_0}{v_0^2-v_1^2}+\frac{4}{\pi}n_0\frac{\nu_{eff}}{\omega_p}\frac{1}{v_0+v_1}.$$

The integration constant in Eq. (18) was determined from the condition that the current in the beam be conserved

$$\int_{v_1}^{v_0} vf\,dv=\text{const},$$

$v_1(z)$ is the lower limit of the distribution function. If the beam distribution function is known the spectrum of the plasma oscillations in resonance with the beam can be found from the quasi-linear Eq. (14)

$$|E_k|^2=2\pi^2\frac{mv^3}{\omega_p\nu_{eff}}\left(\frac{1}{2}\frac{dA}{dz}v^2-\frac{dB}{dz}v+C(z)\right),\quad k=\frac{\omega_p}{v}.$$

We obtain the equations for the quantities $C(z)$ and $\nu_{eff}(z)$ using the following obvious conditions on the spectral density of the oscillations:

$$|E_k|^2(v=v_1)=0,\ |E_k|^2(v=v_0)=0,$$

$$\frac{1}{2\pi}\int|E_k|^2\,dk=16\pi n_0T_e\left(\frac{\nu_{eff}}{\omega_p}\frac{m}{M\alpha^3}\right)^{1/2}.$$

Using these conditions we can write the formula for the spectral density of the energy of the plasma oscillations in the form

$$|E_k|^2=-2\pi^2\frac{m}{\omega_p\nu_{eff}}\frac{dB}{dz}v^3\frac{(v_0-v)(v-v_1)}{v_0+v_1} \tag{19}$$

and obtain the following set of equations for ν_{eff} and v_1:

$$\frac{\nu_{eff}}{\omega_p}=\pi\frac{n_1}{n_0}\frac{v_0v_1}{(v_0-v_1)^2}, \tag{20}$$

$$-\frac{1}{\nu_{eff}}\frac{d\nu_{eff}}{dz}=24\pi\frac{\omega_pT}{m(v_0-v_1)^3}\left(\frac{\nu_{eff}}{\omega_p}\frac{m}{M\alpha^3}\right)^{1/2}. \tag{21}$$

We solve this set of equations in two limiting cases: small spread in the beam $\Delta v=v_0-v_1\ll v_0$ (initial stage of the relaxation) and a strongly spread out beam $\Delta v\sim v_0$ (state close to the plateau). In the first case we find from (20) and (21) an equation for

$$\frac{1}{v_0}\frac{d\Delta v}{dz}=12\pi^{3/2}\left(\frac{n_1}{n_0}\frac{m}{M\alpha^3}\right)^{1/2}\frac{T}{m(\Delta v)^3}\omega_p. \tag{22}$$

The solution of this equation

$$\frac{\Delta v}{v_0}\approx2\left[\frac{3\pi^{1/2}}{2}\frac{z}{l_{NL}^{(2)}}\right]^{1/4} \tag{23}$$

determines the way the beam spreads out under the conditions when the modulational instability is important. The energy of the oscillations changes rather slowly with distance ($E_0^2\propto z^{-1/4}$; section indicated by the number 3 in Fig. 3).

At large $z\sim l_{NL}$, when the spread in the beam $\Delta v\sim v_0$ the solution of (21) can also be found easily:

$$\nu_{eff}\approx\omega_p\frac{M\alpha^3}{m}\left(\frac{v_0}{12\pi\omega_pz}\frac{mv_0^2}{T_e}\right)^2. \tag{24}$$

The energy of the resonance oscillations in this regime changes inversely proportional to z (section 4 in Fig. 3). The beam distribution function is given by the relation

$$f(z,v)=\frac{2n_1}{v_0}\left[1-\frac{v_0}{36\pi}\left(\frac{1}{v}-\frac{2}{v_0}\right)\left(\frac{l_{NL}^{(2)}}{z}\right)^2\right]. \tag{25}$$

Finally, when $z>l_{NL}$ the plasma noise is damped so that it vanishes and a plateau is established on the distribution function.

One possible application of the theory expounded here is the problem of the relaxation of fast fluxes of electrons in different problems of plasma astrophysics. We have noted in Fig. 1 the parameters of one such phenomenon (shaded region)—the currents of high-energy (10 to 20 keV) electrons penetrating into the ionosphere in auroral regions. In that case the fast electrons traverse a distance of more than 10^8 cm in the ionospheric plasma without noticeable loss of energy which is, apparently, connected with the moderation of the collective relaxation rate caused by the development of the modulational instability. Indeed, an estimate using Eq. (13) gives a value for the relaxation length $l_{NL}\sim10^{10}$ cm which exceeds the quasi-linear relaxation length by six orders.

4. The relaxation process for relativistic energies is basically similar to the relaxation of non-relativistic beams considered above. We restrict ourselves therefore in the present section to a brief exposition of the main results which characterize the relaxation of a relativistic beam in a plasma. We shall consider the case of the relaxation of a beam with a large angular spread $\Delta\theta\equiv v_\perp/c>mc^2/\mathscr{E}$ ($\mathscr{E}$ is the beam energy) which is of most interest for applications (first of all for plasma heating). In that case the relaxation is connected with the development of the kinetic beam instability.

The quasi-linear theory of the relaxation of relativistic beams was considered in[14–16]. The basic conclusions of the quasi-linear theory are the following. Owing to the anisotropy of the relativistic masses the fast developing transverse thermal spread stabilizes the pumping of "sloping" oscillations and the instability spectrum is in the case of a relativistic beam nearly one-dimensional $k_\perp/k \lesssim \Delta\theta$. Diffusion with respect to longitudinal momenta is the basic feature in the field of these oscillations and the angular spread stays approximately constant. The quasi-linear relaxation length changes as compared to the non-relativistic case (4) by a factor $\mathscr{E}(\Delta\theta)^2/mc^2$:

$$l_{QL}=\frac{c}{\omega_p}\frac{n_0}{n_1}\frac{\mathscr{E}}{mc^2}(\Delta\theta)^2\Lambda\frac{T_e}{mc^2}. \tag{26}$$

The role of the parameter ε which measures the importance of the non-linear effects connected with the induced scattering of the oscillations by the ions is now played by the quantity

$$\varepsilon_{rel}\approx 10\frac{M}{m}\left(\frac{T_e}{mc^2}\right)^3\left(\frac{mc^2}{\mathscr{E}}\right)^2\frac{1}{(\Delta\theta)^2}. \tag{27}$$

The theory of the relaxation of a relativistic beam including induced scattering by ions was constructed in[17]. If $\varepsilon_{rel}\ll 1$ the spectral transfer of the oscillations as a result of scattering by ions, as in the non-relativistic case, stabilizes the beam instability at a level of noise in resonance with the beam given by the relation

$$\Lambda^{-1}\gamma_i(E_0^2)=\gamma_b,\quad \gamma_b=\omega_p\frac{n_1}{n_0}\frac{mc^2}{\mathscr{E}}\frac{1}{(\Delta\theta)^2}$$

(γ_b is the beam instability growth rate, γ_i the growth rate for the transfer of the oscillations due to the ions (Eq. (6)).

We thus find

$$\left[\frac{E_0^2}{16\pi}\right]_{WT}=n_0T_e\frac{T_e}{\mathscr{E}}\frac{n_1}{n_0}\frac{M}{m}\frac{\Lambda}{(\Delta\theta)^2}. \tag{28}$$

At the same time the induced scattering leads to an isotropization of the spectrum of the plasma oscillations as the result of which the main quasi-linear effect of the action of the oscillations on the beam becomes diffusion in the angle θ in momentum space. The relaxation length increases by a factor $\varepsilon^{-1}\Lambda^{-1}$ as compared to its quasi-linear value and turns out to be equal to

$$l_{WT}=\frac{c}{\omega_p}\left(\frac{\mathscr{E}}{mc^2}\right)^3\left(\frac{mc^2}{T_e}\right)^2\frac{n_0}{n_1}\frac{m}{M}(\Delta\theta)^4. \tag{29}$$

for a beam with an angular spread $\Delta\theta$.

For typical parameters of relativistic beams used for plasma heating the quantity $\varepsilon_{rel}\sim 10^{-2}$ to 10^{-3} so that for the collective relaxation of such beams the non-linear effects of the induced scattering by ions are important.

The limit of applicability of the theory based upon the weak turbulence equations is, as in the non-relativistic case, connected with the appearance of a dissipation mechanism for the noise in resonance with the beam, caused by the development of the modulational instability and collapse. One can easily find the limit in terms of the level of the energy of resonance oscillations established under the action of induced scattering (Eq. (28)) and it has the following form (see[8]):

$$\frac{n_1}{n_0}=\Lambda^{-1}\frac{m}{M}\frac{\mathscr{E}}{mc^2}(\Delta\theta)^2. \tag{30}$$

Breĭzman and Ryutov[8] have suggested that above the threshold (30) the modulational instability creates such an efficient mechanism for the dissipation of the plasma noise excited by the beam, that when the beam density increases the energy of that noise becomes frozen at a level corresponding to the appearance of the modulational instability (9). As in the non-relativistic case, the relaxation length of the beam as function of its density then shows a plateau:

$$l_{NL}^{(1)}=\frac{c}{2\pi\omega_p}\left(\frac{\mathscr{E}}{T_e}\right)^2(\Delta\theta)^2. \tag{31}$$

Knowing the rate of the dissipation produced by the collapse (the quantity ν_{eff} given by Eq. (1)) we are able to find the upper limit of the plateau on the relaxation length:

$$\frac{n_1}{n_0}=\frac{M\alpha^3}{m}\frac{\mathscr{E}}{mc^2}\left(\frac{T_e}{mc^2}\right)^2(\Delta\theta)^2. \tag{32}$$

At large beam densities the energy of the oscillations in the resonance region of the spectrum can be found from Eq. (13) corresponding to the compensation of the pumping energy in the region by the beam dissipation produced by the modulational instability. The role of the induced scattering by ions is in that regime negligibly small and the spectrum of the oscillations in resonance with the beam which corresponds to pumping when the modulational instability develops is nearly one-dimensional.[2)] The diffusion of the beam particles for such a spectrum of oscillations is described by the equation

$$c\frac{\partial f}{\partial z}=e^2\frac{\partial}{\partial p_\parallel}\left[\int dk\,|E_k|^2\,\delta\left(kc\left(1-\frac{\theta^2}{2}\right)-\omega_p\right)\frac{\partial f}{\partial p_\parallel}\right]. \tag{33}$$

The relaxation length determined from Eqs. (13) and (33) decreases, as in the non-relativistic case, slowly ($\propto n_1^{-1/2}$) with increasing beam density and is considerably longer than the relaxation length found from the weak turbulence Eqs. (26) and (29):

$$l_{NL}^{(2)}=\frac{c}{2\pi\omega_p}\left(\frac{n_0}{n_1}\frac{M\alpha^3}{m}\right)^{1/2}\left(\frac{\mathscr{E}}{mc^2}\right)^{5/2}\frac{mc^2}{T_e}(\Delta\theta)^3. \tag{34}$$

In experiments on the triggering of a pulsed thermonuclear reaction in a plasma target by a relativistic beam, conditions (30) and (32) correspond to current densities in the beam which are approximately equal to 10^9 and 10^{11} A/cm^2, so that for the actual parameters of the beams in those experiments the collective relaxation of the beam is described by the weak turbulence theory. At the same time in experiments on heating of a gaseous plasma the corresponding magnitudes of the limiting current densities are appreciably lower and the protraction of the collective relaxation caused by the modulational instability therefore becomes important.

[1)]The turbulence occurring as a result of the modulational instability of the plasmons trapped in the collapsing cavities occurs mainly at short lengths $k\gtrsim \lambda_D^{-1}(W/n_0T)^{1/2}$. The energy of such a turbulence in the long-wavelength region of the spectrum in resonance with the beam is small in the ratio of the corresponding phase volumes.

$$\Delta=\left(\frac{T_e}{mv_0^2}\right)^{1/2}\frac{T_e}{T_{\perp b}}\frac{\Delta v}{v_0}\left(\frac{n_0T}{W}\right)^{3/2}$$

($T_{\perp b}$ is the transverse temperature in the beam). Assuming that $\Delta \ll E_0^2/16\pi W$ we shall in the present paper not take into account the effect of the trapped plasmons on the dynamics of the beam.

2)According to[18] the development of the collapse leads to the excitation of rather strong sound turbulence in the plasma. However, the conversion of plasma noise in resonance with the beam into density fluctuations which are produced by sound does not contribute greatly to the total energy balance as the characteristic growth rate of such a process cannot exceed the growth rate of the modulational instability $\gamma_M \sim \omega_p(mW/Mn_0T)^{1/2}$, and hence remains considerably smaller than ν_{eff}.

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The establishment of a stationary turbulence spectrum due to induced scattering of waves by particles

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We consider the problem of the evolution of the turbulence spectrum, taking into account the processes of creation, destruction, and induced scattering of waves. We show that any initial distribution of waves relaxes to a stationary distribution. We estimate the relaxation time.

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In studying plasma turbulence one often has to deal with the following equation for the occupation numbers $n(\mathbf{k}, t)$:

$$\frac{\partial}{\partial t}n=2\gamma_{\mathbf{k}}n+n\int A(\mathbf{k};\mathbf{k}')n(\mathbf{k}';t)\,d^3\mathbf{k}'+\varepsilon_{\mathbf{k}}. \qquad (1)$$

The problems of the spectra of the turbulence excited by beam or parametric plasma heating (see, e.g., [1–4]) and also some problems in non-linear optics,[5,6] in particular, reduce to the solution of this equation. The first term on the right-hand side of Eq. (1) describes the induced emission and absorption processes of waves (depending on the sign of the growth rate $\gamma_{\mathbf{k}}$), and the second the processes of induced scattering of waves by particles. The scattering probability is characterized by the kernel $A(\mathbf{k}, \mathbf{k}')$. The actual form of the kernel is unimportant for what follows. Essential is only that the number of quanta is conserved in the scattering. A formal expression of this fact is the antisymmetry of the kernel with respect to the interchange of the arguments $\mathbf{k}$ and $\mathbf{k}'$:

$$A(\mathbf{k};\mathbf{k}')=-A(\mathbf{k}';\mathbf{k}).$$

The third term on the right-hand side of (1) is the intensity of the thermal noise source.

We assume that Eq. (1) has a stationary solution and we consider the problem of how this stationary solution

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