Phase transitions in gauge and spin-lattice systems

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A simple recursion equation giving an approximate description of critical phenomena in lattice systems is proposed. The equations for a d-dimensional spin system and a $2d$-dimensional gauge system coincide. An interesting consequence is the zero transition temperature in the two-dimensional Heisenberg model and four-dimensional Yang-Mills model; this corresponds to asymptotic freedom in field theory.

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INTRODUCTION

Second-order phase transitions possess the remarkable property of universality. In the critical region the micro-structure of the system is unimportant, and only its dimensionality and internal-symmetry group are important. Increasing the internal symmetry makes a phase transition more difficult—the critical dimensionality d_c above which an ordered phase exists is increased. Thus, $d_c = 1$ for the discrete group SO(1) (the Ising Model), while $d_c = 2$ for the continuous groups SO(n) (the Heisenberg model). Recently, gauge systems in which the parameters of the group can depend on the position in the lattice have been introduced[1]. There is reason to expect that $d_c = 4$ for continuous gauge groups. Such a system corresponds to relativistic field theory with a gauge-invariant spectrum of states. Here the noninvariant objects (quarks) are confined within the invariant ones (hadrons) by long-range forces[1].

A recursion equation describing critical phenomena in a gauge system was proposed in the author's paper[2].

Below, using the same methods, we shall obtain a recursion equation for spin systems. As in the gauge equation, the dependence on the dimensionality of space is separated out in explicit form. Here there is a remarkable analogy between a d-dimensional spin system and a 2d-dimensional gauge system: the recursion equations for them coincide. From this follow, in particular, the values $d_c = 2$ for the discrete gauge group and $d_c = 4$ for a continuous gauge group. The transition temperature and critical indices, which, in the general case, are determined by solving the recursion equation numerically, can be expanded in powers of $d - d_c$.

The zeroth and first terms of the expansions for the indices are exact. In this respect our equation resembles Wilson's equation[3], which gives the first terms of the expansion in 4 − d for spin systems.

1. THE EXACT EXCURSION EQUATIONS

We shall consider the two-dimensional Heisenberg model and introduce for it a Z-functional—a partition function with pinned spins at the boundary; by definition, the couplings between the spins at the boundary appear with half weight in the energy. As was remarked by Berezinskiĭ[4], Z-functionals are multiplicative: When neighboring regions are joined into one it is necessary to multiply their Z-functionals and integrate over the spins on the common boundary. It is convenient to consider square regions; then the functional Z_{2L} of a square with side 2L is equal to the averaged product of four Z_L-functionals:

$$Z_{2L}\{S_\Gamma\} = \int \Pi(dS_{int}) \prod_{i=1}^{4} Z_L\{S_{\gamma_i}\}, \quad (1)$$

$$(dS) = d^n S\delta(1-S^2)\Gamma(n/2)\pi^{-n/2}. \quad (2)$$

Here $\{S_\Gamma\}$ is the set of spins on the boundary of the 2L-square, $\{S_{\gamma i}\}$ are the sets of spins on the boundaries of the L-squares, and $\{S_{int}\}$ are the spins on the internal boundaries of the L-squares (see Fig. 1).

Equation (1) is a functional recursion equation, which must be solved with the following initial condition on the unit cell (Fig. 2):

$$Z_a = \exp\left(\frac{\beta}{2}\sum_{i=1}^{4} S_i S_{i+1} + \frac{h}{2}\sum_{i=1}^{4} S_i\right). \quad (3)$$

Analogous equations can be written in d-dimensional space—the number of factors in (1) will then be 2^d. Obviously, we are interested in the free energy in the statistical limit $L \to \infty$. It can be related, in general form, to the solution of the normalized equation. This is done as follows. From the Z-functionals in (1) we separate out their averages over $\{S\}$, i.e., the usual partition functions:

$$Z_L\{S\} = \langle Z_L\rangle W_L\{S\}. \quad (4)$$

We then obtain for the correlation functional W a normalized equation of the form

$$W_{2L} = \hat{R}(W_L)/\langle\hat{R}(W_L)\rangle, \quad (5)$$

and we can express the partition function in terms of the correlation functional by the recursion relation

$$\langle Z_{2L}\rangle = \langle Z_L\rangle^{2d}\langle R(W_L)\rangle. \quad (6)$$

We then find the free energy $-\beta F = \lim L^{-d} \ln\langle Z_L\rangle$:

$$-\beta F(\beta, h) = \sum_{k=1}^{\infty} 2^{-kd} \ln\langle\hat{R}(W_{2^{k-1}})\rangle + \ln\langle Z_a\rangle. \quad (7)$$

As is usual in the recursion approach[3], the singularities of the free energy are associated with unstable

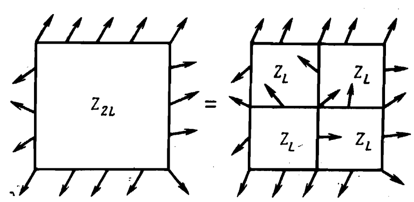

FIG. 1. Exact recursion equations for the correlation functional of the two-dimensional Heisenberg model. The arrows indicate the distribution of spins on the sides of the squares. Averaging is performed over the internal spins.

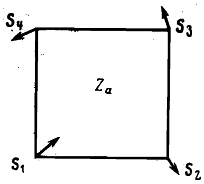

FIG. 2. Correlation functional of a unit cell.

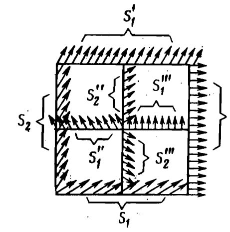

FIG. 3. Approximate recursion equation. Spin fluctuations along the sides of the squares are not taken into account.

fixed points of the equation for the correlation functional:

$$W_L \to W_*\{S\} + (\beta-\beta_c)L^{1/\nu}\Phi_1\{S\} + hL^{d-\Delta}\Phi_2\{S\}. \tag{8}$$

The critical indices ν and Δ are determined from the equation linearized about W_*: Φ_1 and Φ_2 are its eigenfunctions, with eigenvalues $2^{1/\nu}$ and $2^{d-\Delta}$ respectively. After this, the free energy has the well-known scaling form

$$F(\beta, h) = (\beta-\beta_c)^{\nu d} f(h(\beta-\beta_c)^{(\Delta-d)\nu}) + \text{regular terms} \tag{9}$$

The singular terms are determined by the large distance $2^k \sim (\beta - \beta_c)^{-\nu}$ or $2^k \sim h^{-1/(d-\Delta)}$ in the sum (7).

2. LOW-TEMPERATURE APPROXIMATION

Up to now our arguments have been exact and have not differed, in essence, from the general Kadanoff-Wilson scheme[3].

This scheme acquires practical significance if a successful way of approximating the functional equations by integral equations is found. The well-known approximation of Wilson is valid for $d = 4 - \epsilon$, when the spin fluctuations are close to being Gaussian.

We propose another approximation, which is valid for $d = d_c + \epsilon$, when the transition temperature is low.

In the low-temperature approximation we shall neglect fluctuations of the spins along the side of a square (with $d = 2$, for the present) and shall seek the Z-functional in the form of a product of functions of the average spins of opposite sides:

$$Z_L = F_L(S_1, S_1')F_L(S_2, S_2'). \tag{10}$$

After integrating over the four average internal spins in (1) (Fig. 3), we find

$$F_{2L}(S, S') = (F_L^2)^2_{SS'}. \tag{11}$$

Here F^2 denotes the contraction

$$(F^2)_{SS'} = \int (dS'')F(S, S'')F(S'', S'). \tag{12}$$

In d-dimensional space we shall associate an average spin with each of the 2d faces of a cube and shall seek the Z-functional in the form of a product of d functions $F_L(S_i, S_i)$ of the pairs of spins on opposite faces. Integrating over the spins on the internal faces of the 2^d cubes forming the doubled cube, we find

$$F_{2L}(S, S') = (F_L^2)^{2^{d-1}}_{SS'}. \tag{13}$$

Finally, as in the gauge theory[2], it will be convenient to generalize the equation to the case of an arbitrary change of scale in place of the doubling. Proceeding analogously, when λ^d cubes are joined together we find the equation

$$F_{\lambda L}(S, S') = (F_L^\lambda)^{\lambda^{d-1}}_{SS'}, \tag{14}$$

where F^λ is the λ-fold contraction.

For noninteger λ this contraction can be defined by means of an eigenfunction expansion on the unit sphere.

For the case when there is no magnetic field, so that F depends only on the scalar product (SS′), this expansion has the form

$$F_L = \sum_p f_p(L)\chi_p(SS')N_p. \tag{15}$$

For the Fourier coefficients $f_p(L)$ we obtain from (14) the equation

$$\chi_p(1)f_p(\lambda L) = \int (dS) \left[\sum_q f_q^\lambda(L)\chi_q(SS')N_q\right]^{\lambda^{d-1}} \chi_p(SS'). \tag{16}$$

This equation goes over into the gauge recursion equation of[2], if in (16) we replace

$$\lambda \to \lambda^2, \quad L \to L^2, \quad d \to d/2. \tag{17}$$

The spherical functions $\chi_p(z)$ appearing in (15) and (16) coincide for $n = 2$ and 4 with the characters of the groups U(1) and SU(2), respectively. Thus, the XY-model is analogous to photodynamics, i.e., to a lattice Abelian gauge theory, and the four-component Heisenberg model is analogous to the Yang-Mills lattice theory.

These analogies are not accidental. Polyakov has shown by field-theory methods that in four-dimensional photodynamics a phase transition without ordering occurs, as in the two-dimensional XY-model, while in the two-dimensional Heisenberg model there are, at zero temperature, singularities associated with asymptotic freedom, as in the Yang-Mills theory.

In the framework of our approximation the analogy goes further and gives a relation between the critical indices, e.g.,

$$\Delta_{\text{gauge}}(2d)/\Delta_{\text{spin}}(d) = \nu_{\text{spin}}(d)/_{\text{gauge}}(2d) = 2. \tag{18}$$

3. THE ISING MODEL

We shall start from the Ising model, in which the recursion equation (14) can be investigated analytically to completion.

For the coefficients of the normalized correlation functional

$$w_L = 1 + SS'\,\mathrm{th}\,\beta(L) = e^{\beta(L)SS'}/\mathrm{ch}\,\beta(L) \tag{19}$$

the equation reduces to the following:

$$\mathrm{th}[\beta(\lambda L)\lambda^{1-d}] = \mathrm{th}^\lambda\beta(L), \quad \beta(a) = \beta. \tag{20}$$

Linearizing the equation near the fixed point β_c, we find the index $\Delta' = d - \nu^{-1}$ in (8):

$$\Delta' = (\ln\lambda)^{-1}\ln\left[\frac{\mathrm{sh}\,2\beta_c}{\mathrm{sh}\,2\beta_c\lambda^{1-d}}\right]. \tag{21}$$

The second index Δ can be found if we seek the solution in the form

$$w_L = \frac{1}{\mathrm{ch}\,\beta_c}\exp(\beta_c SS' + hL^{d-\Delta}(S+S')) \tag{22}$$

and linearize (14) in h. We then obtain

$$\Delta=1+2\beta_c(\lambda^{1-d}-1)/\ln\lambda. \quad (23)$$

The dependence of the transition point and critical indices on the model parameter λ indicates the inexactness of our approximation.

It is interesting that the indices do not change when λ is replaced by λ^{-1}, whereas the transition temperature acquires a vector λ^{d-1}. From this point of view, the choice $\lambda = 1$ is optimal (the apparent singularities at $\lambda = 1$ cancel in all the quantities). Numerical values of the parameters for $\lambda = 1$ for the two-dimensional model are given in the table.

The critical temperature coincides with the exact answer, the dimension Δ of the order parameter is out by 5% and the dimension $\Delta' = d - \nu^{-1}$ of the energy density is out by 25%. Of course, for the two-dimensional Ising model such a crude approximation is of no value. There exist calculations by Kadanoff, also based on the recursion approach, giving an accuracy of better than 1%. Unfortunately, it has not been possible to generalize these calculations—they use the discreteness of the group in an essential way.

Our equations for the Ising model should become exact for $d \to 1 + \epsilon$, when the transition temperature tends to zero. In this case we find

$$2\beta_c=\frac{\ln\lambda}{1-\lambda^{-\varepsilon}}+\frac{1-\lambda^2}{3(1-\lambda^{-\varepsilon})}e^{-4\beta_c}+O(e^{-6\beta_c}), \quad (24)$$

$$\Delta'=1+\frac{2}{3}\frac{\lambda^2-1}{\ln\lambda}e^{-4\beta_c}+O(e^{-6\beta_c}), \quad (25)$$

$$\Delta=\frac{1}{3}\frac{\lambda^2-1}{\ln\lambda}e^{-4\beta_c}+O(e^{-6\beta_c}). \quad (26)$$

The first terms of the ϵ-expansion ($2\beta_c = \epsilon^{-1}$, $\Delta' = 1$, $\Delta = 0$) do not depend on λ, and so it may be hoped that they are exact. We have not succeeded in finding a proof of this hypothesis. It is curious that the first terms of the ϵ-expansion give a fair approximation to the two-dimensional Ising model.

Concerning the gauge-model analog, the following is known[5].

For $d = 2$ it reduces to the one-dimensional Ising model and is described exactly by our equation. For $d = 3$ the free energy reduces to that of the three-dimensional Ising model with $\beta_c = 0.76$ and $\nu = 0.62$. Finally, the transition temperature is known for $d = 4$: $2\beta_c = \ln(1 + \sqrt{2})$. Our equations ((20) with the replacement (17) for $\lambda = 1$) give $\beta_c = 0.9$ and $\nu = 1.05$ for $d = 3$, and $2\beta_c = \ln(1 + \sqrt{2})$ and $\nu = 0.66$ for $d = 4$.

Summarizing, we can say that, by comparison with other recursion methods, our equation describes discrete groups rather crudely. The advantages of our equation will become apparent in the case of continuous groups, where the other methods are inapplicable.

4. CONTINUOUS SYMMETRY GROUPS

In systems with continuous symmetry the critical dimensionality $d_c = 2$. We shall seek a solution of Eq. (19) for $d = 2 + \epsilon$ in Gaussian form:

$$F_L(\cos\theta)\to\exp(A(L)-\beta(L)\theta^2/2) \quad (27)$$

with a low effective temperature $\beta^{-1}(L)$.

Integrating by the method of steepest descents and retaining terms $\sim\beta$ and ~ 1, we find a relation for $\beta(L)$:

$$\beta(\lambda L)=\lambda^{\varepsilon}\beta(L)+{}^1/_6(n-2)\lambda^{\varepsilon}(1-\lambda)+O(\beta^{-1}). \quad (28)$$

	$\lambda=1$	Onsager's solution
$2\beta_c$	$\ln(1+\sqrt{2})=0.881$	$\ln(1+\sqrt{2})$
Δ'	$\sqrt{2}\ln(1+\sqrt{2})=1.246$	1
Δ	$1-\ln(1+\sqrt{2})=0.119$	1/8

From this we find the transition temperature:

$$\beta_c=\frac{n-2}{6}\frac{1-\lambda}{\lambda^{-\varepsilon}-1}\to\frac{n-2}{6\varepsilon}\frac{\lambda-1}{\ln\lambda}+O(1) \quad (29)$$

and the critical index ν:

$$\nu=\varepsilon^{-1}+O(1). \quad (30)$$

The second critical index Δ corresponds to the anisotropic solution

$$F_L\to\exp(A-\beta_c\theta^2/2+hL^{d-\Delta}(S+S')(1+\alpha\theta^2)). \quad (31)$$

Linearizing Eq. (11) in the magnetic field h and integrating by the method of steepest descents, we obtain

$$\alpha=1/12, \quad (32)$$

$$2^{-\Delta}=1-(n-1)/12\beta_c+O(\beta_c^{-2}). \quad (33)$$

Substituting β_c for $\lambda = 2$ from (29) into this, we find the first term of the ϵ-expansion:

$$\Delta=\frac{n-1}{n-2}\frac{\varepsilon}{2}+O(\varepsilon^2). \quad (34)$$

The exact values of the linear terms in the ϵ-expansions for the indices have been found by Polyakov by the methods of chiral field theory.

Our results (30) and (34) coincide with the exact results. This is not surprising, inasmuch as the transition temperature vanishes linearly with ϵ, and our model becomes exact in the low-temperature limit. The proportionality coefficient $\epsilon\beta_c$ in (29) depends on λ, and differs from the exact value:

$$\varepsilon\beta_{c\,\text{exact}}\to(n-2)/2\pi. \quad (35)$$

For $\lambda = 1$ the difference will amount to 5%.

But what happens in two-dimensional models? The power temperature singularities as $\epsilon \to 0$ go over into exponential singularities, e.g.,

$$r_c\sim\left(1-\frac{\beta}{\beta_c}\right)^{-\nu}\to\exp\left(\frac{\beta\nu}{\beta_c}\right)\to\exp\left(\frac{2\pi\beta}{n-2}\right). \quad (36)$$

The power dependences on the spatial scale L go over into logarithmic dependences:

$$\beta(L)\to\beta-\frac{n-2}{2\pi}\ln\frac{L}{a}, \quad (37)$$

$$hL^{d-\Delta}\to hL^2\left[\frac{\beta(L)}{\beta}\right]^{(n-1)/2(n-2)} \quad (38)$$

It is not difficult to verify this by considering the recursion equations for $\epsilon = 0$ in the Gaussian approximation. The zero transition temperature of the two-dimensional Heisenberg model means that ordering is absent at all temperatures, in accordance with Hohenberg's theorem.

The XY-model, in which the corresponding group SO(2) is Abelian, represents a special case. For $\epsilon = 0$, $n = 2$, the transition temperature and critical index Δ in (35) and (34) remain indeterminate. The correlation length (36) is infinite at sufficiently low temperatures, when the Gaussian approximation (27) is applicable. The corrections to the Gaussian approximation are exponentially small, since they are connected with the edges of the range $|\varphi| < \pi$ of integration over the angles of the two-dimensional spin vector.

If we extend the integration to $|\varphi| < \infty$, the Gaussian form with any β will be a fixed point. Linearizing Eq. (11) for n = 2 near the degenerate Gaussian point

$$F_L=\exp\left(A-\frac{\beta}{2}\theta^2+\frac{\mathbf{h}(\mathbf{S}+\mathbf{S}')}{|\mathbf{S}+\mathbf{S}'|}L^{2-\Delta}\Psi(\theta)\right), \tag{39}$$

we arrive at the integral equation

$$2^{-\Delta}\Psi(\theta)=\sqrt{\frac{\beta}{\pi}}\int_{-\infty}^{+\infty}d\varphi\exp\left(-\beta\left(\varphi-\frac{\theta}{2}\right)^2\right)\cos\frac{\varphi-\theta}{2}\Psi(\varphi), \tag{40}$$

the solution of which can be found in the form of a double series in θ^2 and $T \equiv \beta^{-1}$:

$$\Psi=1-\frac{\theta^2}{24}\left(1+\frac{7T}{90}-\frac{\theta^2}{80}+\dots\right), \tag{41}$$

$$\Delta\ln 2=\frac{T}{12}-\frac{1}{6}\left(\frac{T}{12}\right)^2+\dots \tag{42}$$

Solving Eq. (11) numerically for n = 2 (without a magnetic field) shows that the initial function $F_0 = \exp(\beta\cos\varphi)$ goes over after several iterations to the Gaussian form $\exp(A - \tilde{\beta}\varphi^2/2)$ with $\tilde{\beta} = \beta + O(1)$. The Gaussian form is held for a very long time for $\beta \gtrsim 1$, and then moves away rapidly to F = 1, which corresponds to the disordered phase. For $\beta > \beta_c \approx 1.7$, the Gaussian form remains throughout the entire interval chosen (100 iterations). This means that for $\beta > \beta_c$ the correlation length is effectively infinite, i.e., the region $\beta > \beta_c$ is on a phase-transition line. This corresponds to the curve $T_c(\epsilon)$ (Fig. 4) obtained by solving Eq. (16) numerically for $\lambda = 2^{1/(d-1)}$, when this equation simplifies in the Fourier representation.

We recall that, according to the analogy between gauge and spin systems, all that has been said above can be carried over to four-dimensional photodynamics. These results agree with the general conclusions of Berezinskiĭ for the XY-model[4] and of Polyakov for photodynamics.

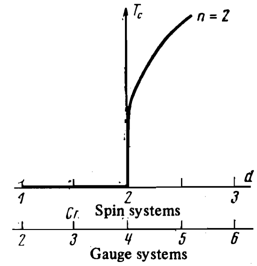

FIG. 4

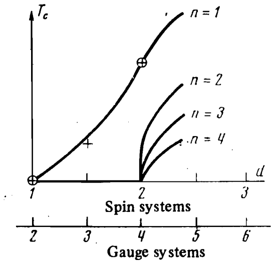

FIG. 5

FIG. 4. Qualitative dependence, found from the recursion equations, of the transition temperature on the dimensionality of space for an SO(2) spin system (XY-model) and a U(1) gauge system (photodynamics).

FIG. 5. Qualitative dependence, found from the recursion equations, of the transition temperature on the dimensionality of space for gauge and spin systems with SO(n) symmetry. The crosses correspond to exact results [5] for the discrete gauge group $SO(1) = Z_2$; the circles correspond to exact results for the Ising model.

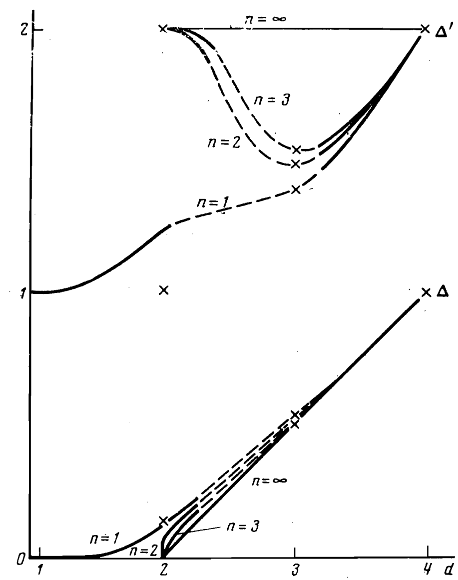

FIG. 6. Qualitative dependence of the dimensions of the spin (Δ) and energy density (Δ') on the dimensionality of space for the generalized SO(n) Heisenberg model. The behavior for d → 1 and d → 2 is found from the recursion equations, and for d → 4 from the ϵ-expansion; the crosses correspond to the known indices of the two- and three-dimensional models.

CONCLUSION

The results of this paper are summarized in Figs. 5 and 6. Figure 5 enables us to understand intuitively how phase transitions cease at the critical dimensionality of space. Figure 6 shows what happens to the spectrum of dimensions in this case. The behavior of the spectrum for $d = 4 - \epsilon$ is known from Wilson's work[3]. The upper curves correspond to the dimension $\Delta' = d - \nu^{-1}$ of the energy-density operator. The dimensions of composite operators are not given in this figure. These dimensions, for $d = 2 + \epsilon$, can also be found by linearizing the recursion equations. As regards practical applications of the model, the greatest interest evidently lies in the equation of state in two-dimensional magnets, which can be found by numerical solution of the recursion equations. In this case it is possible to take into account the spin-space anisotropy, which should suppress fluctuations of one or two components of the spin.

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158