Generation of gravitational waves by a strong electromagnetic wave scattered by electrons in a magnetic field

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(Submitted December 20, 1974)
Zh. Eksp. Teor. Fiz. 68, 1969–1974 (June 1975)

The problem of generation of harmonics of a gravitational wave (multiples of ω_0) by an electromagnetic wave of frequency ω_0 and amplitude E_0, when the latter is scattered by electrons in an external magnetic field, is solved in the linear approximation of general relativity theory. An interpretation of the effect is presented. It is shown that if the dimensions of the region of the magnetic field are larger than $R_{\text{char}} \sim cE_0/\omega_0 B$ at $\omega > e_0 B/mc$ (B is the magnetic field, m the electron mass) or $R_{\text{char}} \sim c/\omega_0$ at $\alpha\omega_0 < e_0 B/mc$ ($\alpha\mathcal{E}/mc^2$, $\mathcal{E}$ is the energy of the electron), then the main contribution to the effect comes from the process of linear (in the wave amplitudes) conversion of the electromagnetic waves emitted by the electron into gravitational waves when the former propagate in the magnetic field. In the special cases $B = 0$ and $E_0 = 0$ the results obtained here agree with the results of previous papers.

PACS numbers: 04.40., 04.30.

In connection with the preparation for some experiments on the detection of gravitational waves it seems interesting to investigate the problem of generation of gravitational waves by electromagnetic waves when the latter are scattered by electrons in a static magnetic field.

We consider an electron in a static homogeneous magnetic field $\mathbf{B}$. Let a strong circularly polarized electromagnetic wave be incident on the electron, such that the wave vector $\boldsymbol{\kappa}$ is directed along the vector $\mathbf{B}$. The motion of the electron in a coordinate system with the Oz axis along $\mathbf{B}$, for particular initial conditions, has the form:

$$x=\bar{R}\cos\omega_0 t,\quad y=g\bar{R}\sin\omega_0 t,\quad z=0;$$
$$\bar{R}=c\gamma/(\alpha\omega_0-g\omega_B). \tag{1}$$

The electron energy is related to the wave parameters in the following manner:

$$\mathscr{E}=mc^2\alpha,\quad \alpha^2=1+\frac{\gamma^2}{(1-g\omega_B/\alpha\omega_0)^2}. \tag{2}$$

Here $\gamma = e_0E_0/mc\omega_0$, $\omega_B = e_0B/mc$ and $g = \pm 1$ defines the direction of rotation of the electric field strength vector of the wave. We shall solve the problem neglecting radiation reaction (electromagnetic and gravitational) on the charge.

We first clarify the role of various mechanisms in the process of generation of gravitational waves (GW). The fundamental equation for the components Ψ^{ik} of the GW potentials is of the form (unperturbed galilean metric)

$$\Box\Psi^{ik}=\frac{16\pi G}{c^4}T^{ik},\qquad \Box=\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}, \tag{3}$$

i, k = 0, 1, 2, 3 with the condition $\partial\Psi^{ik}/\partial x^k = 0$. In Eq. (3) G is the gravitational constant, $T^{ik}(r, t)$ are the components of the energy-momentum tensor of the system: charge + magnetic field ($\mathbf{B}$) + plane electromagnetic wave of the field of the moving charge ($\mathbf{E_c}$, $\mathbf{H_c}$). In the sequel we consider only the spatial components $\Psi^{\bar\alpha\bar\beta}$ and correspondingly $T^{\bar\alpha\bar\beta}$[1]. The structure of $T^{\bar\alpha\bar\beta}$ is as follows:

$$T^{\bar\alpha\bar\beta}=T^{\bar\alpha\bar\beta}_{(p)}+T^{\bar\alpha\bar\beta}_{(f)}=T^{\bar\alpha\bar\beta}+\frac{1}{4\pi}\langle(E^{\bar\alpha}E_3^{\bar\beta}+H^{\bar\alpha}H_3^{\bar\beta}+B^{\bar\alpha}H_3^{\bar\beta}+B^{\bar\alpha}H^{\bar\beta})\rangle, \tag{4}$$

$\bar\alpha$, $\bar\beta$ = 1, 2, 3 and the brackets $\langle\ldots\rangle$ denote symmetrization of the expression in parentheses, $T^{\bar\alpha\bar\beta}_{(p)}$ are the components of the energy-momentum tensor of the particle and $T^{\bar\alpha\bar\beta}_{(f)}$ are the components of the energy momentum tensor of the fields, including that of the moving charge.

The values of $\Psi^{\bar\alpha\bar\beta}$ far from the charge are determined by the corresponding Fourier components $T^{\bar\alpha\bar\beta}(\mathbf{k})$ ($\mathbf{k}$ is the propagation vector of the GW). In its turn:

$$T^{\bar\alpha\bar\beta}_{(f)}(\mathbf{k})=\frac{1}{(2\pi)^3}\int f_1(\mathbf{k}-\mathbf{p})f_2(\mathbf{p})\,d\mathbf{p},$$

where $f_1(\mathbf{k}-\mathbf{p})$ are the Fourier transforms of a field of the wave or of the magnetic field and $f_2(\mathbf{p})$ are the Fourier transforms of the field strengths of the wave emitted by the charge moving under the influence of the electromagnetic fields.

It is easy to show that on account of the conservation laws the last term in parentheses of Eq. (4) does not contribute to the intensity of the gravitational radiation, i.e., there is no direct conversion of the electromagnetic wave (EW) into a GW in a magnetic field without participation of the electrons if the propagation vector $\boldsymbol{\kappa}$ of the EW is directed along $\mathbf{B}$. If the EW is strong, the Fourier components of the field of the electron contain high-order harmonics of the fundamental frequency ω_0, and as usual, each field ($\mathbf{E_C}$ or $\mathbf{H_C}$) consists of two parts:

$$\mathbf{E}_C=\mathbf{E}_N+\mathbf{E}_W,\quad \mathbf{H}_C=\mathbf{H}_N+\mathbf{H}_W, \tag{5}$$

where $\mathbf{E}_N$ and $\mathbf{H}_N$ are the parts of the fields depending on the electron velocity (non-wave, or near-zone fields) and $\mathbf{E}_W$ and $\mathbf{H}_W$ are the radiated fields of the electron (free fields in the wave zone).

We write out the explicit form of the Fourier expansions of the fields[5]:

$$\mathbf{E}_C^{\nu'}(p)=4\pi i\left[\frac{\mathrm{P}}{p^2-(\nu'\omega_0/c)^2}+i\pi\frac{\nu'}{|\nu'|}\delta\left(p^2-\left(\frac{\nu'\omega_0}{c}\right)^2\right)\right]\left(\frac{\nu'\omega_0}{c}\mathbf{j}^{\nu'}-\mathbf{p}\rho^{\nu'}\right),$$

$$\mathbf{H}_C^{\nu'}(p)=4\pi i\left[\frac{\mathrm{P}}{p^2-(\nu'\omega_0/c)^2}+i\pi\frac{\nu'}{|\nu'|}\delta\left(p^2-\left(\frac{\nu'\omega_0}{c}\right)^2\right)\right][\mathbf{p}\times\mathbf{j}^{\nu'}];$$

$$\mathbf{j}^{\nu'}=e_0\oint\frac{d\omega_0\tau}{2\pi}\mathbf{v}(\tau)\exp\{i\nu'\omega_0\tau-i\mathbf{p}\mathbf{r}_0(\tau)\},$$

$$\rho^{\nu'}=e_0\oint\frac{d\omega_0\tau}{2\pi}\exp\{i\nu'\omega_0\tau-i\mathbf{p}\mathbf{r}_0(\tau)\},$$

where P denotes principal value of the integral.

It can be seen from Eq. (4) that contributions to the intensity of the gravitational radiation come from terms proportional to the stresses of the fields **E** and **B** and the fields (5). Let us consider this question in more detail.

1. The contribution from the tensions of the fields (5). This term is proportional to

$$\langle (E^{\bar\alpha}E_C^{\bar\beta}+H^{\bar\alpha}H_C^{\bar\beta})\rangle$$

and consists of two parts. However, it is easy to show that a nonvanishing contribution comes only from terms proportional to

$$\langle (E^{\bar\alpha}E_N^{\bar\beta}+H^{\bar\alpha}H_N^{\bar\beta})\rangle,$$

whereas the contribution from the second part proportional to

$$\langle (E^{\bar\alpha}E_W^{\bar\beta}+H^{\bar\alpha}H_W^{\bar\beta})\rangle,$$

vanishes on account of the conservation laws. Thus one of the mechanisms for generation of GW is the process of nonlinear (in the amplitude E_0) conversion of EW into GW when the former is scattered on the near-zone field of the moving charge. It follows from the conservation laws that contributions to the ν-th harmonic of the GW come from the $\nu' = \nu + 1$ and $\nu' = \nu - 1$ harmonics of the near-zone fields.

2. The contributions from the stresses of the magnetic field **B** and from the fields of the charge. In this case both terms in (5) yield a nonzero contribution. However, the main effect here will be the effect of the linear (in the wave amplitude) conversion of the electromagnetic radiation of the electron (the field H_W) into gravitational radiation in the magnetic field [1].

Let us estimate the ratios of the two contributions 1 and 2. Let, for instance $\gamma > 1$, $\omega_0 > \omega_B$. The contributions will be identical for $E_0E_N \sim BH_W$ or $e\alpha E_0/R^2 \sim e_0\alpha\omega_0 B/Rc$. Hence

$$R_{\text{char}} \sim cE_0/\omega_0 B. \tag{6}$$

In the region $R < R_{\text{char}}$ the process of nonlinear conversion of the incident EW into a GW on the near-zone field dominates, and for $R > R_{\text{char}}$ the main contribution comes from the process of linear transformation of the "wave field" of the charge into GW in the magnetic field.

A quantitative calculation confirms the above analysis. We list only the final results. The intensity of the gravitational radiation is calculated by means of the Landau-Lifshitz formula [2] and is of the form [3]:

$$\frac{dI}{d\Omega}=\frac{G}{2\pi c^5}\sum_{\nu=1}^{\infty}(\nu\omega_0)^2\sum_{s=1,2}\left|T_{(s)}^{\nu}\left(\frac{\nu\omega_0}{c}\mathbf{n}\right)\right|^2. \tag{7}$$

Here ν is the order of the harmonic GW, $\mathbf{n} = \mathbf{R}_0/R_0$ is the direction of the observation point, $d\Omega = \sin\theta\, d\theta d\varphi$, θ and φ are the spherical angles of $\mathbf{n}$ (θ is counted from the direction of **B**) s = 1, 2 is the polarization index of the GW; s = 1 characterizes the polarization in the direction $2^{-1/2}(\mathbf{e}_\theta\mathbf{e}_\theta - \mathbf{e}_\varphi\mathbf{e}_\varphi)$, s = 2—in the direction $2^{-1/2}(\mathbf{e}_\theta\mathbf{e}_\varphi + \mathbf{e}_\varphi\mathbf{e}_\theta)$, where $\mathbf{e}_\theta$ and $\mathbf{e}_\varphi$ are the spherical unit vectors. Further

$$T_{(1)}^{\nu}=mc^2\alpha\left\{\frac{1-\beta^2\sin^2\theta+\cos^2\theta}{\sin^2\theta}J_\nu(x)-\beta^2\frac{1+\cos^2\theta}{x}J_\nu'(x)\right.$$
$$\left.-\frac{\omega_B}{\alpha\omega_0}\frac{\beta\sin\theta}{2\nu}J_\nu'(x)\right\}+\frac{e_0c\beta E_0}{\omega_0}\left\{\frac{1+\cos^2\theta}{x}J_\nu'(x)\right.$$
$$\left.+\left(1-\frac{\nu^2}{x^2}(1-\cos\theta)\right)J_\nu(x)\right\}+i(R\omega_B mc)\cos\theta J_\nu(x), \tag{8}$$

$$T_{(2)}^{\nu}=2mc^2\beta\,\mathrm{ctg}\,\theta\left\{J_\nu'(x)-\frac{1}{x}J_\nu(x)-g\frac{\omega_B}{\alpha\omega_0}\frac{\sin\theta}{4\nu\beta}J_\nu(x)\right\}+\frac{e_0cE_0}{2\omega_0}\cdot$$
$$\cdot\left(2\,\mathrm{ctg}\,\theta J_\nu(x)+xJ_\nu'(x)\frac{1-\cos\theta}{\sin\theta}\right)+i(R\omega_B mc)\beta\sin\theta J_\nu'(x), \tag{9}$$

where $x = g\nu\beta\sin\theta$, R is the distance traveled by the electromagnetic wave emitted by the charge and β equals

$$\beta=(\alpha^2-1)^{1/2}/\alpha=\gamma\omega_0/|\alpha\omega_0-g\omega_B|. \tag{10}$$

We note that for B = 0 we obtain from (8) and (9)

$$T_{(1)}^{\nu}=mc^2\alpha\,\mathrm{ctg}\frac{\theta}{2}\mathrm{ctg}\,\theta J_\nu(x),\quad T_{(2)}^{\nu}=mc^2\alpha\beta\,\mathrm{ctg}\frac{\theta}{2}J_\nu'(x), \tag{11}$$

$$\frac{dI}{d\Omega}=\frac{G}{2\pi c^5}(mc^2\alpha)^2\mathrm{ctg}^2\frac{\theta}{2}\sum_{\nu=1}^{\infty}(\nu\omega_0)^2(\beta^2J_\nu'^2(x)+\mathrm{ctg}^2\theta J_\nu^2(x)).$$

The term with $\nu = 1$ which gives the largest contribution for $\gamma \ll 1$ yields, after dividing (11) by the flux density of EW $I_0 = cE_0^2/4\pi$, the quasiclassical limit ($\hbar \to 0$) of Voronov's formula [4].

If $E_0 = 0$, but β is considered a constant independent of γ, one can obtain from (7)—(9) the results of Gal'tsov et al. [3] where the emission of gravitational waves in a magnetic field has been calculated. Expressing the quantity γ in terms of the electron energy α and the frequencies ω_0, ω_B according to Eq. (10) and substituting the expression for γ into (8), (9), we convince ourselves that the first terms in (8), (9) which describe the contributions from the mass stresses cancel against the corresponding terms in $T_{(f)}^{\bar\alpha\bar\beta}$.

If the magnetic field satisfies the condition $\omega_B < \alpha\omega_0$ but the wave is strong, then

$$\frac{dI}{d\Omega}=\frac{G}{2\pi c^5}(mc^2\alpha)^2\left(\mathrm{ctg}^2\frac{\theta}{2}+\frac{R^2\omega_B^2}{c^2\alpha^2}\sin^2\theta\right)\cdot \tag{12}$$
$$\cdot\sum_{\nu=1}^{\infty}(\nu\omega_0)^2(\beta^2J_\nu'^2(x)+\mathrm{ctg}^2\theta J_\nu^2(x)).$$

The intensity of gravitational radiation emitted into a solid angle $d\Omega$ is proportional to the intensity of the electromagnetic radiation emitted by the charge. It follows from (12) that for $\gamma \sim \alpha \gg 1$ the main contribution to the intensity of the gravitational radiation comes from angles $\theta = \pi/2 \pm \Delta\theta$ with $\Delta\theta \sim 1/\alpha$ and high harmonics $\nu \sim \alpha^3 \sim \gamma^3$.

The first term in (12) ($\sim\cot^2(\theta/2)$) describes the effect of nonlinear conversion of the incident EW into a GW when the former is scattered on the near-zone field of the charge and the second term describes the linear conversion of the wave field of the charge into GW in the magnetic field. Both contributions become comparable for $R \sim c\alpha/\omega_B \sim cE_0/\omega_0 B$, in agreement with (6).

In the other limiting case $\omega_B > \alpha\omega_0$ the main contribution to the intensity of the gravitational radiation comes from the term with $\gamma = 1$, corresponding to nonrelativistic motion of the electron, although the wave is strong ($\gamma > 1$). The quantities α and β for this case are of the order $\alpha \sim 1$, $\beta \sim \gamma\omega_0/\omega_B$ and

$$\frac{dI}{d\Omega}=\frac{G}{2\pi c^5}\frac{(mc^2)^2}{4}\gamma^2\omega_0^2(1+\cos^2\theta)\left[\mathrm{ctg}^2\frac{\theta}{2}+\frac{1}{4}\sin^2\theta\left(1+4R^2\frac{\omega_0^2}{c^2}\right)\right]. \tag{13}$$

Consequently, in a strong magnetic field ($\omega_B > \alpha\omega_0$) the contributions from the linear and from the nonlinear conversions of electromagnetic waves into gravitational waves become equal for $R_{\text{char}} \sim c/\alpha\omega_0$ independently of the ratio E_0/B.

All the results may be reformulated in terms of the effective cross section of the appropriate reaction; for this purpose one must divide the intensity formulas by the flux density of the incident EW:

$$d\sigma = \frac{1}{I_0}\frac{dI}{d\Omega} = \frac{4\pi}{cE_0^2}\frac{dI}{d\Omega}. \qquad (14)$$

For the case $\omega_B/\alpha\omega_0$ and $\gamma > 1$ we determine $d\sigma$:

$$d\sigma = 4\pi\frac{G}{c^3}\frac{\alpha^2}{\gamma^2}\frac{W_{el}}{\omega_0^2}\left(\operatorname{ctg}^2\frac{\theta}{2} + \frac{R^2\omega_B^2}{c^2\alpha^2}\sin^2\theta\right), \qquad (15)$$

where W_{el} is the intensity of the electromagnetic radiation of the charge in the fields under consideration[5]. Taking into account the fact that the main contribution to the intensity comes from angles $\theta = \pi/2 \pm 1/\alpha$ in (15), the integration over the angles and summation over ν yields the following reaction cross section:

$$\sigma = \frac{8\pi}{3}\frac{Ge_0^2}{c^4}\frac{(\alpha^2-1)\alpha^4}{\gamma^2}\left(1 + \frac{R^2\omega_B^2}{c^2\alpha^2}\right). \qquad (16)$$

In general, the integration over the angles in Eq. (15) at small γ and in Eq. (13) for arbitrary γ leads to a divergent result. This divergence is similar to the divergence of the Rutherford cross section for fast particles at small angles and is characteristic for fields of the Coulomb type[6].

For $\omega_B > \alpha\omega_0$ the differential cross section of the reaction has the form

$$d\sigma = \pi\frac{Ge_0^2}{c^4}(1+\cos^2\theta)\left[\operatorname{ctg}^2\frac{\theta}{2} + \frac{1}{4}\sin^2\theta\left(1 + 4R^2\frac{\omega_0^2}{c^2}\right)\right]. \qquad (17)$$

We note that without taking into account the second terms which describe the contribution of the magnetic field to the process under consideration, Eqs. (16), (17) have a different structure than in the analogous problem of scattering of EW without conversion into GW[7]. The reason for this difference is related to the fact that in the problem of pure scattering of EW the intensity of the scattered wave is proportional to $|\mathbf{j}_1\cdot\mathbf{E}|^2$ ($\mathbf{j}_1$ is the current induced by the incident EW) whereas in the case under consideration

$$dI/d\Omega \sim |\mathbf{E}\mathbf{E}_\mathbf{N}|^2$$

and does not depend on the velocity of the electron in the wave. Thus, for $\omega_B > \alpha\omega_0$ the current $j_1 \sim E\omega_0/\omega_B$ and consequently the scattering cross section of EW must contain the additional factor $(\omega_0/\omega_B)^2$ compared to our Eq. (17). Correspondingly, for a similar reason in Eq. (16) in the approximation $\beta \to 1$ there appears the additional coefficient α^2. The ratio between the cross sections of pure scattering and of the process considered here in the case $\omega_B < \alpha\omega_0$ and $\gamma \gg 1$ equals

$$\frac{\sigma_{gr}}{\sigma_{em}} = \frac{Gm^2}{e_0^2}\alpha^2\left(1 + \frac{R^2\omega_B^2}{c^2\alpha^2}\right). \qquad (18)$$

The differential and total cross sections of the reactions considered here contain the resonance point. In order to determine the cross sections for arbitrary relations between $\alpha\omega_0$ and ω_B including the resonance point, it is necessary to take into account the radiation reaction of the electromagnetic wave scattered by the electron.

In the quoted work of Zel'dovich and Illarionov[7] such a problem has been solved for the case of pure scattering of EW on plasma electrons. If one neglects the contribution of the longitudinal (along $\boldsymbol{\kappa}$) electric field of the plasma to the process under consideration, then the parameters which enter the final results should be considered equal to[7]

$$\frac{\omega_B}{\omega_0} = \alpha\left[1 \pm \left(\frac{\gamma^2}{\alpha^2-1} - \frac{\alpha^6}{k^2}\right)^{1/2}\right], \quad \alpha = \left[\frac{1+(1+4s\gamma^2)^{1/2}}{2}\right]^{1/2}$$

where

$$k = \frac{3\lambda}{2r_0}, \quad r_0 = \frac{e_0^2}{mc^2}, \quad \lambda = \frac{c}{\omega_0}, \quad s = \frac{\alpha^2(\alpha^2-1)}{\gamma^2},$$

and the plus sign is taken to the right of the resonance, the minus sign to its left. Then the formulas obtained in that way will have the interpretation of the corresponding reaction cross sections for given α, γ, ω_B, including the resonance point. The admissibility of such a generalization is not sufficiently reliable in our problem, since the magnitude of the contribution from the stresses of the plasma electric field may not be small on account of the fact that this quantity will depend essentially on the size of the region occupied by the plasma field. We also note that the value of that field has been determined by Zel'dovich and Illarionov[7].

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Translated by M. E. Mayer
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