Existence of a hexagonal relief on the surface of a dielectric fluid in an external electrical field

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The transformation of a horizontal surface of a dielectric fluid into an immobile corrugated surface following application of an external electric field is studied. The destruction of the plane boundary as a result of development of instability in fields exceeding a critical value brings the system to a stable stationary state. It is shown that at a small supercriticality the state possesses a hexagonal structure and is characterized by a hard excitation regime.

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INTRODUCTION

It is known[1,2] that a plane surface of a dielectric fluid becomes unstable in a vertical electric field when the field intensity E exceeds a certain value E_c determined by the parameters of the medium, by the dielectric constant ε, by the density ρ, and by the surface-tension coefficient α. The surface of a ferromagnetic fluid in a magnetic field is just as unstable. The instability of a plane boundary was first observed in the case of a dielectric fluid.[3,4] Cowley and Rosensweig[5] have subsequently observed a similar instability for a ferromagnetic fluid. In particular, they have established that at slight excesses above the instability threshold, the plane surface of a ferromagnetic fluid turns into an immobile corrugated surface in the form of a hexagonal lattice, analogous to the Bénard cells in convection slightly above critical.

This phenomenon constitutes a unique two-dimensional phase transition. The order parameter in this case is the deviation $\eta(\mathbf{r}_\perp)$ of the surface from plane. As shown in this paper, in the dynamic description $\eta(\mathbf{r}_\perp, t)$ is a generalized coordinate, while the quantity $\rho\Psi$, where Ψ is the hydrodynamic potential on the surface, is the generalized momentum. The Hamiltonian then coincides with the free energy of the system in the external electric field. It can be expanded in powers of the order parameter. The fundamental point is that this expansion contains terms of third order in η. We call attention to the fact that in the theory of phase transitions, as noted by Landau,[6] the presence of analogous terms in the expansion of the free energy should lead to a first-order phase transition of the crystallization type. As applied to our case, this corresponds to a hard excitation regime.

Let us see now what the surface-relief structure should be. Standing waves whose wave vectors that are nearly equal in modulus build up during the linear stage of the instability slightly above its threshold. During the nonlinear stage, owing to the cubic terms, three standing waves with angle $\pi/3$ between them turn out to be coupled. The fourth-order terms do not lead to such a rigorous correlation between the excited waves. One can therefore expect that in the case of weak excess above critical the new stationary stage would have a hexagonal structure, by virtue of the two-dimensionality of the transition. Naturally, a solution of this type can be realized only if it is stable. As shown in this paper, a solution of hexagonal type is characterized by a hard excitation regime and is linearly stable up to a certain supercriticality $h_1 = (E_1/E_c)^2 - 1$. There exists in this case a field region $0 < h < h_2$ where there are no other stable solutions. At supercriticalities exceeding h_2 ($h_2 < h_1$), one more solution becomes stable, in the form of a quadratic lattice. The realization of one of these structures in the interval $h_2 < h < h_1$ depends on the manner in which the given value of h is reached (from above or from below). Transitions from one state to another are hard.

1. VARIATIONAL PRINCIPLE AND CANONICAL VARIABLES

Let us consider the surface oscillations of an ideal dielectric fluid in an external electric field $\mathbf{E}$ in a uniform gravitational field $\mathbf{g}$. We introduce a coordinate frame with z axis parallel to the vector $\mathbf{g}$. In this frame, the shape of the surface is given by the function $z = \eta(r_\perp)$, and the normal to the surface is

$$\mathbf{n}=[1+(\nabla\eta)^2]^{-1/2}(-\nabla\eta,\ 1).$$

The kinematic condition

$$\frac{\partial\eta}{\partial t}+\mathbf{v}\nabla\eta=v_z \tag{1}$$

connects the quantity η with the velocity of the fluid. The latter is determined from the equation

$$\rho\left(\frac{\partial v_i}{\partial t}+(\mathbf{v}\nabla)v_i\right)=\frac{\partial\sigma_{ik}}{\partial x_k}+\rho g_i,$$
$$\sigma_{ik}=-\left(\varepsilon-\rho\left(\frac{\partial\varepsilon}{\partial\rho}\right)_T\right)\frac{E^2}{8\pi}\delta_{ik}+\frac{\varepsilon E_i E_k}{4\pi}-p\delta_{ik}, \tag{2}$$

where σ is the stress tensor.[7] To form a closed system, Eqs. (1) and (2) are supplemented by the electrostatic equations

$$\mathbf{E}=-\nabla\varphi,\quad \operatorname{div}(\varepsilon\nabla\varphi)=0$$

and by the boundary conditions

$$(\sigma_{ik}^1-\sigma_{ik}^2)n_k=\alpha\operatorname{div}\frac{\nabla\eta}{[1+(\nabla\eta)^2]^{1/2}}n_i,$$
$$\varphi_1=\varphi_2,\quad E_{n1}=\varepsilon E_{n2}\quad\text{at}\quad z=\eta,\ v\to0\quad\text{as}\quad z\to-\infty.$$

The subscripts 1 and 2 pertain here to the vacuum and to the medium, respectively.

In the study of surface waves it suffices to consider only potential flows for which $\mathbf{v}=\nabla\Phi$, where Φ is determined from the equation $\nabla^2\Phi=0$ with the boundary conditions

$$\rho\left[\frac{\partial\Phi}{\partial t}+\frac{(\nabla\Phi)^2}{2}\right]_{z=\eta}+\rho g\eta=\alpha\,\mathrm{div}\frac{\nabla\eta}{[1+(\nabla\eta)^2]^{1/2}}-\frac{(\varepsilon-1)^2E_{n1}^2}{4\pi\varepsilon}+\frac{(\varepsilon-1)^2E_\tau^2}{8\pi}. \tag{3}$$
$$\nabla\Phi=0\quad\text{as}\quad z\to-\infty.$$

Zakharov has shown previously[8] that Eqs. (1) and (3) in the absence of electric fields have a Hamiltonian structure, and the quantities $\Psi=\rho\Phi(\eta,\mathbf{r}_\perp,t)$ and $\eta(\mathbf{r}_\perp,t)$ are canonically conjugate:

$$\frac{\partial\eta}{\partial t}=\frac{\delta H}{\delta\Psi},\quad\frac{\partial\Psi}{\partial t}=-\frac{\delta H}{\delta\eta}.$$

This result remains in force also in the presence of the field. In this case the Hamiltonian

$$H=H_1+H_2+H_3,$$
$$H_1=\int\rho\frac{v^2}{2}d\mathbf{r},\quad H_2=\int\left\{\frac{\rho g\eta^2}{2}+\alpha[(1+(\nabla\eta)^2)^{1/2}-1]\right\}d\mathbf{r}_\perp,$$
$$H_3=-\int d\mathbf{r}_\perp\int_{-\infty}^{\eta}dz\frac{\varepsilon E_2^2}{8\pi}-\int d\mathbf{r}_\perp\int_{\eta}^{\infty}dz\frac{E_1^2}{8\pi},$$

coincides with the free energy of the system in an external electric field, while the canonical variables coincide with the previous ones. This fact is verified directly by varying the Hamiltonian H. The only nontrivial variation is that of H_3 with changing boundary. The variation is carried out in this case in analogy with the procedure used, for example, in the book of Landau and Lifshitz[7] in the derivation of the stress tensor σ_{ik}.

It should be noted that the equations of motion of the surface waves are Hamiltonian also under more general assumptions: at an arbitrary dependence $\varepsilon=\varepsilon(E^2)$, in the presence of arbitrary geometry, for an interface between two dielectric fluids, etc. It is clear that after replacing ε by μ and E by H, all the foregoing applies equally well to a ferromagnetic fluid in an external field.

2. STATIONARY SOLUTIONS

We consider henceforth a situation wherein the external homogeneous electric field is parallel to the vector $\mathbf{g}$. In this case, as is well known,[2] there exists a certain threshold field E_c, above which a plane boundary is unstable. We investigated such a regime, which occurs above the threshold of this instability. We consider first once more briefly the linear stage of the instability of the plane boundary. This problem is easiest to formulate and to solve within the framework of the Hamiltonian approach. It suffices for this purpose to take the Fourier transform with respect to $\mathbf{r}_\perp$ and to expand the Hamiltonian H in powers of the canonical variables, retaining only the quadratic terms

$$H_0=\int d\mathbf{k}\left[\frac{k}{2\rho}|\Psi_k|^2+\frac{\rho\omega_k^2}{2k}|\eta_k|^2\right],$$

$$\omega_k^2=gk+\frac{\alpha k^3}{\rho}-\frac{(\varepsilon-1)^2E^2}{\varepsilon(\varepsilon+1)4\pi\rho}k^2,$$

where $\omega_\mathbf{k}^2$ is the square of the frequency of the small oscillations. It follows from this directly that at

$$\frac{E^2}{8\pi}>\frac{E_c^2}{8\pi}=\frac{\varepsilon(\varepsilon+1)}{(\varepsilon-1)^2}(\rho g\alpha)^{1/2}$$

the frequency $\omega_\mathbf{k}^2<0$, and the plane boundary is unstable, the maximum growth rate being possessed by the oscillations with wave number k_0 equal to $(\rho g/\alpha)^{1/2}$.

The nonlinear stage of development of the instability is determined by the following terms of the expansion of the Hamiltonian H:

$$H_{int}=\int\prod_i d\mathbf{k}_i\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\left[V_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}\eta_{\mathbf{k}_1}\Psi_{\mathbf{k}_2}\Psi_{\mathbf{k}_3}-\frac{1}{3!}u_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}\eta_{\mathbf{k}_1}\eta_{\mathbf{k}_2}\eta_{\mathbf{k}_3}\right]$$
$$+\int\prod_i d\mathbf{k}_i\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)\left[F_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4}\eta_{\mathbf{k}_1}\eta_{\mathbf{k}_2}\Psi_{\mathbf{k}_3}\Psi_{\mathbf{k}_4}+\frac{1}{4!}T_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4}\eta_{\mathbf{k}_1}\eta_{\mathbf{k}_2}\eta_{\mathbf{k}_3}\eta_{\mathbf{k}_4}\right]+\ldots, \tag{4}$$

where the matrix elements have the symmetry that follows from (4). An explicit expression for the coefficients u and T will be presented later on, whereas the coefficients V and F have been calculated by Zakharov.[8]

The nonlinear terms lead to a limitation of the amplitude of the oscillations. However, the transition to some stationary state is impossible, since the system is Hamiltonian—the damping $\gamma_\mathbf{k}$, which is due to the viscosity

$$\frac{\partial\Psi_\mathbf{k}}{\partial t}+\gamma_\mathbf{k}\Psi_\mathbf{k}=-\frac{\delta H}{\partial\eta_\mathbf{k}^*}$$

is essential. It is obvious that $\Psi_\mathbf{k}=0$ for the new stationary state, and the stationary relief of the surface is determined from the equation

$$\frac{\delta(H_2+H_3)}{\delta\eta_\mathbf{k}^*}=\frac{\rho\omega_k^2}{k}\eta_\mathbf{k}-\frac{1}{2}\int u_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}\eta_{\mathbf{k}_1}\eta_{\mathbf{k}_2}\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)d\mathbf{k}_1d\mathbf{k}_2$$
$$+\frac{1}{3!}\int T_{-\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}\eta_{\mathbf{k}_1}\eta_{\mathbf{k}_2}\eta_{\mathbf{k}_3}\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)d\mathbf{k}_1d\mathbf{k}_2d\mathbf{k}_3=0. \tag{5}$$

We confine ourselves in the expansion of the Hamiltonian to terms of fourth order inclusive, which in any case presupposes smallness of the angle of inclination of the surface $|\nabla\eta|\ll1$. As will be shown below, for stationary solutions with hard excitation regimes, this assumption calls, besides the natural condition that the excess of the external field over the threshold be small, also that the ratio of the matrix elements u/T be small. In this case this reduces to the condition

$$\chi=(\varepsilon-1)/(\varepsilon+1)\ll1,$$

which we assumed to be satisfied.

We change over in (5) to the dimensionless variables

$$k/k_0\to k,\quad k_0^3\eta_\mathbf{k}\to\eta_\mathbf{k}.$$

In terms of the new variables, the expressions for the square of the frequency and the matrix elements near the threshold take the form

$$\omega_k^2 = k(k-1)^2 - 2hk^2,$$

$$u_{k_1k_2k_3} = \chi\left[2(k_1k_2+k_1k_3+k_2k_3) - \sum_i k_i^2\right],$$

$$T_{k_1k_2k_3k_4} = \sum_{i<j} |\mathbf{k}_i+\mathbf{k}_j|^3 - \sum_i k_i^3 - [(\mathbf{k}_1\mathbf{k}_2)(\mathbf{k}_3\mathbf{k}_4) + (\mathbf{k}_1\mathbf{k}_3)(\mathbf{k}_2\mathbf{k}_4) + (\mathbf{k}_1\mathbf{k}_4)(\mathbf{k}_2\mathbf{k}_3)].$$

In particular, at $|\mathbf{k}_i| = 1$ we have

$$u_{k_1k_2k_3} = u = 3\chi,$$
$$T_{k_1-k_1k_2-k_2} \equiv T_\theta = 16|\cos {}^1/_2\theta|^3 + 16|\sin {}^1/_2\theta|^3 - \cos 2\theta - 10, \qquad (6)$$

θ is the angle between the vectors $\mathbf{k}_1$ and $\mathbf{k}_2$.

We proceed now to a study of the stationary solutions of (5). We call attention first to the fact that in the case of weak supercriticality $(h = (E/E_c)^2 - 1 \ll 1)$ the unstable waves are standing waves with wave vectors in a narrow layer $h^{1/2}$ near $k = k_0 = (\rho g/\alpha)^{1/2}$. If we confine ourselves only to the interaction of the excited waves, then the third-order terms near the threshold take the form

$$H_{int}^{(3)} = -\frac{1}{3!}u\int \eta_{k_1}\eta_{k_2}\eta_{k_3}\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\prod_i d\mathbf{k}_i + V\int \eta_{k_1}\Psi_{k_2}\Psi_{k_3}\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\prod_i d\mathbf{k}_i.$$

In view of the momentum conservation $\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3=0$ and $|\mathbf{k}_i| \approx 1$, the wave vectors $\mathbf{k}_i$ form, with good accuracy, an equilateral triangle. This means that three waves, with angle $\pi/3$ between their wave vectors, are coupled. One can therefore expect the stationary state to have a hexagonal structure in the case of weak supercriticality.

For a solution of this type, the reciprocal-lattice vectors q_i should be chosen with modulus equal to unity, corresponding to a maximum growth rate of the plane boundary. This state corresponds to the relief

$$\eta(\mathbf{r}_\perp) = 2\eta_3(\cos \mathbf{q}_1\mathbf{r}_\perp + \cos \mathbf{q}_2\mathbf{r}_\perp + \cos \mathbf{q}_3\mathbf{r}_\perp),$$
$$\mathbf{q}_1+\mathbf{q}_2+\mathbf{q}_3 = 0. \qquad (7)$$

Taking the Fourier transform of (7), we obtain from (5) the dependences of the amplitude η_3 on the supercriticality (Fig. 1):

$$\eta_{3\,\mathrm{I},\,\mathrm{II}} = \frac{u}{T_0+4T_{\pi/3}} \pm \left[\left(\frac{u}{T_0+4T_{\pi/3}}\right)^2 + \frac{4h}{T_0+4T_{\pi/3}}\right]^{1/2}, \qquad (8)$$

where T_0 and $T_{\pi/3}$ (see (6)) are equal respectively to 5 and $6\sqrt{3} - \frac{15}{2}$.

The first solution in (8) is characterized by a hard excitation regime with a discontinuity in the order parameter

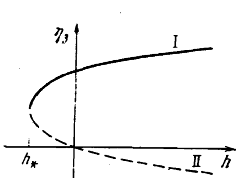

FIG. 1.

$$\eta_c = \frac{2u}{T_0+4T_{\pi/3}} = \frac{6}{24\sqrt{3}-25}\chi,$$

from which we get directly the applicability criterion: $\chi \ll 1$. We set aside, however, the question of the values of ε at which the solution will still retain a hexagonal structure. To this end it is necessary at least to satisfy the condition $T_0 + 4T_{\pi/3} > 0$, where the matrix elements T take into account the dependence on χ. This inequality is violated at $\varepsilon > \varepsilon_c = 2.1$. One should therefore expect in the experiment the formation of a hexagonal structure at $\varepsilon < \varepsilon_c$. As a rule, this condition is satisfied for ferromagnetic fluids and cannot be realized in the case of a boundary between two dielectric fluids with close values of ε_1 and ε_2. In the latter case, after making the substitutions

$$\varepsilon \to \frac{\varepsilon_2}{\varepsilon_1}, \quad E_0^2 \to \varepsilon_1 E_0^2, \quad \rho \to \rho_1+\rho_2, \quad g \to \frac{\rho_2-\rho_1}{\rho_2+\rho_1}g$$

the problem is fully equivalent to that considered above.

As to the second solution (the lower part of the branch), it is of the "supersoft" type, and the amplitude for it near threshold is proportional to h.

In addition to the considered periodic structure, two other structures[1)] are possible, in the form of a one-dimensional lattice

$$\eta(\mathbf{r}_\perp) = 2\eta_1 \cos \mathbf{q}\mathbf{r}_\perp \qquad (9)$$

and a quadratic lattice

$$\eta(\mathbf{r}_\perp) = 2\eta_2(\cos \mathbf{q}_1\mathbf{r}_\perp + \cos \mathbf{q}_2\mathbf{r}_\perp), \qquad (10)$$

where $\mathbf{q}_1 \cdot \mathbf{q}_2 = 0$.

It is obvious that the third-order terms make no contributions for the last two solutions.[2)] Both solutions are characterized by a soft excitation regime. For the first of them, the amplitude is

$$\eta_1 = 2(h/T_0)^{1/2},$$

and for the second

$$\eta_2 = 2[h/(T_0+2T_{\pi/2})]^{1/2}.$$

3. STABILITY AND COLLECTIVE OSCILLATIONS

Let us investigate the stability of the obtained stationary solutions. To this end we linearize the equations of motion against the background of the stationary relief. For surface-shaped perturbations of the form $\xi_k e^{-i\Omega t}$ we obtain

$$\Omega^2\xi_k = k(k-1)^2\xi_k - 2hk^2\xi_k - ku\int \eta_{k_1}\xi_{k_2}\delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})d\mathbf{k}_1 d\mathbf{k}_2 + \frac{k}{2}\int T_{-kk_1k_2k_3}\eta_{k_1}\eta_{k_2}\xi_{k_3}\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3-\mathbf{k})d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3. \qquad (11)$$

In the derivation of this equation we have left out terms proportional to $V_{k_1k_2k_3}$ and $F_{k_1k_2k_3k_4}$ (4), which can be easily seen to have an additional smallness relative to $|\nabla\eta|$. The most dangerous from the point of view of stability are, as before, perturbations with wave numbers on the order of unity, $|k-1| \sim \sqrt{h}$, to which we shall henceforth confine ourselves. Outside this wave-vector interval, the frequency Ω remains equal to the frequency ω of the oscillations of the plane boundary, and such oscillations are therefore stable.

1. We consider first the stability of the hexagonal structure. For perturbations with wavelengths $k-1 \sim \sqrt{h}$ it is necessary to distinguish between two cases. In the first case the wave vector **k** lies in the interval between the reciprocal-lattice vectors (nonresonant oscillations), and in the second the vector **k** is close to one of the reciprocal-lattice vectors (resonant oscillations).

In the first case none of the vectors of the type $\mathbf{k}+\mathbf{q}_i$ has a length close to unity, and the natural modes are plane waves $\xi \sim \exp(i\mathbf{kr}_\perp)$. Their dispersion relation takes the form

$$\Omega^2=(k-1)^2-2h+(T_\theta+T_{\theta+\pi/3}+T_{\theta-\pi/3})\eta_3^2, \tag{12}$$

where θ is the angle between the vectors **k** and $\mathbf{q}_1$. The square of the frequency is minimal when the wave vector **k** has a unit length and makes an angle $\pi/6$ with one of the reciprocal-lattice vectors:

$$\Omega_{min}^2=-2h+(2T_{\pi/6}+T_{\pi/2})\eta_3^2.$$

It follows therefore that the "supersoft" stationary solution ($\eta \sim h$) is unstable at small h, while to the contrary the "hard" stationary solution is stable with respect to nonresonant oscillations.

Expression (12) for the square of the frequency is valid up to angles $\theta_{\min} \sim \chi \ll 1$. At smaller angles third-order terms become resonant—among the vectors of the type $\mathbf{k}+\mathbf{q}_i$ there appear vectors with lengths on the order of unity. In this case the eigenfunctions of Eq. (1) constitute a combination of six waves:

$$\xi_{\mathbf{k}}=\sum_i c_i\delta(\mathbf{k}-\mathbf{q}_i-\boldsymbol{\varkappa}), \quad |\varkappa|\ll 1.$$

The system of equations for the amplitudes c_i takes the form

$$\Omega^2 c_1=[(\varkappa\mathbf{q}_1)^2-2h+(2T_{\pi/3}+T_0)\eta_3^2]c_1 + \frac{1}{2}T_0\eta_3^2 c_{-1}-u\eta_3(c_{-2}+c_{-3})+T_{\pi/3}\eta_3^2(c_2+c_{-2}+c_3+c_{-3}). \tag{13}$$

Analogous equations for the remaining five coefficients are obtained by cyclic permutation of the subscripts.

Equations (13) break up into two systems: for even $\alpha_n=\frac{1}{2}(c_n+c_{-n})$ and for odd $\beta_n=(c_n-c_{-n})/2i$ combinations of the coefficients ($n=1,2,3$). For the odd perturbations we have

$$\Omega^2\beta_i=(\varkappa\mathbf{q}_i)^2\beta_i+u\eta_3\sum_1^3\beta_n.$$

From these follows the dispersion relation

$$\Omega^6-3\Omega^4(^1/_2\varkappa^2+u\eta_3)+3\Omega^2\varkappa^2(u\eta_3+{}^3/_{16}\varkappa^2) -{}^9/_{16}u\eta_3\varkappa^4-{}^1/_{32}\varkappa^6(1+\cos 6\varphi)=0, \tag{14}$$

where φ is the angle between the reduced wave vector $\varkappa$ and one of the reciprocal-lattice vectors. The extrema of the square of the frequency as functions of φ lie at the points $\cos 6\varphi=\pm 1$ (we note that at these points Eq. (14) can be solved exactly). The minima of the square of the frequency with respect to $\varkappa$ lies at the point $\varkappa=0$. We present expressions for the frequencies of the three branches of the odd oscillations at small $\varkappa(\varkappa^2 \ll u\eta_3)$:

$$\Omega_1^2={}^1/_4\varkappa^2, \quad \Omega_2^2={}^3/_4\varkappa^2, \quad \Omega_3^2=3u\eta_3+{}^1/_2\varkappa^2.$$

Two branches of the odd modes are acoustic and one branch is optical. The different speeds of sound (1/2 and $\sqrt{3}/2$) correspond to different values of the compressibility of the system in the different directions, along the principal diagonal of the hexagon and perpendicular to its side.

From the expression for the square of the frequency of the optical branch Ω_3^2 it follows that the stationary solution II is unstable at $h>0$, since $u\eta_3$ is negative at $h>0$. The solution I with hard excitation regime relative to the odd modes is stable.

We turn now to the even perturbations. The equations for the coefficients α_i take the form

$$\Omega^2\alpha_i=[(\varkappa\mathbf{q}_i)^2+2(T_0+T_{\pi/3})\eta_3^2-4h]\alpha_i-(^1/_2T_0\eta_3^2-2h)\sum_1^3\alpha_n.$$

Just as for the odd modes, the squares of the frequencies are minimal at $\varkappa=0$. We present their explicit expressions at small $\varkappa$:

$$\Omega_{4,5}^2=2(T_0+T_{\pi/3})\eta_3^2-4h+{}^1/_2\varkappa^2\pm{}^1/_4\varkappa^2,$$
$$\Omega_6^2={}^1/_2(T_0+4T_{\pi/3})\eta_3^2+2h+{}^1/_2\varkappa^2.$$

It is seen that the solution I at small h is stable relative to the even oscillations. When the external field becomes smaller than the critical one ($h<0$), the hard state retains stability up to a value h_* determined from the condition $\Omega_6^2(\varkappa=0)=0$, or $h_*=-u^2/4(T_0+4T_{\pi/3})$, corresponding to the vertex of the parabola in Fig. 1. Thus, hysteresis should be observed when the external field is decreased after reaching the maximum.

We note that the state corresponding to section II of Fig. 1 is always unstable, namely at $h<0$ relative to oscillations with frequency Ω_3 and at $h<0$ relative to oscillations with frequencies Ω_6. When speaking of a hexagonal lattice we shall therefore always bear in mind solution I.

The smallest stability margin is possessed by even oscillations with frequencies $\Omega_{4,5}(\varkappa=0)$. The threshold below which the hexagonal lattice becomes unstable is determined from the condition $\Omega_{4,5}^2=0$ and is equal to

$$h_1=\frac{2(T_0+T_{\pi/3})}{(2T_{\pi/3}-T_0)^2}u^2=25.6u^2. \tag{15}$$

2. We shall show that a stationary solution in the form of a one-dimensional lattice (9) is unstable with respect to perturbations propagating at an angle θ to the reciprocal-lattice vector (external instability). Let θ be close in magnitude to neither of the values $\pm\pi/3$ or $\pm 2\pi/3$, so that the third-order terms are not resonant.

In this case the dispersion relation is

$$\Omega^2=(k-1)^2-2h+T_0\eta_1^2.$$

It follows from (6) that the maximum increments are possessed by perturbations with $\theta=\pm\pi/2$ and $k=1$:

$$-\Omega^2=\Gamma^2_{max}={}^2/_5(23-16\sqrt{2})h>0.$$

3. We now investigate the stability of the quadratic lattice (10). We consider first nonresonant oscillations, the wave vector **k** of which is not close to any of the vectors $\mathbf{q}_i$ of the reciprocal lattice. In addition, we assume that none of the angles between the vector **k** and the vectors $\mathbf{q}_i$ is close to $\pi/6$. For perturbations of this type we obtain from (11) an expression for the square of the frequency

$$\Omega^2=(k-1)^2-2h+(T_\theta+T_{\pi/2-\theta})\eta_2^2.$$

It is minimal at $k=1$ and $\theta=\pi/4$:

$$\frac{\Omega^2_{min}}{h}=-2+\frac{16}{16\sqrt{2}-13}\left[2(4\sqrt{2}-1)\cos\frac{\pi}{8}-5\right]\approx 0.92.$$

Thus, the quadratic lattice is stable with respect to these perturbations. In the case when the wave vector of the perturbation is close to one of the reciprocal-lattice vectors ($\mathbf{k}=\mathbf{q}_i+\varkappa$, $|\varkappa|\ll 1$), the eigenfunctions of Eq. (11) constitute a combination of four waves:

$$\xi_\mathbf{k}=\sum c_i\delta(\mathbf{k}-\mathbf{q}_i-\varkappa),$$

the amplitudes c_i of which obey the equations that follow from (11).

Just as in the case of a hexagonal lattice, the perturbations can be even or odd. Odd perturbations are acoustic oscillations with frequencies

$$\Omega^2=\varkappa^2\cos^2\varphi,\quad \Omega^2=\varkappa^2\sin^2\varphi.$$

while the even perturbations are optical with frequencies

$$\Omega^2=T_0\eta_2^2+{}^1/_2\varkappa^2\pm[(2T_{\pi/2}\eta_2^2)^2+{}^1/_4\varkappa^4\cos^2 2\varphi]^{1/2}.$$

where φ is the angle between the vectors $\varkappa$ and $\mathbf{q}_1$. All these oscillations are stable, in particular

$$\Omega^2_{min}=(T_0-2T_{\pi/2})\eta_2^2=(23-16\sqrt{2})\eta_2^2>0.$$

Finally, we consider perturbations with wave vector **k** in the vicinity of the unit vector $\mathbf{k}_0$ making an angle $\pi/6$ with one of the reciprocal-lattice vectors (for the sake of argument—with $\mathbf{q}_1$)

$$\mathbf{k}_1=\mathbf{k}_0+\varkappa,\quad |\varkappa|\ll 1.$$

In this case the vector $\mathbf{k}_2$, which is equal to $\mathbf{k}_1+\mathbf{q}_2$, also has a modulus close to unity and the perturbations are combinations of two waves

$$\xi_\mathbf{k}=c_1\delta(\mathbf{k}-\mathbf{k}_1)+c_2\delta(\mathbf{k}-\mathbf{k}_2),$$

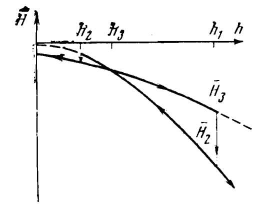

FIG. 2. Dependence of the average energy density $\bar H$ on the supercriticality h for a hexagonal ($\bar H_3$) and quadratic ($\bar H_2$) lattice. The solid and dashed lines correspond to stable and unstable states, respectively.

the amplitudes c_i of which satisfy the equations

$$\Omega^2c_1=[\varkappa^2\cos^2(\varphi-\pi/6)-2h+(T_{\pi/6}+T_{\pi/3})\eta_2^2]c_1-u\eta_2c_2,$$
$$\Omega^2c_2=[\varkappa^2\cos^2(\varphi+\pi/6)-2h+(T_{\pi/6}+T_{\pi/3})\eta_2^2]c_2-u\eta_2c_1.$$

The squares of the frequencies are

$$\Omega_\pm^2=-2h+(T_{\pi/6}+T_{\pi/3})\eta_2^2+{}^1/_4\varkappa^2(2+\cos 2\varphi)\pm[(u\eta_2)^2+{}^3/_{16}\varkappa^4\sin 2\varphi]^{1/2}.$$

We see therefore that the square of the frequency Ω^2_- is minimal at $\varkappa=0$ and is negative at small supercriticalities $h<h_2$, where h_2 is determined from the condition $\Omega^2_{-min}=0$:

$$h_2=\frac{T_0+2T_{\pi/2}}{(2T_{\pi/3}+2T_{\pi/6}-2T_{\pi/2}-T_0)^2}u^2\approx 0.465u^2.$$

Thus, a quadratic lattice is unstable for small h, and the only stable stationary solution in the region $0<h<h_2$ is the hexagonal relief. In the region $h_2<h<h_1$, both the hexagonal and the quadratic lattices are stable. Comparing the average energy densities for these structures, we can answer the question of the nonlinear instability. For the quadratic lattice the average energy per unit area is

$$\bar H_2=-\frac{8h^2}{T_0+2T_{\pi/2}},$$

and for the hexagonal lattice

$$\bar H_3=-\frac{1}{T_0+4T_{\pi/3}}[(u\eta_3+4h)^2-4h^2]$$

(see Fig. 2). They become comparable at a supercriticality

$$h_3=\frac{u^2}{T_0+4T_{\pi/3}}\frac{a-1}{(a-2)^2}\approx 5.77u^2;$$

here

$$a=\left(1+2\frac{T_0+4T_{\pi/3}}{T_0+2T_{\pi/2}}\right)^{1/2}.$$

In strong fields, the quadratic lattice is energywise favored in strong fields and the hexagonal in weak fields. However, if the external field increases adiabatically slowly, then the hexagonal structure, which is produced at $h=0$, is preserved up to a supercriticality h_1, at which the structure is destroyed and a hard transition to quadratic cells takes place. In the reverse slow decrease of the field to the subcriticality h_2, a quadratic lattice will be observed.

It should be noted that the numerical value of h_1 must be approached with caution because of the random numerical smallness in the denominator of (15) for the threshold h_1, but one can hope the qualitative picture of the transitions to remain valid.

We note also that in the presence of dissipation imaginary increments appear in the expressions for the squares of the frequencies of the natural oscillations, but all the conclusions concerning the stability and the expression for the thresholds remain in force.

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1)One more structure with triangular symmetry is possible, but in our case it coincides with the hexagonal structure (7) and is obtained from the latter by shifting the origin (see[9]).

2)Generally speaking, the third-terms make a definite contribution to the fourth-order terms, but this contribution is small in terms of the parameter χ^2.

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Features of the volt-ampere characteristics and oscillations of the electric potential in superconducting channels

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The physical nature of the supercritical resistive current states in narrow superconducting channels and the connection of this phenomenon with relaxation processes of "mixing" of the electron and hole components of the normal-excitation spectrum are discussed. The exact solutions of the kinetic equations in the vicinity of the singular points of the structure of the resistive state are investigated and the effective boundary conditions at these points for the macroscopic equations of the structure are found. The solutions of these equations for large currents of the order of the upper critical current j_{c2} and the volt-ampere characteristics of a long channel are constructed. The role of the principle of minimum entropy production in the formation of the structure of the resistive state is noted. At low currents the static structure is found to be unstable, generally speaking. The physical reasons for the instability are analyzed together with the corresponding manifestations of the nonstationarity in the resistive state.

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It is well known[1] that the peculiar diamagnetic properties of a superconductor are, in a certain sense, a more fundamental characteristic than the infinite conductivity. Significant in this respect is, e.g., the explanation of the nature of the dissipative current states in bulk type-I and type-II superconductors. From a microscopic point in view[2,3] the electric fields that arise in these superconductors on passage of a transport current have, in essence, an induction origin. They are associated with the dynamics of the magnetic fluxes in the superconductor and with the acceleration of the superconducting condensate in the vortex electric fields:

$$\partial \mathbf{p}_s/\partial t = e\mathbf{E}, \quad \operatorname{rot} \mathbf{p}_s = -e\mathbf{H} \tag{1}$$

($\mathbf{p}_s$ is the condensate momentum per electron; $\hbar = c = 1$).

A different situation arises in narrow superconducting "channels" connected to a current source. In view of the small transverse dimensions of the samples the dissipative current states observed experimentally in them are not explained by vortex mechanisms[2] or by the structure of the intermediate state,[3] and, thus, a new physical aspect of superconductivity is manifested here—a singularity in the response of the superconductor to a nonequilibrium longitudinal electric field.

As already noted,[4] the question of the nature of the resistive states in narrow channels abuts primarily upon the study of the Cooper instability in the normal current state at below-critical temperatures $T < T_c$. Unlike a condensate-accelerating vortex electric field, which is associated with the change of magnetic flux and (in accordance with Anderson's theorem[5] on violation