The stability of the polarization of colliding beams

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The depolarizing effect of a colliding beam on the polarization of particles in storage rings, due to the stochastic mixing of the particle trajectories in an external field (owing, for example, to quantum fluctuations in the synchrotron radiation), is studied. In particular, a limitation on the maximum number of colliding-beam particles that do not destroy the radiative polarization of the forward beam is obtained for an electron (positron) beam.

1. As is well known, electrons (positrons) get polarized under the action of the synchrotron radiation when they move for long periods in storage rings [1–6]. The direction \( \mathbf{n} \) of the equilibrium polarization (i.e., the direction of the axis of precession of the spin of a particle moving in the storage ring along a closed equilibrium trajectory) in a nonuniform field varies along the particle orbit, repeating itself every other revolution [6,7]. The degree of equilibrium polarization and the time required for its establishment depend on the proximity to spin resonances. The role of the depolarizing effects becomes more and more important as we approach resonances. It is usually possible in storage rings to choose the particle energy such that the “dangerous” resonances are sufficiently far away. The electrons will then undergo self-polarization to a degree of polarization close to unity.

The situation changes in storage rings with colliding beams. Because of the strong nonlinearity of the collective field of the colliding bunches, the spin resonances decrease sufficiently slowly in strength as their number increases. The frequencies of the orbital motion become dependent on the amplitudes of the betatron and synchrotron oscillations of the particle. It is evident that resonant harmonics always exist under these conditions, and the depolarizing influence of the quantum fluctuations in the synchrotron radiation should be investigated in greater detail. In order for a region of stability of the radiative polarization to exist, the resonance density, which is proportional to the number of colliding-beam particles, should not be too high. It follows from the results of the investigation that we can, by properly choosing the parameters of the motion of the beams, obtain polarized electrons with a number of colliding-beam particles corresponding to the maximum radiative emittance of the storage device.

In a preceding paper [8] we have, by solving the equation for the polarization density, obtained formulas allowing the determination of the establishment time for, and the degree of, the equilibrium polarization under steady-state conditions for the motion of a beam in a storage ring with an arbitrary field without restrictions on the proximity of the spin resonances. In particular, the strength of the depolarizing action of the resonant harmonics that is connected with the quantum jumps in the momentum during the radiation emission is characterized by the inverse time \( \tau_r^{-1} \):

\[
\tau_r^{-1} = \pi \sum \left| \omega_k \right|^2 \delta(v - v_k),
\]

where \( \nu \) is the frequency of spin precession about the direction of the precession axis \( \mathbf{n} \), \( \nu_k = k_\| + k_\perp \nu' \) is an integral combination of the frequencies of the orbital motion \( \nu' \)'s, the \( \nu'_k \)'s are the frequencies of the betatron and synchrotron oscillations \( k_\| \) and \( k_\perp \)'s are whole numbers, the \( w_k \)'s are the resonating harmonics' powers due to the deviation of the particle from the equilibrium orbit in the inhomogeneous field, the angle brackets denote averaging over the distribution and equilibrium motion of the particles in the beam, and \( \delta(v - v_k) \) is a delta function. The quantities \( w_{k, \nu} \), \( \nu_k \), and \( v_k \) are functions of the amplitudes (the action variables) of the synchrotron and betatron oscillations.

The degree \( \xi \) of equilibrium polarization depends on the relation between the strengths of the depolarizing and polarizing mechanisms of the action of the synchrotron radiation. If \( \tau_r \) and \( \tau_p \) are the time and degree of equilibrium polarization for \( w_{k, \nu} = 0 \), then there gets established during the time \( \tau_p \tau_r \) (or \( \tau_r \tau_p ) \) an equilibrium polarization of degree

\[
\xi = \xi_p \nu' (\tau_p + \tau_r).
\]

The formula (1) is the starting point in the present paper, and it can be obtained on the basis of simple arguments, which clarify the physical meaning. The quantum fluctuations in the particle momentum during the emission of radiation give rise to a stochastic straying of the detuning \( \nu_k = \nu - v_k \) as a result of the mixing of the particle trajectories in the inhomogeneous field. It then becomes possible for the resonances to be transmitted with velocity \( \nu_k \) equal to

\[
\left| \nu_k \right| \sim \left| \nu \right|^{1/ \gamma},
\]

where \( \nu_p \) is the radiative trajectory-mixing time (or the order of the decrement of the radiative damping: \( \tau^{-1} \sim \gamma e^{|m|/|w_k|} \)), \( \gamma = (1 - m)^{-1/2} \) is the relativistic factor, and \( m \) and \( \nu \) are the electron charge, mass, and acceleration respectively. Under conditions of rapidity of transmission \( (w_{k, \nu}^2 < |\nu_k|) \), the change in the component of the particle-spin vector \( \mathbf{s} \) along the direction \( \mathbf{n} \) is equal to \( \xi \approx \xi_p \gamma \nu' \)

\[
\Delta s_{\|} = (s - s_{\|}) \left( 2 \pi |w_k| |\epsilon_k| \right)^{1/2} \cos (\theta_k + \alpha / 2),
\]

where \( w_{k, \nu} \), \( \epsilon_k \), and \( \Phi_k \) are the values of the power, the transmission velocity, and the phase of the spin precession at the instant when \( \epsilon_k = 0 \). Let \( f(t, \gamma) \) be the stationary distribution function of the particles over the action variables \( \gamma \) of the orbital motion. The number of passages of a resonance per unit time is equal to \( \xi_p \gamma \nu \nu_{p, \nu} \). Then we obtain for the mean rate \( \xi \) of change of \( \xi \) the expression

\[
\xi = - \int d\gamma f(t, \gamma) |\epsilon_k| \delta(v - v_k) \frac{\partial}{\partial \epsilon_k} (\Delta s_{\|})^{1/2} \sim \pi |w_k| |\delta(v - v_k)| \xi_p \gamma \nu' \gamma,
\]

which, after summing over the resonances, agrees with (1). Notice that since in deriving (1) we did not use the
radiative character of the particle-trajectory mixing, formula (1) describes the depolarizing influence in the resonance of any agency that destabilizes the orbital motion.

2. Let us consider the simplest case when the motion is such that the direction $n$ of the equilibrium polarization is almost constantly along the orbit and deviates little from the direction of the driving magnetic field. In the linear approximation in the deviations from the equilibrium plane orbit, the expression for $w_k$ can be reduced to the form

$$w_n = -\frac{\alpha}{\pi^2} \exp \left(-\pi^2 |\Psi_n| \right)_n$$

where $R$ is the radius of the storage ring (the length of the equilibrium orbit divided by $2\pi$), $n$ is the vertical deviation from the equilibrium plane orbit for which $n = \eta_2 = \text{const}$, $\psi_k$ is an integral combination of the phases $\varphi_{\alpha}$ of the orbital motion ($\psi_k = \varphi_{k}$), the symbol $(\ldots)_1$ denotes averaging over the phases of the motion, and $\alpha(\theta)$ is a periodic function of the generalized azimuth $\theta$ of the particle motion:

$$\alpha = -\sqrt{\frac{n^4+1}{n^4-1}} \exp \left[-\pi \int (K-1) d\theta \right].$$

Here $K(\theta)$ is the curvature of the orbit in units of $\frac{2\pi}{\nu}$.

\[ R^{-1}(Kd\theta = 2\pi) \text{ and } \nu = \gamma G, \ G \text{ being the ratio of the anomalous magnetic moment of the electron to the normal moment. In an azimuthally symmetric storage ring we have } \alpha = -\nu^3 = \text{const.} \]

As is well known, the beam's truly vertical dimension, which is due to the quantum fluctuations in the radiation, is determined by the relation between the vertical oscillations and the radial and phase oscillations. Therefore, in describing the perturbing influence of a colliding bunch on the vertical deviations, it is, generally speaking, necessary to take into account, besides the direct action of the $z$ component $F_z$ of the force of the colliding bunches, that action of the radial component $F_x$ that is due to the $z-x$ coupling. We shall neglect the action of the force $F_x$ assuming that the steady-state amplitudes of the $z$-oscillations are given: allowance for $F_x$ is not essential to the elucidation of the basic characteristics of the effect of the colliding bunches on the polarization and greatly complicates all the formulas. The equation describing the vertical deviation then has the form

$$d^2z/d\theta^2 + g_z F_z(x, z, \theta) = 0.$$  \hspace{1cm} (3)

Let us consider the case of a head-on encounter between electrons and positrons, when the equilibrium orbits of the bunches in the interaction region coincide. Then the effective potential $V$ can be written in the form

$$V_{\text{eff}} = -\frac{eV}{2\pi}, \ V = \frac{-2\pi N_{e}}{2} \ln \left(\frac{1}{\eta_{b}}\right),$$

where $N$ is the total number of particles of the colliding beam and $\eta_{b}$ is the classical electron radius. The functions $\sigma$ and $g$ describe the density distribution of the counterbunches about the equilibrium orbit, and are normalized in the following fashion:

$$\int_{0}^{\frac{\pi}{2}} \sigma(x) \phi(\theta) d\theta = 1.$$  \hspace{1cm} (4)

Let us choose the distributions of $\sigma$ and $g$ in the form

$$\sigma = \frac{1}{2\pi} \exp \left(-\frac{x^2}{\lambda^2} - \frac{z^2}{\alpha^2} \right),$$

$$g(\theta) = \frac{2}{\alpha} \sum_{n} \delta \left(\theta - \frac{2n\pi}{\alpha} \right).$$

where $2\alpha$ and $2\lambda$ are the radial and vertical dimensions of the colliding bunches at the places of encounter ($x \pm 2\lambda$) and $p/2$ is the number of bunches ($p$ is the number of places of encounter in the storage ring). Let us assume that the magnetic system of the storage ring has a period of $\theta_0 = 2\pi / p$.

Let us begin with the study of one-dimensional resonances due to the colliding beam:

$$\nu \approx \nu_n + \nu_0, \ \left(\nu_n \approx 0, 1 \right).$$

The maximum number $n_{\text{max}}$ starting from which the harmonics $|k_z| > k_{\text{max}}$ can be neglected is found from the equation

$$\left(\tau_{\text{max}}\right)^{-1} = \frac{2 \pi}{\nu} \chi\frac{1}{\gamma\beta R} \left(\frac{x_0}{2\lambda} \right)^{1/2},$$

where $\chi$ is the Compton wavelength, $R_\infty = |\nu_0|^{-1}$ is the radius of curvature of the trajectory at the sections with magnetic fields, and $(\tau d)_{\text{min}}$ is the minimum value of $\tau d$. The harmonic $w_k$ and the particle vertical-oscillation frequency shift $\Delta \nu_k$ due to the colliding beam determine through the formula (1) the quantity $\tau d$.

For a Gaussian distribution over the amplitudes $a_z$ of the vertical oscillations:

$$f(a_z) = \left(\sqrt{\pi a_z} \right)^{-1} \exp \left(-a_z^2/\pi a_z^2 \right).$$

Then (5) (see the Appendix)

$$\frac{1}{\left(\tau_{\text{max}}\right)^{-1}} = \frac{4}{\pi} \pi N_e R_\infty A \int_{|\nu|}^{\infty} \left(\frac{a_z}{\nu}\right)^{1/2} \frac{B}{a_z^2 + 2a_z^2} d\nu,$$

Here

$$A = \frac{1}{\pi} \int_{-\nu}^{\nu} \frac{1}{1 - \exp \left[-2\pi (v - \nu)^2 \right]} d\nu,$$

$$B = \frac{x_0}{x_0^2 + 2a_z^2},$$

where

$$\nu = \nu_0, a_z = a_{\alpha} \sqrt{1 + 2\alpha^2}, \ \text{and} \ \alpha = \frac{a_z}{x_0} \exp \left[-2\pi^2 |\Psi_n| \right] _n.$$
When conditions are sufficiently close to the resonances with \( |k_z| < k_{z,\text{max}} \), the beam is depolarized. The "dangerous" frequency interval \( \Delta \nu \) occupied by the resonance \( \nu \approx k_0 p + k_z \nu_z \) is a function of the number \( k_z \):

\[
|\Delta \nu| \ll (k_{z,\text{max}})^2 \Delta \nu_{\text{max}}.
\]

Here \( \Delta \nu_{\text{max}} \) is the maximum shift introduced into the frequency \( \nu_z \) by the colliding bunch (see the Appendix):

\[
\Delta \nu_{\text{max}} = \frac{2N_r R |j|}{\pi \tau_1 (x_0^2 + z_0^2)}.
\]

In order for \( \nu \)-value regions free from "dangerous" intervals to exist, it is clearly sufficient that

\[
\sum_{k_z = 1}^{k_{z,\text{max}}} |\Delta \nu| \ll (k_{z,\text{max}})^2 \Delta \nu_{\text{max}} < \frac{p}{\nu_z}.
\]

The condition (8), together with Eq. (7), imposes limitations on the maximum possible number \( N_{\text{max}} \) of particles in the colliding beam that does not destroy the polarization, limitations which are roughly the same as those which are dangerous in the orbital motion. The conditions limiting the maximum possible number \( N \) of particles (the radiant emittance) for which the encounters are still stable can be written down on the basis of the well-developed theory of motion near nonlinear resonances.

As is evident from (7) and (8), \( N_{\text{max}} \) strongly depends on the ratio \( (a_0')/z_0 \) increasing as this ratio decreases. A decrease in the modulus of the Floquet solutions (which are \( \beta \) functions) at the places of encounter leads, as is well known, to a gain in the radiant emittance of the storage device (owing to increases in the beam densities at the places of encounter). To increase the radiant emittance of a storage device with polarized beams, it is also advantageous to decrease \( |f_z|/|f_0| \) (the degree of equilibrium polarization is less sensitive to changes in \( |f_z|/|f_0| \)). In fact, the increase in the force \( |f_z|/|f_0| \) of action of the colliding bunch, the betatron-frequency shift and the harmonics of the vertical deviation, which determine the power \( |w_k| \) of the spin resonances, are less sensitive to changes in the \( \beta \) functions at the places of encounter.

The inclusion of resonances with radial and synchrotron particle oscillations leads to an increase in the number of "operating" resonances. The condition (6) is then replaced by (usually, \( \Delta \nu_{2,\text{max}} \gtrsim \Delta \nu_{1,\text{max}} \gtrsim \Delta \nu_{\text{max}} \)) the condition

\[
\Delta \nu_{2,\text{max}} \gtrsim \Delta \nu_{1,\text{max}} \gtrsim \Delta \nu_{\text{max}} < \frac{p}{\nu_z}\] (9)

which imposes a stronger limitation on \( N_{\text{max}} \) (\( k_{z,\text{max}} \) and \( k_{x,\text{max}} \) are the maximum numbers of the "operating" resonances with radial and synchrotron oscillations). The equations for \( k_{z,\text{max}} \) and \( k_{x,\text{max}} \) can be obtained in similar fashion. In allowing for the slow synchrotron oscillations, we must, generally speaking, take into consideration, besides the modulation of the betatron harmonics \( w_{k_z}, w_{k_x}, k_z, k_x \) of the perturbation propagating perpendicularly to \( n \), the synchrotron modulation of \( w \cdot n \) and the betatron frequencies \( \nu_z \) and \( \nu_x \).

The formula (1) of course allows us to carry out a more detailed investigation with allowance for all the real factors determining the properties of the orbital motion. However, because of the above-noted complex dependence of \( w_k \) and \( \epsilon_k \) on the amplitudes of the oscillations, it is more expedient to carry out such investigations for specific storage devices. The main results, however, remain valid independent of the structure of the electromagnetic field acting on the colliding beams inside the storage ring.

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APPENDIX

The expression for the vertical component \( F_z(x, \theta) \) of the force exerted by the colliding bunches at \( x = 0 \) is obtained from (4) (it is given in the form (5)):

\[
F_z = \frac{4N_r R}{\tau_2} \int dt \frac{z \exp(-i\xi \theta)}{[z^2((\xi - t)^2 + z^2)]^{1/2}}\]

The frequency shift \( \Delta \nu_z \) induced by the colliding beam is computed from the formula (11-13):

\[
\Delta \nu_z = \frac{|f_z|/a_z}{(2\pi)^2} \int \left[ \frac{d\theta}{2\pi} \cos \psi \cos \epsilon, F_z(x, \cos \psi, 0) \right] \]

The expression for \( w_k \) can be reduced with the aid of (2) and (3) to the form

\[
\text{For even } k_z, \quad w_k = 0 ; \quad \text{while for odd } k_z, \quad w_k = \frac{N_{\text{r},z}}{\pi} A \left[ \frac{2a_z^{-1}}{a_z^2} \right]^2 \exp(-ik_z \phi) \] (A.1)

where \( L_n(t) \) is the Bessel function of imaginary argument. The asymptotic form of the functions \( L_n(t) \) for \( m \gg 1 \) is known (11-13):

\[
L_n(t) = \exp \left[ \frac{m^2t^2}{2(m^2t^2)_{1/n}} \right] \left[ 1 + \frac{m^2}{2(m^2t^2)_{1/n}} \right]^{-n}.
\]

The dominant contribution in the integral over \( t \) in (A.1) lies, for \( |k_z| \gg 1 \), in the vicinity of \( t = a_z^{-1}/2z_k^2 \). Thus, the final expression for \( |w_k| \) has the form

\[
L_n(t) = \left[ \frac{N_{\text{r},z}}{\pi} \frac{|f_z|/a_z}{(2\pi)^2} \right] \left[ \frac{2a_z^{-1}}{a_z^2} \right]^2 \exp \left[ \frac{m^2}{2(m^2t^2)_{1/n}} \right] \left[ 1 + \frac{m^2}{2(m^2t^2)_{1/n}} \right]^{-n}.
\]

For a Gaussian amplitude \( (a_0') \) distribution, we obtain the following expression for \( \tau_{d2} \):

\[
\tau_{d2} = \frac{\pi N_{\text{r},z}}{|f_z|} \exp(-\xi/|a_z|) \frac{1}{|f_z|} (|a_z|/|\Delta \nu|)^{1/2}.
\]

(A.2)

(here \( a_z \) is the value of the amplitude for which \( \epsilon_k = 0 \)). Investigation of (A.2) for its extremal values shows that the minimum value of \( \tau_{d2} \) is determined by the formula (6).

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In motion across a unidirectional magnetic field, $n$ is obviously directed along the field, and does not depend on the azimuth of the particle motion.

We measure all the frequencies in units of the equilibrium frequency of revolution of the particles of the storage element.

The formulas for $r_0$ and $T_0$ are given in [1].

The formula for $r_p$ can be found in [1-4].

Since $F_z$ is an odd function of $z$, only resonances with odd numbers $k_z$ are possible. Resonances of even parity arise when the coupling between the vertical and radial oscillations is taken into account.

The condition (9) alters if the frequency of the synchrotron oscillations is small compared to the spread $\Delta \omega_k$ of the frequencies of the motion. Then instead of one distinct resonance in the band $\Delta \omega_k$, there appears a series of synchrotron resonances of roughly the same power, over which the appropriate expressions must clearly be summed.