The collapse of electromagnetic waves in a plasma

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The effect of longwave electromagnetic perturbations on the collapse of Langmuir waves is studied.[1] It is shown that during the initial stage both Langmuir and electromagnetic waves collapse in an isotropic plasma. In distinction from the isotropic plasma, in a plasma with a weak magnetic field there is a possibility for the collapse of long-wave electromagnetic waves. The collapse leads to the formation of filaments collapsing in a direction perpendicular to the magnetic field.

INTRODUCTION

The fundamental nonlinear mechanisms of weak turbulence of high-frequency (HF) waves in a plasma—both electromagnetic waves and Langmuir waves—lead to condensation of the turbulence spectra in the long-wave region. This raises the question of the mechanism of energy dissipation for long-wave HF waves. One of the important mechanisms of energy dissipation for HF waves is the collapse of waves, first considered by Zakharov[1] for Langmuir turbulence. This mechanism consists in the formation of regions of lowered density-cavities—as a result of the action of the HF waves on the plasma. After a finite time the cavities collapse, leading to a dissipation of the energy of the HF waves. This phenomenon can be considered as a nonlinear stage of instability of a "cold" Langmuir gas, discovered by Vedenov and Rudakov[2] (cf. also[3]).

The purpose of the present paper is to clarify the influence of electromagnetic perturbations on the collapse, both in an isotropic plasma and in a plasma with a weak magnetic field. We consider the case when the plasma can be regarded as having no inertia—static. It is shown that as a result of the development of instability of the HF waves with small k (Sec. 2) the formation of a collapse occurs in an isotropic plasma (Sec. 3). Both the potential (Langmuir) and nonpotential (longwave) electromagnetic waves are subject to collapse. In distinction from the isotropic plasma, where the influence of nonpotential perturbations on potential ones is important only in the static region, it is shown in the present paper (Sec. 2) that in a plasma with a magnetic field the influence of the nonpotential waves is always important. One should note that the longwave HF oscillations can no longer be subdivided into potential and nonpotential ones even in a sufficiently weak magnetic field ($\beta = 8\pi nT/H_0^2 \gtrsim 1$), (cf., e.g.,[4]). For such waves in a plasma with $\beta \gtrsim 1$ one can, as a rule, neglect the thermal pressure of the electrons. This allows one to show that as a result of the development of the instability of the "cold" photon gas (Sec. 4) a collapse is produced in a plasma with $\beta \gtrsim 1$. The development of the collapse leads to the formation of filaments collapsing in a direction perpendicular to the magnetic field.

1. THE FUNDAMENTAL EQUATIONS

In an isotropic plasma there exist two types of HF oscillations, electromagnetic and Langmuir waves, having respectively the dispersion laws:

$$\omega_1=(\omega_p{}^2+k^2c^2)^{1/2},\quad \omega_2=\omega_p(1+{}^3/_2k^2r_D{}^2).$$

The latter exist only in the region $kr_D < 1$, where their nonlinear damping (Landau damping by electrons and collision damping) is small. These oscillations involve mainly the electrons, whose motion is described by hydrodynamic equations ($\omega/k > v_{Te}$). The basic mechanism of interaction of the HF waves is their interaction with low-frequency motions of the plasma. One can qualitatively understand the structure of this interaction. As the HF waves propagate the average characteristics of the plasma vary slowly. This leads to a change of the frequency of the HF waves, by the amount $\Delta\omega \approx (\omega_p^2/2\omega)\delta\tilde{n}/n_0$ on account of a change of the density, and $\Delta\omega \sim kv_d$ on account of the Doppler effect, where v_d is the drift velocity of the electron gas. The main nonlinear mechanism, it turns out, is the scattering of HF waves on low-frequency density fluctuations $\delta\tilde{n}$; the Doppler effect which leads to the dying out of oscillations turns out to be small. This allows one to go over to a simplified description, based on averaging with respect to the "fast time" $1/\omega_p$. Thus, the HF oscillations are described by the linearized system of hydrodynamic equations for the electron fluid and the Maxwell equations

$$\frac{\partial \mathbf{v}_e}{\partial t}=-\frac{e}{m}\mathbf{E}-3v_{Te}^2\nabla\frac{\delta n_e}{n_0},$$
$$\operatorname{div}\mathbf{E}=-4\pi e\delta n_e,\quad \operatorname{rot}\mathbf{E}=-\frac{1}{c}\frac{\partial H}{\partial t},\qquad (1)$$
$$\operatorname{rot}\mathbf{H}=-\frac{4\pi e}{c}(n_0+\delta\tilde{n})\mathbf{v}_e+\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}.$$

Here v_e and δn_e are the velocity and density variation of the electrons in the HF oscillations.

The equations (1) do not take into account the electronic nonlinearities due to Doppler effect on electrons and to scattering on forced fluctuations of the density and velocity with frequencies $\sim 2\omega_p$. The characteristic time scales τ of these processes are:

$$\tau^{-1}\sim\omega_p(kr_D)^2W/nT,$$

where W is the energy density of the HF waves. We assume henceforth that only faster processes are considered.

Let us consider longwave oscillations ($kc \ll \omega_p$), when the frequencies of all the HF waves are close to the plasma frequency ω_p. Following Zakharov[1], we introduce the slowly varying quantity $\mathbf{E}_1$ (denoted in the sequel by $\mathbf{E}$) by means of the formula

$$\mathbf{E}={}^1/_2(\mathbf{E}_1e^{-i\omega_pt}+\mathbf{E}_1{}^*e^{i\omega_pt}).$$

Assuming $\delta\tilde{n}$ to be time-independent and neglecting the second derivative of $\mathbf{E}$ with respect to t, we obtain

$$i\mathbf{E}_t-\frac{c^2}{2\omega_p}\operatorname{rot}\operatorname{rot}\mathbf{E}+\frac{3}{2}\omega_pr_D{}^2\nabla\operatorname{div}\mathbf{E}=\omega_p\frac{\delta\tilde{n}}{2n_0}\mathbf{E}.\qquad (2)$$

To close Eq. (2), we take into account the influence of the HF waves on the average characteristics of the plasma. As is well known, the HF waves exert an HF force $F = -\nabla u$ (where $u = e^2|E|^2/4m\omega_p^2$) on the "drift" electrons. As a result of this action there occurs a charge separation in slow motions, i.e., there appears a polarization potential $\tilde{\rho}$ that causes the action of the HF waves to be transmitted also to the ions. From the kinetic equations for slow motions[5]

$$\frac{\partial f_e}{\partial t}+(\mathbf{v}\nabla)f_e+\frac{1}{m}\nabla(e\tilde{\varphi}-u)\frac{\partial f_0^e}{\partial \mathbf{v}}=0,$$
$$\frac{\partial f_i}{\partial t}+(\mathbf{v}\nabla)f_i-\frac{1}{M}\nabla e\tilde{\varphi}\frac{\partial f_0^i}{\partial \mathbf{v}}=0, \quad (3)$$
$$\Delta\tilde{\varphi}=-4e\int(f_i-f_e)\,d\mathbf{v}$$

we then determine the response

$$\delta\tilde{n}=\int f_e\,d\mathbf{v}$$

to the HF force. As a result we obtain

$$\delta\tilde{n}_{\varkappa\Omega}=G_{\varkappa\Omega}u_{\varkappa\Omega},$$
$$G_{\varkappa\Omega}=-\frac{\varkappa^2}{4\pi e^2}\frac{|\varepsilon_i+1|^2\varepsilon_e+|\varepsilon_e|^2(\varepsilon_i+1)}{|\varepsilon|^2},$$
$$\varepsilon_\alpha=\frac{4\pi e^2}{m_\alpha\varkappa^2}\int\frac{\varkappa\partial f_0^\alpha/\partial\mathbf{v}}{\Omega-\varkappa\mathbf{v}}d\mathbf{v},\quad \varepsilon=\varepsilon_e+\varepsilon_i+1.$$

The basic physical information on the interaction of HF waves is determined by the structure of the Green's function G. In the simplest case (in the static limit $\Omega/\kappa < v_{Ti}$) G is a constant

$$G=-n_0/(T_e+T_i),$$

and the system (2)—(3) reduces to a single equation for E:

$$i\mathbf{E}_t-\frac{c^2}{2\omega_p}\operatorname{rot}\operatorname{rot}\mathbf{E}+\frac{3}{2}\omega_p r_D^2\nabla\operatorname{div}\mathbf{E}+\frac{e^2|E|^2\mathbf{E}}{8m\omega_p(T_e+T_i)}=0. \quad (4)$$

In particular, this equation leads to Eq. (1.2) of[1] for $\mathbf{E} = -\nabla\psi$, an equation that describes the Langmuir oscillations. However, as is easily seen, the Langmuir waves also generate nonpotential oscillations, namely electromagnetic waves.

In the other, hydrodynamic limit ($\Omega/\kappa > v_{Ti}$) we have

$$G=\frac{n_0}{M}\frac{\varkappa^2}{\Omega^2-\varkappa^2c_s^2}, \quad (5)$$

where $c_s = (T_e/M)^{1/2}$ is the speed of ionic sound.

We note that in distinction from the static approximation, where the nonlinearity is inertia free, in the hydrodynamic stage the inertia of the plasma is essential.

2. THE INSTABILITY OF A MONOCHROMATIC HF WAVE

We consider the problem of stability of a Langmuir wave of finite amplitude. From the fact of instability or stability of the monochromatic wave one can deduce conclusions about the behavior of sufficiently wide wave packets. In particular, from the fact that a monochromatic wave with k = 0 is unstable one can predict the evolution of the isotropic intensity spectrum.

Before discussing the stability problem we carry out some transformations. We carry out in (2) and (3) a Fourier transformation in the coordinates and change to the new variables a_k (cf.[6])

$$\mathbf{E}_k=i(2\pi\omega_p)^{1/2}\sum_\lambda \mathbf{s}_{k\lambda}a_{k\lambda},$$

where $s_{k\lambda}$ are the polarization unit vectors for the HF waves. For electromagnetic waves, the vectors $s_{k\lambda}$ satisfy the conditions

$$\mathbf{s}_{k1}=\mathbf{s}_{k2}^*,\quad \mathbf{s}_{k\lambda}\mathbf{s}_{k\lambda_1}^*=\delta_{\lambda\lambda_1},\quad \mathbf{k}\mathbf{s}_{k\lambda}=0,$$

and $s_{k3} = \mathbf{k}/k$ for the Langmuir waves. In the new variables Eq. (4) is of the form

$$\frac{\partial a_{k\lambda}}{\partial t}+i\Omega_{k\lambda}a_{k\lambda}=-i\frac{\omega_p}{2n_0^2}\sum_{\lambda_1}\int(\mathbf{s}_{k\lambda}\mathbf{s}_{k_1\lambda_1}^*)\,\delta n_\varkappa a_{k_1\lambda_1}\delta_{k-k_1-\varkappa}\,dk_1\,d\varkappa. \quad (6)$$

In these equations the eigenfrequencies are

$$\Omega_{k\,1,2}=\Omega_k=\frac{1}{2}\frac{k^2c^2}{\omega_p},\quad \Omega_{k3}=\omega_k=\frac{3}{2}\omega_p k^2 r_D^2$$

Equations (3) and (6) have the exact solution

$$a_{k\lambda}=\frac{A\delta_{\lambda3}}{\omega_p^{1/2}}\delta_{k-k_0}\exp\{-i\omega_{k_0}t\},\quad f_i=f_e=0,\quad \tilde{\varphi}=0,$$

representing a monochromatic Langmuir wave of finite amplitude. Linearizing Eqs. (3) and (6) on the background of this solution we assume for the disturbances

$$\delta a_{k\lambda}\sim\exp\{-i(\Omega+\omega_{k_0})t\}\,\delta_{k-k_0-\varkappa},$$
$$\delta a_{k\lambda}^*\sim\exp\{-i(\Omega-\omega_{k_0})t\}\,\delta_{k-k_0+\varkappa}.$$

Then the following dispersion relation results for Ω:

$$1+G_{\varkappa\Omega}\frac{\omega_p}{4}\frac{W}{n_0^2}\left\{\frac{\cos^2\theta_+}{-\Omega+\omega_{k_0+\varkappa}-\omega_{k_0}}+\frac{\cos^2\theta_-}{\Omega+\omega_{k_0-\varkappa}-\omega_{k_0}}+\frac{\sin^2\theta_+}{-\Omega+\Omega_{k_0+\varkappa}-\omega_{k_0}}+\frac{\sin^2\theta_-}{\Omega+\Omega_{k_0-\varkappa}-\omega_{k_0}}\right\}=0, \quad (7)$$

where

$$\cos^2\theta_\pm=\frac{(\mathbf{k}_0,\mathbf{k}_0\pm\varkappa)^2}{k_0^2|k_0\pm\varkappa|^2},\quad W=|A|^2.$$

Equation (7) is a natural generalization of Eq. (2.1) in Zakharov's paper[1], when electromagnetic disturbances are taken into account.

It follows from this equation that the Langmuir wave is unstable with respect to the excitation of both Langmuir waves and electromagnetic waves. In the simplest cases these instabilities reduce to decay instabilities of the first[7] and second[8] orders, for which,

$$\omega_{k_0}=\Omega_{k_0-\varkappa,\lambda}+\Omega_s,\quad \Omega_s=\varkappa c_s$$

(this instability exists only in a nonisothermal plasma, $T_e \gg T_i$) and

$$2\omega_{k_0}=\Omega_{k_0-\varkappa,\lambda_1}+\Omega_{k_0+\varkappa,\lambda_2},$$

respectively, and for large amplitudes ($W/nT > n/M$) it leads to a modified decay instability[9], for which $\omega_{k_0} = \Omega_{k_0-\kappa,\lambda}$.

However, a complete investigation of this equation goes beyond the scope of this paper. We note only that similar equations have been treated earlier in[1,9]. Here we limit our attention to the case which is most important for the sequel, namely the stability of a wave with small $k_0 \ll \kappa$; this simplifies considerably the dispersion law (7):

$$1-\frac{\omega_p}{2}\frac{W}{n_0^2}G_{\varkappa\Omega}\left\{\frac{\omega_\varkappa\cos^2\theta}{\Omega^2-\omega_\varkappa^2}+\frac{\Omega_\varkappa\sin^2\theta}{\Omega^2-\Omega_\varkappa^2}\right\}=0. \quad (8)$$

In the static limit this equation reduces to a biquadratic one

$$\Omega^4-\left[\Omega_\varkappa^2+\omega_\varkappa^2-\omega_p\frac{W}{2nT}(\omega_\varkappa\cos^2\theta+\Omega_\varkappa\sin^2\theta)\right]\Omega^2+\Omega_\varkappa^2\omega_\varkappa^2-\frac{\omega_p}{2}\frac{W}{nT}\omega_\varkappa\Omega_\varkappa(\Omega_\varkappa\cos^2\theta+\omega_\varkappa\sin^2\theta)=0.$$

From this equation it follows that instability sets in at

$$\omega_\varkappa\Omega_\varkappa < \frac{\omega_p}{2}\frac{W}{nT}(\Omega_\varkappa\cos^2\theta+\omega_\varkappa\sin^2\theta).$$

In particular, for $\cos^2\theta = 1$ we obtain the condition $W/nT > 3(\kappa r_D)^2$ (cf.[1]) and for $\sin^2\theta = 1$ the condition $W/nT > k^2c^2/\omega_p^2$. The maximal instability increment is obtained for $\theta = 0, \pi$ and $\theta = \pi/2$ (cf.[1]):

$$\gamma_{max}=\omega_p W/4nT. \tag{9}$$

At the maximum with $\cos^2\theta = 1$, a Langmuir wave with $(\kappa r_D)^2 = W/6nT$ is excited and for $\sin^2\theta = 1$, an electromagnetic wave with $(\kappa c/\omega_p)^2 = W/2nT$.

This analysis yields easily a criterion for the static approximation. For this it is necessary that $\gamma_{max}/\kappa_{max} < v_{Ti}$, hence $W/nT > T/Mc^2$ for conversion into an electromagnetic wave and $W/nT < m/M$ for conversion into a Langmuir wave. The first condition is more stringent. For intensities $W/nT > T/Mc^2$ the static approximation is violated for the conversion into an electromagnetic wave. For this process the hydrodynamic description is already valid.

Since the instability regions for conversion into electromagnetic and Langmuir waves are separated in k-space, these instabilities can be considered separately. We note that conversion into electromagnetic waves in Eq. (8) corresponds formally to $\cos^2\theta = 0$, and conversion into Langmuir waves corresponds to $\sin^2\theta = 0$. Since the latter processes are sufficiently well studied in[1], we consider further in the hydrodynamic limit only the instability that leads to conversion into electromagnetic waves. This instability corresponds to a dispersion relation ($k_0 \ll \kappa$):

$$(\Omega^2-\varkappa^2c_s^2)(\Omega^2-\Omega_\varkappa^2)=\frac{1}{2}\Omega_\varkappa\varkappa^2c_s^2\omega_p\frac{W}{nT}\sin^2\theta.$$

It is easy to see that the instability occurs for

$$(\varkappa c/\omega_p)^2<W/nT,$$

and the maximal increment

$$\gamma_{max}^2\approx\omega_{pi}^2\frac{W}{n_0mc^2} \tag{10}$$

is attained for $\sin^2\theta = 1$

$$\left(\frac{\varkappa c}{\omega_p}\right)^4\approx 2\frac{m}{M}\frac{W}{n_0mc^2}.$$

The hydrodynamic stage of this instability holds up to intensities

$$\frac{W}{nT}<\frac{M}{m}\frac{T}{mc^2}$$

(for thermonuclear temperatures this parameter is larger than unity).

Similarly one can consider the problem of stability of an electromagnetic wave. It is clear that the instability increments of such a wave with $k_0 \approx 0$ will coincide with (9) and (10).

It is known[1] that the hydrodynamic stage for the conversion into a Langmuir wave occurs at $W/nT > m/M$. Thus, there exists an intermediate stage of instability, when a "static" conversion occurs into a Langmuir wave and a "hydrodynamic" conversion occurs into an electromagnetic wave. It is easy to see that at this stage one may neglect the influence of the electromagnetic waves on the Langmuir waves. The evolution of the Langmuir waves is described by the equation[1]:

$$\operatorname{div}\left[i\nabla\psi_t+\frac{3}{2}\omega_p r_D^2\Delta\nabla\psi+\frac{\omega_p|\nabla\psi|^2}{32\pi n_0(T_e+T_i)}\nabla\psi\right]=0. \tag{11}$$

The virtual electromagnetic field arising in this case is determined by the condition

$$\frac{c^2}{2\omega_p}\operatorname{rot}\operatorname{rot}\operatorname{rot}\mathbf{E}=-\operatorname{rot}\left\{\frac{\omega_p|\nabla\psi|^2}{32\pi n_0(T_e+T_i)}\nabla\psi\right\},$$

which allows one to integrate Eq. (11):

$$i\nabla\psi_t+\frac{3}{2}\omega_p r_D^2\Delta\nabla\psi+\frac{c^2}{2\omega_p}\operatorname{rot}\operatorname{rot}\mathbf{E}+\frac{\omega_p|\nabla\psi|^2}{32\pi n_0(T_e+T_i)}\nabla\psi=0.$$

It should be said that in the hydrodynamic stage of conversion into a Langmuir wave one can also neglect the influence of the electromagnetic waves.

We now consider an arbitrary distribution of electromagnetic and Langmuir waves. We shall assume that the amplitudes of the waves are sufficiently small, so that the static description (4) applies. Assume that the characteristic scale of variation of $\mathbf{E}$ in Langmuir oscillations is l and in electromagnetic oscillation is L. Then, making use of the results of this section, one can assume that the distribution is unstable for $W/nT > r_D^2/l^2$ and $W/nT > c^2/L^2\omega_p^2$.

3. THE COLLAPSE OF ELECTROMAGNETIC WAVES

We consider the problem of the nonlinear stage of instability of the electromagnetic and Langmuir waves in the static limit (4). We assume that the distribution of electric fields is localized in space, so that $\mathbf{E} \to 0$ for $|\mathbf{r}| \to \infty$.

Introducing the variables

$$t'=t\omega_p,\quad \mathbf{r}'=\sqrt{\frac{2}{3}}\frac{\mathbf{r}}{r_D},\quad \mathbf{E}'=\frac{\mathbf{E}}{4[2\pi n_0(T_e+T_i)]^{1/2}},$$

we rewrite Eq. (4) in dimensionless form (accents have been omitted)

$$i\mathbf{E}_t-\beta\operatorname{rot}\operatorname{rot}\mathbf{E}+\nabla\operatorname{div}\mathbf{E}+|\mathbf{E}|^2\mathbf{E}=0, \tag{12}$$

where $\beta = c^2/v_{Te}^2$.

This equation can also be written in the form

$$\mathbf{E}_t=-i\delta\mathscr{H}/\delta\mathbf{E}^*,$$

where the Hamiltonian is

$$\mathscr{H}=\int\{\beta|\operatorname{rot}\mathbf{E}|^2+|\operatorname{div}\mathbf{E}|^2-|\mathbf{E}|^4/2\}d\mathbf{r}.$$

This form immediately implies the conservation of $\mathscr{H}$. In addition to the Hamiltonian, Eq. (12) has other constants of the motion. Gauge invariance implies the conservation of the number of quasiparticles:

$$\mathscr{N}=\int|\mathbf{E}|^2\,d\mathbf{r},$$

and translation invariance implies conservation of the momentum

$$\mathbf{P}=\int\mathbf{p}\,d\mathbf{r},$$

where the momentum density is

$$p_e=\frac{i}{2}(E_k^*\nabla_e E_k-E_k\nabla_e E_k^*),$$

Finally, rotation invariance leads to conservation of angular momentum

$$\mathbf{M}=\int\{[\mathbf{rp}]+i[\mathbf{EE}^*]\}d\mathbf{r}.$$

The presence of an additional symmetry leads, as is well known, to the appearance of additional constants of the motion. Thus, it is easy to see that for a spherically symmetric distribution the following quantities are conserved

$$\int |E_r|^2 d\mathbf{r} \text{ and } \int |E_\Omega|^2 d\mathbf{r},$$

where E_r and E_Ω are the radial and angular components of the electric field **E**. For a spherically symmetric distribution we introduce the quantity

$$I=\int r^2\left(|E_r|^2+\frac{1}{\beta}|E_\Omega|^2\right)d\mathbf{r}.$$

From (12) follows directly the relation (cf.[1,10])

$$\frac{\partial^2 I}{\partial t^2}=8\mathscr{H}-2\int |\mathbf{E}|^4 d\mathbf{r}<8\mathscr{H}.$$

Integrating this inequality twice we obtain

$$I<4\mathscr{H}t^2+C_1t+C_2, \qquad (13)$$

where C_1 and C_2 are constants. If $\mathscr{H}<0$ the positivity of I implies that the inequality (13) is valid only for some small values of t. For large t the inequality is not valid. This means that the solution of the Cauchy problem with $\mathscr{H}<0$ exists only over a finite time and must lead to the formation of a singularity. This phenomenon, which is analogous to the self-focusing of light (formation of caustics) has received the name of collapse.

As a result of the development of the collapse the amplitude of the electric field increases, which leads to a violation of the static approximation. The subsequent evolution is already described in the framework of Eq. (11). The collapse of electromagnetic waves goes then over into the collapse of Langmuir waves only[1]. We note that the requirement $\mathscr{H}<0$ coincides with the instability condition (9). Thus, the nonlinear stage of instability (9) leads to the formation of collapse.

4. THE INFLUENCE OF AN EXTERNAL MAGNETIC FIELD ON THE COLLAPSE

We now pose the problem of the influence of an external magnetic field H_0 on the collapse. It is clear that in the region $\omega_p \gg \omega_H$ ($\omega_H = eH_0/mc$) the influence of the magnetic field reduces only to a change of the dispersion law of the HF waves, the structure of the interactions remaining the same. In the sequel we impose the following condition on the magnitude of the magnetic field

$$\beta=8\pi n_0T/H_0^2\gg 1.$$

This allows us to restrict ourselves in Eq. (2) to the linear approximation in ω_H/ω_p:

$$i\mathbf{E}_t+\frac{i}{2}[\omega_H\mathbf{E}]-\frac{c^2}{2\omega_p}\,\mathrm{rot\,rot}\,\mathbf{E}+\frac{3v_{Te}^2}{2\omega_p}\nabla\,\mathrm{div}\,\mathbf{E}=\omega_p\frac{\delta n}{2n_0}\mathbf{E}. \qquad (14)$$

It is convenient to go over in (14) to the new variables $\mathbf{E}\rightarrow \mathbf{E}e^{i\omega_H t/2}$. This leads to the appearance of the additional term $-\frac{1}{2}\omega_H\mathbf{E}$ in the equation.

Equation (14) describes three types of oscillations, with the dispersion laws determined by the dispersion equation (cf.[4])

$$\left(\tilde\Omega-\frac{k^2c^2}{2\omega_p}\right)^2\left(\tilde\Omega-\frac{3}{2}\omega_pk^2r_D^2\right)=\frac{1}{4}\omega_H^2\left(\tilde\Omega-\frac{c^2}{2\omega_p}k_\perp^2-\frac{3}{2}\omega_pk_z^2r_D^2\right);$$
$$\tilde\Omega=\Omega-\omega_H/2. \qquad (15)$$

In two important special cases the roots of Eq. (15) are determined for $k_\perp = 0$:

$$\Omega_{1,2}=\pm\frac{\omega_H}{2}+\frac{k^2c^2}{2\omega_p}+\frac{\omega_H}{2},\quad \Omega_3=\omega_p\frac{3}{2}k^2r_D^2+\frac{\omega_H}{2}$$

and for $k_z = 0$:

$$\Omega_1=\frac{k^2c^2}{2\omega_p}+\frac{\omega_H}{2},$$

$$\Omega_{2,3}=\frac{1}{2}\left\{\omega_H+\frac{k^2}{2\omega_p}(c^2+3v_{Te}^2)\pm\left[\omega_H^2+\frac{k^4}{2\omega_p^2}(c^2-3v_{Te}^2)^2\right]^{1/2}\right\}.$$

We note that in the latter equation one may neglect the quantity related to thermal pressure[1)] in the region $3k^2v_{Te}^2/\omega_p < \omega_H$.

Further, we go over in (14), as before, to the variables a_k, by means of the formula

$$\mathbf{E}_k=i(2\pi\omega_p)^{1/2}\sum_\lambda \mathbf{s}_{k\lambda}a_{k\lambda},$$

where $\mathbf{s}_{k\lambda}$ are unit polarization vectors determined from the equation (cf.[4])

$$\hat A\mathbf{s}_{k\lambda}=(\Omega_{k\lambda}-\omega_H/2)\mathbf{s}_{k\lambda},$$

$$\hat A=\begin{pmatrix}\frac{c^2k_z^2+3v_{Te}^2k_x^2}{2\omega_p} & \frac{i}{2}\omega_H & \frac{3v_{Te}^2-c^2}{2\omega_p}k_xk_z\\ -\frac{i}{2}\omega_H & \frac{c^2}{2\omega_p}k^2 & 0\\ \frac{3v_{Te}^2-c^2}{2\omega_p}k_xk_z & 0 & \frac{c^2k_x^2+3v_{Te}^2k_z^2}{2\omega_p}\end{pmatrix}.$$

Here the matrix $\hat A$ is written in a coordinate base with the z axis along the magnetic field, and the vector **k** is placed in the (x, z) plane.

Obviously, the polarization vectors satisfy the orthonormality condition

$$(\mathbf{s}_{\lambda_1}\mathbf{s}_{\lambda_2}^*)=\delta_{\lambda_1\lambda_2}. \qquad (16)$$

In the new variables a_k Eq. (14) has the previous form (6). We pose the problem of stability for a monochromatic HF wave with $\Omega_k = 0$ for $k = 0$. An investigation of the stability of the wave reduces, as before, to the solution of the dispersion equation

$$1+\frac{\omega_p}{4}\frac{W}{n_0^2}G_{\varkappa\Omega}\sum_\lambda\left\{\frac{|(\mathbf{s}_{k_0}^*\mathbf{s}_{k_0+\varkappa,\lambda})|^2}{-\Omega+\Omega_{k_0+\varkappa,\lambda}-\Omega_{k_0}}+\frac{|(\mathbf{s}_{k_0}^*\mathbf{s}_{k_0-\varkappa,\lambda})|^2}{\Omega+\Omega_{k_0-\varkappa,\lambda}-\Omega_{k_0}}\right\}=0. \qquad (17)$$

Here Ω_{k_0} and $\mathbf{s}_{k_0}$ are the frequency and polarization vector of the monochromatic wave.

In the limit of large κ ($\kappa \gg k_0$, $(\kappa r_D)\gg(\omega_H/\omega_p)^{1/2}$) this equation goes over into (8). This implies, in particular, that if the instability limit $W/nT\sim(kr_D)^2$, together with the maximum of the increment (9), is in this region, one may neglect the influence of the magnetic field in the sequel, since the development of the collapse decreases the characteristic scale of variation of the field, corresponding to an increase of the effective k. We therefore consider the intermediate region $k_0 \ll \kappa \ll r_D^{-1}(\omega_H/\omega_p)^{1/2}$.

Making use of the orthogonality (16), we obtain that Eq. (17) can be rewritten in the indicated region in the form

$$\frac{\omega_p}{2}\frac{W}{n_0^2}G_{\varkappa\Omega}\left\{\frac{\Omega_{\varkappa0}\cos^2\theta}{\Omega^2-\Omega_{\varkappa0}^2}+\frac{\Omega_{\varkappa1}\sin^2\theta\cos^2\varphi}{\Omega^2-\Omega_{\varkappa1}^2}+\frac{\Omega_{\varkappa2}\sin^2\theta\sin^2\varphi}{\Omega^2-\Omega_{\varkappa2}^2}\right\}=1.$$

Here θ is the angle between the vectors $\mathbf{s}_{k_0}^*$ and $\mathbf{s}_{k_0}$, while φ is the azimuthal angle between the vectors $\mathbf{s}_{k_0}^*$ and $\mathbf{s}_{\kappa_1}$. It is clear that the maximum of the increment for the conversion into an electromagnetic wave, e.g., with a frequency Ω_{k1}, is situated in the region $\sin^2\theta\approx\cos^2\varphi\approx 1$ (cf. Sec. 2). In the case of applicability of the static approximation, the maximum of the increment for the conversion into a HF wave with frequencies $\Omega_{k\lambda}$ is reached for

$$\Omega_{k\lambda}=\omega_pW/4nT,$$

and the increment itself is

$$\gamma_{max}=\omega_p W/4nT.$$

It is easy to determine the limit of the static approximation:

$$\gamma_{max}/\varkappa_{max}<v_{Ti}.$$

Thus, in the region $\kappa^2c^2/\omega_p \lesssim \omega_H$ this corresponds to the amplitudes $W/nT < T/Mc^2$ (cf. Sec. 2). For large amplitudes the static approximation is violated, the conversion goes over into its hydrodynamic stage.

The instability limit, as shown by calculations, is determined as before:

$$W/nT\geqslant 2\Omega_{\varkappa\lambda}/\omega_p. \tag{18}$$

Let us consider the implications of this instability. As was shown before, one may neglect the quantity related to the thermal pressure in the region $kr_D < (\omega_H/\omega_p)^{1/2}$ and $k_\perp/k_z > v_{Te}/c$. Taking this into account and introducing in (14) dimensionless variables (cf. Sec. 3), we obtain in the static approximation

$$i\mathbf{E}_t+i[\mathbf{qE}]-\beta \operatorname{rot}\operatorname{rot}\mathbf{E}+|\mathbf{E}|^2\mathbf{E}=0, \tag{19}$$

where $q = \omega_H/2\omega_p$.

This equation has the constants of motion

$$\mathscr{H}=\int\{-i(\mathbf{q}[\mathbf{EE}^*])+\beta|\operatorname{rot}\mathbf{E}|^2-|\mathbf{E}|^4/2\}d\mathbf{r},$$

$$\mathscr{N}=\int|\mathbf{E}|^2d\mathbf{r}.$$

Let us consider cylindrically-symmetric solutions of Eq. (19). It is easy to see that the cylindrical symmetry leads to the appearance of additional constants of the motion

$$\int\{|E_\varphi|^2+|E_r|^2\}d\mathbf{r},\quad \int|E_z|^2d\mathbf{r},$$

where E_r, E_φ, and E_z are the cylindrical components of E.

For these solutions we introduce the quantity

$$I=\int r^2|\mathbf{E}|^2d\mathbf{r}.$$

A direct calculation shows that

$$\frac{1}{\beta}\frac{\partial^2 I}{\partial t^2}=8\int\left\{\beta|\operatorname{rot}\mathbf{E}|^2-\frac{|\mathbf{E}|^4}{2}\right\}d\mathbf{r}$$

$$=8(\mathscr{H}+q\mathscr{N})-8q\int|\mathbf{E}|^2d\mathbf{r}+8i\int\mathbf{q}[\mathbf{EE}^*]d\mathbf{r}.$$

It is obvious that the last two terms satisfy the inequality

$$q\int|\mathbf{E}|^2d\mathbf{r}\geqslant i\int\mathbf{q}[\mathbf{EE}^*]d\mathbf{r}.$$

This implies

$$\frac{1}{\beta}\frac{\partial^2 I}{\partial t^2}\leqslant 8(\mathscr{H}+q\mathscr{N}).$$

Thus, if $\mathscr{H}+q\mathscr{N}<0$ at the initial instant, then, reasoning as above, we arrive at the conclusion that at some instant there appears a singularity, i.e., a collapse. The condition $\mathscr{H}+q\mathscr{N}<0$ coincides with the instability criterion (18). Thus, we can say that the development of an instability of the "cold" photon gas, (18) leads to collapse. One should note that in a collapse, in a magnetic field there occurs the formation of filaments flopping (collapsing) to the center. As a result of this the amplitude at the center increases and the static approximation breaks down. During the subsequent stage of development of the collapse the inertia of the medium becomes important.

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*[rp] ≡ r × p.

[1)]It follows from (15) that for such k the pressure is significant only in the region $k_\perp/k_z < v_{Te}/c$.

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