Nonlinear interaction between differently polarized waves traveling in opposite directions in a ring laser

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The nonlinear polarization effects in a gas laser with a ring resonator possessing polarization anisotropy and located in zero magnetic field are investigated on the basis of the semiclassical vector theory. It is shown that because their polarization vectors are different and the active medium possesses nonlinear anisotropy, the opposing waves are generated at different frequencies and intensities. The difference between the frequencies of the opposing waves increases from zero in proportion to the intensity of generation. Expressions for the frequencies and intensities of generation of the waves are derived as functions of the parameters of the resonator and the active medium.

INTRODUCTION

The polarization characteristics of ring resonators have recently been intensively studied (see the bibliography in[1,2]). It has been shown that there exist in each direction two running-wave-polarization eigenstates with different eigenvalues, which determine the frequencies and losses. In the absence of a magnetic field, the eigenvalues for the opposing waves are equal. The eigenvectors belonging to one and the same eigenvalue are different for the opposite directions when the losses in the radiation from the resonator are taken into account. Thus, in a real resonator without a magnetic field, the frequencies of the opposing waves are equal, while their polarizations are different.

Upon the application of a magnetic field to the active medium[3,4] or to the resonator elements, in which the polarizations of the opposing waves are nonlinear, there occurs, owing to the magneto-optical effects, a splitting of the opposing-wave frequencies. The magnitude of the difference frequency is proportional to the magnetic field, but does not depend on the generation intensity.

In the present paper we show that in the absence of a magnetic field there arises as a result of the nonlinear interaction with the active medium a polarization-induced difference between the frequencies of the opposing waves that is proportional to the generation intensity. A qualitative explanation of this new effect consists in the following. In the case of an elliptically polarized running wave, the active medium, which is isotropic in the absence of an electromagnetic field, becomes anisotropic because of the nonlinear interaction between the circular components of the field via the common σ- and π-transition sublevels. The polarization anisotropies in the saturation and nonlinear dispersion of the medium lead to the deformation of the polarization state of the field and to a change in the nonlinear shift of the frequency of generation of the wave. Since the opposing-wave polarizations, which are determined by the properties of the resonator, are, as a rule, different in the region where the active medium is located, the saturation and nonlinear dispersion of the medium turn out to be different for them. This determines the differences between the intensities, between the deformations of the polarizations, and between the nonlinear generation-frequency shifts of the opposing waves. The last effect gives rise to a change in the difference frequency[1].

The problem is solved under the assumption that there obtains a single-mode regime in which there is generated in each direction a monochromatic running wave with a definite polarization state possessing the least losses. It is assumed that the resonator possesses a sufficiently large amplitude or phase anisotropy, so that the perturbation introduced by the active medium has little effect on the polarization state.

The polarization-induced frequency difference can be measured directly if its value exceeds the width of the opposing-wave-frequency synchronization region. In the opposite case, it can be distinguished when the sign of the frequency "support" produced by rotation or by a nonreciprocal Faraday element changes.

1. THE RING RESONATOR WITH AN ARBITRARY POLARIZATION ANISOTROPY

1. The polarization properties of a resonator are conveniently described with the aid of the Jones matrix method[5], in which to each polarization element corresponds a square matrix of rank two. The matrix $\hat{P}$ of a series of elements is obtained by multiplying the matrices of the individual elements in the order in which they act on the traveling-wave vector:

$$\hat{P} = \hat{A}_N \ldots \hat{A}_2 \hat{A}_1.$$  

If by chance the indicated series of elements forms a closed ring resonator, then we can find the eigenvectors $\mathbf{q}$ and eigenvalues $\lambda$, which determine the frequencies and losses in the resonator:

$$\hat{P}\mathbf{q} = (P_{ij}) \mathbf{q} = \lambda \mathbf{q}, \quad \lambda^3 - 3\lambda^2 + \text{Det} \hat{P} = 0.$$  

(1)

The matrices of any arbitrarily arranged elements can be represented in the form of a product of three very simple matrices, two of which—the matrices for the partial polarizer

$$\hat{K}(k) = \begin{pmatrix} k & 0 \\ 0 & 1/k \end{pmatrix}$$  

(2)

and the linear phase plate

$$\hat{\Phi}(\varphi) = \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix}$$  

(3)

are diagonal and the third—the rotation matrix

$$\hat{S}(\alpha) = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}$$  

(4)

is nondiagonal. Here and below we use the notation: $s_\alpha = \sin \alpha$ and $c_\alpha = \cos \alpha$.

2. Let us find the relations between the resonator matrices $\hat{P}_r$ and $\hat{P}_i$ for the opposing waves in the co-


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ordinate systems connected with the directions of propagation of the waves. In constructing them, we must remember that the order in which the resonator elements act on the wave vector of one wave is the reverse of the order in which they act on the counterwave vector:

$$\hat{P}_r = -\sum_{i=0}^{n} A_i, \quad \hat{P}_c = -\sum_{i=0}^{n} A_i.$$  

If one of the resonator elements is a Faraday rotator that rotates the wave vector through an angle $\theta$, i.e., $A_k = S(\theta)$, then in constructing the matrix $P_k'$ we must set $A_k = S(-\theta)$, which corresponds to the presence of a preferred magnetic-field defined direction. Below we shall consider the case when $h = 0$. The results can easily be generalized to the $h \neq 0$ case with the aid of the above-indicated rule for allowing for magnetic rotation.

Using the properties of the product of transposed matrices $B^T C = (CB)^T$, we obtain

$$\hat{P}_c' = \left( \prod_{i=0}^{n} A_i \right)^T.$$  

For the polarization-element matrices (2)-(4), we have the following equality, obtained with the aid of the reflection matrix:

$$\tilde{A} = \hat{A} A^{-1}, \quad \tilde{P} = \hat{P} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  

From the equality (5) we obtain

$$\tilde{P}_c' = \left( \hat{P} \prod_{i=0}^{n} A_i^{-1} \right) = \tilde{P} \circ \tilde{P}_c.$$  

It is convenient to use the resonator matrices for the opposing waves when they are written in one coordinate system, connected, for example, with the direction:

$$\hat{P}_c' = \tilde{P} \circ \tilde{P}_c.$$  

3. It is evident from the equality (6) that

$$Sp \hat{P}_r = Sp \hat{P}_c', \quad Det \hat{P}_r = Det \hat{P}_c'$$  

and, in accord with the Eq. (1) for the eigenvalues, we obtain

$$\lambda_{i}^{(1)} = \lambda_{i}^{(2)} = \lambda_{i}^{(-h)} (k=1, 2).$$  

If there is a nonreciprocal Faraday rotator in the resonator, then

$$\lambda_{i}^{(1)} (h) = \lambda_{i}^{(2)} (-h) (k=1, 2).$$  

The determinants of the polarization-element matrices (2)-(4) are real and, therefore, since

$$Det \hat{P} = \prod_{i=0}^{n} A_i, \quad Det \hat{P}_r = Det \hat{P}_c'$$  

and, in accord with the Eq. (1) for the eigenvalues, we obtain

$$\lambda_{i}^{(0)} = \lambda_{i}^{(1)} \lambda_{i}^{(-h)} (k=1, 2).$$  

The above-considered properties of the resonator eigenvalues and eigenvectors indicate that in the absence of nonreciprocal Faraday rotators in the resonator it is impossible to produce a frequency difference for, and bring about losses of, the opposing waves (see formula (8)), but that the opposing-wave polarization states are nonetheless nonreciprocal.

2. COMPUTATION OF THE FREQUENCIES AND INTENSITIES OF GENERATION OF THE DIFFERENTIALLY POLARIZED OPPOSING WAVES

1. Let us consider the single-mode regime of generation in a gas ring laser. The field vector in angular coordinates has the form

$$E = E_0 e^{i \phi} e^{i \theta} e^{i \phi} \exp(-i (\omega t - kz)) + E_0 e^{i \phi} e^{i \theta} \exp(-i (\omega t + kz)) + c.c.$$  

where $E_0$ is the total wave intensity in the direction $n = (r, \theta, \phi)$; $e_r$ and $e_\theta$ are the complex components of the normalized polarization vector:

$$|e_r|^2 + |e_\theta|^2 = 1.$$  

To determine the frequencies and intensities of the running waves, we must solve the system of stationary equations of generation for the field vectors of the opposing waves in an angular basis:

$$\hat{G} \left( e_r, e_\theta \right) = \mu_1 \left( e_r, e_\theta \right), \quad \hat{G} \left( e_r, e_\theta \right) = \mu_2 \left( e_r, e_\theta \right).$$
where \( \mu_k = -i(a_k - v - \sigma) \), \( n = (r, l) \) \((k = 1, 2; i = \pm k) \),
\[
Q = \eta_0 - L(R|e_n|^2 + |e_{n'}|^2) - L(I|e_n|^2 + |e_{n'}|^2),
\]
and
\[
Q_{\mu} = M_{\mu} - W|e_n|^2 e_n^*.
\]

The elements of the matrix \( \tilde{Q} \) for the counterwave are obtained by means of the substitution \( r = 1 \).

For the field (14), the complex interaction coefficients take, in the absence of a magnetic field, the form
\[
A_{\mu} = 3\eta_0 \left( \frac{1}{T_n} \right) (i|\eta_0|/2)^{1/2} |e_n|^2 e_n^*.
\]

The real parts of the interaction coefficients are even, while the imaginary parts are odd, functions of the detuning of the generation frequency \( \omega \) relative to the center \( \omega_0 \) of the amplification contour.

2. The complex Q-factor matrix \( \tilde{M} \) of the resonator determines the losses, the frequencies, and the linear relation between the angular components of the resonator field. For waves propagating in different directions, the matrices \( M_1 \) and \( M_2 \) are different. To construct the matrix \( \tilde{M} \) acting on the field-vector components in the eigenvector representation. In this representation, it is diagonal, and is expressible in terms of the resonator eigenvalues \( \lambda_{1,2} \) (9):
\[
\tilde{M} = \begin{pmatrix} M_{11} & 0 \\ 0 & M_{22} \end{pmatrix} - \tilde{M} + \Delta \tilde{M}^T,
\]
where \( \tilde{M} \) is the unit matrix.

The elements \( \tilde{M}_{0k} \) can, with allowance for (9), be represented in the form
\[
\tilde{M}_{0k} = - \tilde{M}_{0k} \left( \ln|i| - i(-1)^{(k-1)} \frac{\Phi}{2} \right) \quad (k=1,2)
\]
and do not, in accord with (8), depend on direction \( L_0 \) is the optical length of the resonator).

The Q-factor matrix \( \tilde{M} \) can be obtained from \( \tilde{M} \) by means of the following transformation:
\[
M_{1} = A_1 M_0 A_1^T, \quad M_{2} = A_2 M_0 A_2^T,
\]
where \( \tilde{A} \) is the matrix that transforms the field-vector components from the natural to the angular basis and that is expressible in terms of the eigenvector components in the angular basis. The components of the matrix \( \tilde{A} \) are of the form
\[
(\tilde{A})_{nm} = \phi_{nm} \quad (n = r, i; m, k = 1, 2).
\]

Since the eigenvectors for the opposite directions are different (see (12)), the transformation matrices \( \tilde{A} \) and \( \tilde{A}_l \) are also different (this is the cause of the difference between the matrices \( \tilde{M} \) and \( \tilde{M}_l \)). The Q-factor matrix \( \tilde{M} \) is, in accord with (17) and (19), equal to
\[
\tilde{M} = \tilde{M} + \frac{\Delta \tilde{M}^T}{\tilde{M}}.
\]

Here \( \Delta \tilde{M} \) is the determinant (19) of the matrix \( \tilde{M} \). The components of the matrix \( \tilde{M} \) are of the form
\[
\tilde{M}_{ij} = \begin{pmatrix} 1 \quad 0 \\ 0 \quad 1 \end{pmatrix}, \quad \tilde{M} = \begin{pmatrix} 1 \quad 0 \\ 0 \quad 1 \end{pmatrix},
\]
and do not change, while \( \epsilon \) and \( \phi \) do not change, while \( \epsilon \) and \( \phi \) do not change, while \( \epsilon \) and \( \phi \) do not change, while \( \epsilon \) and \( \phi \) do not change, while \( \epsilon \) and \( \phi \) do not change. As will be shown below, the difference between the eigenvectors, which define the polarization state of the opposite-wave fields in the resonator, determines the differences in frequency and intensity of generation of the opposing waves.

3. We shall seek the solution to Eqs. (15) under the condition that the gain exceed the losses only for the type of oscillation with the highest Q factor, i.e., under the condition that
\[
-\text{Re} M_{nl} > \text{Re} M_{nl}, \quad n = r, l,
\]
Equations (15) have been written in the weak-field approximation (i.e., under the assumption that \( I_0 << 1 \)), and are valid up to the third order in the field. Accordingly, in order not to exceed the accuracy of the matrix element, \( \tilde{Q} \), should not be of order higher than the second in the field. This means that in fulfilling the condition for smallness of the nonlinear deformation of the polarization state
\[
|M_{nl} - M_{nl}| > \alpha + \text{Re} M_{nl},
\]
we can replace the components \( n, l \) of the normalized field vector in (16) by the components \( q_{nl} \) [Eq. (12)] of the eigenvector of the linear problem. The vectors \( q_{nl} \) and \( q_{nl}^* \) correspond to the eigenvalue with the highest Q factor \( M_{nl} \). We obtain
\[
Q_{nl} = Q_{nl} - I_l (R + L_{nl}) - I_l (E + \Delta L_{nl}),
\]
(23)
and
\[
Q_{nl} = Q_{nl} - I_l / \text{Re} M_{nl},
\]
where
\[
R = R_l, L = L_l, \Delta = \alpha + \text{Re} M_{nl},
\]
The solution of (15) with the matrix elements (23) is valid up to terms linear in the intensities \( I_0 \) and \( I_1 \), i.e., up to and including terms linear in \( \alpha + \text{Re} M_{nl} \), pump excess over the threshold.

The replacement of the elements given in (16) by the elements given in (23), which do not depend on the components of the eigenvector of Eq. (15), allows us to determine the eigenvalues and eigenvectors of the
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with the aid of the relations 

\[ \omega_{n} = \nu + \alpha - \text{Im} \mu_{n}, \quad \alpha + \text{Re} \mu_{n} = 0, \]

which are the stationarity conditions for the amplitudes of the opposing waves (14). In accord with the condition (21) for single-mode generation, \( \mu_{n} = \mu_{0n} + \mu_{1n} \), where \( \mu_{1n} \) is the nonlinear change, proportional to the intensities \( I_{p} \) and \( I_{s} \) in the eigenvalue. Neglecting terms of order \( \mu_{2} \), we obtain from Eq. (24) that 

\[ \mu_{n} = \frac{M_{n}}{M_{n} - M_{s}} |\text{Sp} \hat{Q}_{n} - \text{Sp} \hat{M}_{n}| - \text{Det} \hat{Q}_{n} = \text{Det} \hat{M}_{n}. \]

Using the formulas (17), (20), and (26), we find the expressions for \( \Delta \mu = \mu_{p} - \mu_{l} \) and \( \mu = \mu_{p} + \mu_{l} \): 

\[ \Delta \mu = f(V + f_{1} f_{2}), \quad \mu = f(G_{0} + f_{1} f_{2}) \]

where 

\[ f = \frac{1}{2} \left[ \begin{array}{c} f_{1} + 1 \text{Re} \left( G_{0} a - G_{0}^{*} \right) \left( V a^{*} - V a \right) \end{array} \right], \]

\[ \omega = \frac{1}{2} \left[ \begin{array}{c} f_{2} - 1 \text{Im} \left( G_{0} a - G_{0}^{*} \right) \left( V a^{*} - V a \right) \end{array} \right]. \]

Of greatest interest are the expressions for the differences between the intensities and the frequencies of the opposing waves: 

\[ \delta = (V v - V v') / \Gamma', \]

\[ \Delta \omega = \omega_{n} = \text{Im} \Delta \mu = f \left( V v - V v' \right) / \Gamma', \]

The frequency difference \( \Delta \omega \) is proportional to the intensity, and depends on the opposing-wave polarization states, which are determined by the parameters of the ring resonator, and on the nonlinear-interaction coefficients. The expression (29) contains terms that are even, as well as terms that are odd, functions of the generation-frequency detuning relative to the center of the amplification contour.

The frequency difference \( \Delta \omega \) vanishes in the case of linear polarizations of the opposing waves (\( \epsilon_{1} = \epsilon_{2} = 0 \), \( \pm \pi/2 \)) with an arbitrary difference between the azimuths, in the case of circular polarizations (\( \epsilon_{1} = -\epsilon_{2} = \pm \pi/4 \), and in the case when the sum of the ellipticities \( \epsilon_{1} + \epsilon_{2} \) vanishes; in all these cases \( a^{*} = a = 0 \).

Let us consider the case when the polarizations of the opposing waves differ little from each other (i.e., when \( \Delta \epsilon, \Delta \varphi < 1 \)). We obtain, correct to second order in smallness in the polarization parameters \( \Delta \epsilon \) and \( \Delta \varphi \), the expression 

\[ \Delta \omega = -2\Delta \epsilon \left( \begin{array}{c} \Gamma_{n} \left( V v - V v' \right) / \Gamma' \end{array} \right) - 2\Delta \varphi \left( \begin{array}{c} 1 + \left( V v - V v' \right) / \Gamma' \end{array} \right). \]

The coefficient attached to \( \Delta \epsilon \) is an odd, while the coefficient attached to \( \Delta \varphi \) is an even, function of the detuning, the odd component of \( \Delta \omega \) being a quantity of first order in smallness, the even component a quantity of second order. Notice that \( \Delta \omega \) is, as a rule, different from zero at the line center.

In the expression (30), the ellipticity parameter \( \epsilon \) has an arbitrary value. Let us compare the nonlinear frequency shifts \( \Delta \omega_{l} \) and \( \Delta \omega_{c} \) for nearly linear (\( \epsilon \ll 1 \)) and nearly circular (\( \pi/4 - \epsilon = \beta \ll 1 \)) polarizations:

\[ \Delta \omega_{l} = \Delta \omega_{c} = \frac{4 \epsilon_{l} \Delta \epsilon}{\Phi_{l}} \int \Delta \omega_{l} = \Delta \omega_{c} = \frac{4 \epsilon_{c} \Delta \epsilon}{\Phi_{c}} \]

where 

\[ \Phi_{l} = \text{Im} \left( \left( -L - W \right) \left( -L' - W' \right) \right), \quad \Phi_{c} = \text{Im} \left( \left( -L + W \right) \left( -L' + W' \right) \right), \]

\[ \sigma_{c} = \Delta \omega_{c} \sqrt{N_{l} / 2N_{th}} \]

is the amplification factor at the line center, and 

\[ \alpha = \frac{\omega}{\omega_{max}} \]

is the relative excess of the amplification factor at the generation frequency over its threshold value. In the pure isotope, 

\[ \Delta \omega_{l} = \Delta \omega_{c} = \frac{4 \epsilon_{l} \Delta \epsilon}{\Phi_{l}} \left( \begin{array}{c} 1 + f' \end{array} \right)^{2} \left( \frac{2K_{l} + 1}{K_{l} - 1} \right)^{2} \]

where 

\[ K_{l} = K'_{l} + K''_{l}, \quad \Delta \omega = \frac{2}{\tau_{l} + \tau'_{l}} \]

In a 50% isotopic mixture, 

\[ \Delta \omega_{l} = \frac{4 \epsilon_{l} \Delta \epsilon}{\Phi_{l}} \left( \begin{array}{c} 1 + f' \end{array} \right)^{2} \left( \frac{2K_{l} + 1}{K_{l} - 1} \right)^{2} \]

where 

\[ f = \frac{\omega_{l} + \omega_{c} - (\omega_{l} + \omega_{c})}{\Phi_{l}} \]

It can be seen from the expressions for \( \Delta \omega_{l} \) and \( \Delta \omega_{c} \) that they are both of second order in smallness in the polarization parameters, but that in the case of nearly circular polarization the frequency difference \( \Delta \omega_{c} \) contains only terms that are odd functions of the detuning, and usually vanishes at the line center.

The nonlinear deformation of the polarization states can be obtained by comparing the eigenvectors of the matrices \( Q \) and \( M \) and turns out, in accord with the condition (22), to be small (proportional to \( \epsilon \)).
The complex coefficients of the interaction between circularly polarized running waves in an active monoisotopic gaseous medium are given by:

\[ a - ia = a_0 (1 - (1 - 2y^2) \eta^2) - i \chi (1 + 2y) \eta, \]
\[ R = a_0 (1 + (1 - 2y^2) \eta^2) + i (1 - 2y) \eta', \]
\[ F = 2K_R, \]
\[ L = 4a_0 \left( \frac{\omega}{\omega_{is}} \right)^2 \left\{ 1 + \frac{2\gamma_1 \gamma_2}{(\gamma_1 + \gamma_2) \omega_{is}} - i (1 + 2y^2) \right\}, \]
\[ L_1 = 4a_0 \left( \frac{\omega}{\omega_{is}} \right)^2 \left\{ 2K_1 \frac{\gamma_1 \gamma_2}{(\gamma_1 + \gamma_2) \omega_{is}} + K_1 - i (1 + 2y) \right\}, \]
\[ L_2 = 4a_0 \left( \frac{\omega}{\omega_{is}} \right)^2 \left\{ 2K_2 \left( 1 + \frac{\gamma_1 \gamma_2}{(\gamma_1 + \gamma_2) \omega_{is}} \right) - K_2 - i (2K_1 + K_2) \right\}, \]

where
\[ 2y = \frac{\Delta \omega_{is}}{\omega_{is}} = 1, \]
\[ \chi = \frac{1 - 2y f(y)}{\gamma_1 + \gamma_2}, \]
\[ F(y) = e^{-2y} \int e^{yt} dt, \]
\[ \Delta \omega_{is} \text{ is the isotope line shift}, \]
\[ K_1 = K + K', \quad K_2 = \frac{2(K' \gamma_1 + K'' \gamma_2)}{\gamma_1 + \gamma_2}, \quad \eta = \frac{\gamma_1}{\gamma_2} \ll 1. \]

For Ne\(^{20} : \text{Ne}^{22}, \lambda = 0.6328 \mu, \]
\[ y = 0.5, \quad \chi = 0.42, \quad K_1 = 0.24, \quad K_2 = 0.32. \]

Note added in proof (February 20, 1974). Our attention has been drawn to G. S. Kruglik and É. G. Pestov's paper (Zh. Prikl. Spektrosk. 16, 308 (1972)), in which the influence of the polarization of the opposing waves on their competition in the ring laser is discussed. The expression (11) obtained in their paper for the polarization-induced frequency splitting differs from our formula (29) in that it depends on the detuning, the total angular momenta \( j_a \) and \( j_b \), and the difference between the azimuths of the ellipses of polarization of the opposing waves.

1) The possible existence of such an effect was suggested by S. A. Gordon.
2) The authors thank V. A. Sokolov for computing the coefficients (A.1).