Exact theory of propagation of ultrashort optical pulses in two-level media

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It is shown that the equation $\sigma_{\xi\tau}=-\sin\sigma$, which arises in many branches of physics and in particular describes the propagation of ultrashort optical pulses in a two-layer medium without dissipation, can be solved exactly under the condition $\sigma(\pm\infty, \tau)=0 \pmod{2\pi}$ by reduction to the inverse scattering problem for a certain differential operator. Furthermore explicit solutions can be obtained, describing the interaction of so-called 2π pulses (solitons) and 0π pulses (double solitons). It is shown that in the interaction of solitons and double solitons their amplitudes and speeds do not change, but the phases and the coordinates of the centers make a jump. It is also shown that the only soliton collisions that occur are binary collisions between solitons, between double solitons, or between a soliton and a double soliton.

The equation

$$\frac{\partial^2\sigma}{\partial\xi\,\partial\tau}\equiv\sigma_{\xi\tau}=-\sin\sigma,\qquad -\infty<\xi,\ \tau<\infty, \tag{1}$$

arises in many branches of physics. In particular, it describes the propagation of ultrashort optical pulses in a two-level medium without dissipation. The dimensionless quantities ξ and τ are connected with the space and time variables x, t in the following way:

$$\xi=(\Omega/c)x,\qquad \tau=(t-x/c)\Omega;$$
$$\Omega=(\alpha c)^{1/2},\qquad \alpha=2\pi n_0\omega_0 P^2/\hbar c,$$

where c is the phase velocity of light in the medium. Here ω_0 is the carrier frequency of the incident pulse, which is a plane wave propagated along the x axis, n_0 is the density of two-level atomic systems in the medium, and P is the dipole matrix element for the transition between the upper and lower levels of the system.

In our case the electromagnetic field is of the following form:

$$E(x,t)=\bar{E}(x,t)\cos(k_0x-\omega_0t),$$

where the connection of σ with the amplitude E is given by $\sigma_t=P\bar{E}/\hbar$, it being assumed that

$$\omega_0\bar{E}\gg\bar{E}_t,\qquad k_0\bar{E}\gg\bar{E}_x.$$

The detailed derivation of Eq. (1) and its application to the description of optical pulses are given in a paper by Lamb,[1] to which we refer the reader. We point out that Eq. (1) has been known for a rather long time in connection with the theory of surfaces of constant negative curvature.[2] It also arises in the theory of dislocations,[3] in some models of field theory,[4-6] in the theory of superconductivity,[7-9] and in nonlinear mechanical models of wave propagation.[10] Up to the present, however, no complete analytical description of the solutions of Eq. (1) has been given, and the purpose of many authors has been only to find particular solutions of the equation. In the present paper we shall give an exact description of the general solution of Eq. (1) under the condition:

$$\int_{-\infty}^{\infty}A(\xi,\tau)\,d\xi=0\pmod{2\pi}.\qquad A=\sigma_\xi, \tag{A}$$

which arises naturally in applications. Under the condition (A) Eq. (1) can be solved exactly by the method of the inverse problem, if we first identify solutions of Eq. (1) that differ from each other only by an integer multiple of 2π.

We note that the possibility of solving Eq. (1) by the method of the inverse problem was pointed out by Lamb,[11] but he did not develop the corresponding formalism. In this case we can use the substitution $A=\sigma_\xi$ to reduce Eq. (1), with the boundary conditions $\sigma(\xi,\tau)\to 0 \pmod{2\pi}$ for $|\xi|\to\infty$, to the equation

$$A_t(x,t)=-\sin\Big(\int_{-\infty}^{x}A(x',t)\,dx'\Big),\qquad x=\xi, \tag{2}$$

where to make the argument more intuitive we shall from now on give the variable $t=\tau$ the meaning of time. It is assumed that A(x, t) satisfies the condition (A).

The method of the inverse problem was discovered by Kruskal, Greene, Gardner, and Miura[12] and was applied by them to the well-known Korteweg-de Vries equation. Later Zakharov and Shabat[13,14] applied the method to the equation

$$iu_t+u_{xx}+\varkappa u|u|^2=0. \tag{3}$$

Furthermore, Faddeev and Zakharov have proposed a mechanical interpretation of the method of the inverse solution as applied to the Korteweg-de Vries (KdV) equation, using an invariant description of infinite-dimensional mechanical systems.[15] The mechanical interpretation of the method of the inverse solution has also been applied (cf. [16]) to Eq. (3) in the case $\kappa<0$ and with boundary conditions different from those of Zakharov and Shabat.[14]

Let us apply the method of the inverse solution to equations of the form $u_t=S(u)$, which can be put (cf. [17]) in the form

$$\partial L/\partial t=[L,M]=LM-ML. \tag{4}$$

Here S is in general a nonlinear operator, and L and M are linear operators containing the set of functions u(x, t) as coefficients. It follows from Eq. (4) that the spectrum of the operator L does not change with time, and the asymptotic characteristics of its eigenfunctions at any instant of time can be easily calculated from their initial values. The reconstruction of the set of functions u(x, t) at an arbitrary time is accomplished by solution of the inverse scattering problem for the operator L.

It is not hard to verify that Eq. (2) can be written in the form (4), where the operators L and M have the following forms:

$$L=\frac{1}{i}\tau_2\frac{d}{dx}+\frac{i}{2}A(x,t)\tau_1,$$

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$$M\psi(x)=-\frac{1}{8}\int_{-\infty}\left[\exp\left(-\frac{i}{2}(\sigma(x,t)+\sigma(x',t))\right)(\tau_0+\tau_3)+\exp\left(\frac{i}{2}(\sigma(x,t)+\sigma(x',t))\right)(\tau_0-\tau_3)\right]\psi(x')dx'$$

and τ_0, τ_1, τ_2, τ_3 are the Pauli matrices

$$\tau_0=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad \tau_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \tau_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\quad \tau_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

We note that the M operator of Eq. (2), is an integral operator, whereas the M operators of the KdV equation and Eq. (3), are differential operators. On the other hand, the L operator for our equation is a special case of the L operator for Eq. (3). We note that in implicit form the operator L already figured in Lamb's work.[11]

A number of papers[13,14,17,18] have brought out the fundamental role of particular solutions of Eq. (3) and the KdV equation, the so-called solitons, which are directly connected with the discrete spectrum of the corresponding operator L; that is, it has been established that for an arbitrary initial condition the asymptotic state is a finite set of solitons. In our case an analogous role is played by particular solutions of Eq. (1):

$$\sigma(x,t)=\pm4\,\mathrm{arctg}\exp[2a(x-t/4a^2+x_0)],\qquad a>0,$$

(x_0 is a real number) which we also call solitons, and by other solutions

$$\sigma(x,t)=-4\,\mathrm{arctg}\left(\frac{a}{|c|}\frac{\cos(2c(x+ct/4|\lambda|^2-\beta))}{\mathrm{ch}(2a(x-at/4|\lambda|^2+x_0))}\right),$$

which we call double solitons.

In the application to the propagation of optical pulses a soliton, or a 2π pulse according to Lamb's classification, plays the role of the pulse for the propagation in an attenuator, associated with self-induced transparency. A double soliton plays the role of a 0π pulse, associated with a more complicated form of self-induced transparency. A soliton is characterized by two parameters—the proper velocity $v=1/4a^2$ and the coordinate x_0 of the center. A double soliton is characterized by four parameters: the proper velocity $1/4|\lambda|^2$, the amplitude $a/|c|$, the coordinate x_0 of the center, and the phase β, where $\lambda=c+ia$. The parameters of the soluble soliton and soliton are independent and can be chosen arbitrarily.

The soliton and double soliton are representatives of an extensive family of solutions of Eq. (1) which can be expressed in explicit form. In the general case such a solution (let us call it an N-soliton) depends on 2N arbitrary real parameters v_j, x_{0j}, $j=1,\dots,k_1$; v_p, x_{0p}, β_p, $a_p/|c_p|$, $p=1,\dots,k_2$, where $k_1+2k_2=N$. At nonidentical velocities v_j and v_p, an N-soliton solution decays as $|t|\to\infty$ into solitons and double solitons, k_1 and k_2 being the numbers of solitons and of double solitons; i.e., the N-soliton solution describes a process of scattering of solitons and double solitons. Only the coordinates of the centers and the phases change in the scattering, the proper velocities and amplitudes remaining the same. Only binary collisions between solitons, between double solitons, and between soliton and double soliton contribute to the changes in the scattering.

In comparison with the KdV equation there is a possibility of formation of bound states of a finite number of solitons and double solitons having identical velocities. For Eq. (3) in the case $\kappa>0$ this fact was established by Zakharov and Shabat.[13] In the N-soliton case, a bound state becomes conditionally periodic with respect to t as $|t|\to\infty$. We note that in this way a double soliton can be interpreted as a bound state of two conjugate "complex solitons" which do not exist separately.

The writer's attention was called to the possibility of solving Eq. (1) by the method of the inverse problem by L. D. Faddeev. He expresses his thanks to L. D. Faddeev for his interest in the present work.

1. THE SCATTERING PROBLEM

The operator L here differs from the operator L in the paper by Zakharov and Shabat[13] only in the choice of basis in spin space. Therefore all the results of [13] on the direct and inverse scattering problems remain valid. We only point out additional properties of the given solutions, which arise because the function A(x, t) is real, and recall the basic facts of scattering theory for the operator L. Throughout most of this section we shall not indicate the dependence on t, since in the scattering problem the time t plays the role of a parameter.

Let us consider the system of equations

$$L\psi=k\psi. \tag{5}$$

For real k we define the Jost functions g(x, k), f(x, k) as solutions of the system (5) with the asymptotic forms

$$g(x,k)=e^{-ikx}\begin{pmatrix}i\\1\end{pmatrix}+o(1)\quad\text{as}\quad x\to-\infty,$$
$$f(x,k)=e^{ikx}\begin{pmatrix}-i\\1\end{pmatrix}+o(1)\quad\text{as}\quad x\to\infty. \tag{6}$$

For real k the pair of functions g(x, k), $\bar g(x,k)=\tau_2 g^*(x,k)$ form a fundamental system of solutions, so that

$$f(x,k)=a(k)\bar g(x,k)+b(k)g(x,k), \tag{7}$$

where the transition coefficients a(k) and b(k) are given by the formulas

$$a(k)=\frac{1}{2i}\{g(x,k),f(x,k)\},\qquad b(k)=\frac{1}{2i}\{f(x,k),\bar g(x,k)\},$$
$$\{g,f\}=g_1f_2-g_2f_1. \tag{8}$$

The Jost functions admit of analytic continuation in the half-plane $\mathrm{Im}\,k>0$, and therefore it follows from Eq. (8) that a(k) can also be analytically continued into the upper half-plane and

$$\lim a(k)=1,\quad |k|\to\infty,\quad \mathrm{Im}\,k\geqslant0. \tag{9}$$

We also note that

$$|a(k)|^2+|b(k)|^2=1. \tag{10}$$

Since A(x) is real, we easily see that

$$a(k)=a^*(-k^*),\quad \mathrm{Im}\,k\geqslant0,\quad b(k)=-b^*(-k),\quad \mathrm{Im}\,k=0. \tag{11}$$

We shall assume that a(k) has no zeroes on the real axis. (We have unfortunately been unable to find in terms of the function A(x) an effective criterion for the absence of zeroes of the function a(x) on the real axis.) It then follows from (9) that in the half-plane $\mathrm{Im}\,k>0$ the quantity a(k) can have only a finite number of zeroes λ_j, $j=1,\dots,N$, which, for greater simplicity and clarity of the resulting formulas, we shall take to be simple zeroes. It follows from Eq. (8) that

$$f(x,\lambda_j)=c_jg(x,\lambda_j),$$

and on the basis of Eq. (6) we conclude that the zeroes of $a(x)$ are eigenvalues of the operator L. From Eq. (11) we find that the numbers λ_j, and also the c_j, $j=1, \ldots, N$, are located symmetrically relative to the imaginary axis.

We shall call the set of quantities

$$\{r(k), \lambda_j, m_j, \quad j=1, \ldots, N\},$$

the scattering data of the operator L; here

$$r(k)=\frac{b(k)}{a(k)}, \quad m_j=c_j \Big/ i\frac{da(\lambda_j)}{dk}, \quad j=1, \ldots, N.$$

We note that $a(k)$ and $b(k)$ can be uniquely reconstructed from the scattering data (cf. [19]).

We shall now say a few words about the inverse problem. Its solution is based on equations of the Marchenko type, which can be obtained by means of the formalism in the work of Faddeev.[19]. From the scattering data we construct the kernel $F(x+y)$:

$$F(u)=\frac{1}{2\pi}\int_{-\infty}^{\infty} r(k)e^{-iku}\,dk+\sum_{j=1}^{N} m_j e^{-i\lambda_j u} \tag{12}$$

and let $K(x, y)$ be a real-valued matrix function satisfying the equation

$$\tau_2 K(x,y)\tau_2+iF(x+y)\tau_1+i\int_{-\infty}^{x} K(x,u)F(u+y)\tau_1\,du=0, \qquad x>y, \tag{13}$$

and the condition

$$[K(x, y), \tau_1]=0. \tag{14}$$

Then the required function $A(x)$ is given by the formula

$$A(x)\tau_0=2[K(x, x), \tau_2/i]\tau_3. \tag{15}$$

We note that the matrix $K(x, y)$ is the kernel of the transformation operator for the solution $g(x, k)$, i.e.,

$$g(x,k)=e^{-ikx}\begin{pmatrix} i \\ 1 \end{pmatrix}+\int_{-\infty}^{x} K(x,u)e^{-iku}\begin{pmatrix} i \\ 1 \end{pmatrix} du.$$

We shall now establish exact formulas for the general solution of Eq. (1). It follows from Eq. (4) that the eigenfunctions of the operator L obey the equation

$$\partial\psi/\partial t+M\psi=0. \tag{16}$$

More exactly, if the initial condition for Eq. (16) satisfies (5), then the solution of Eq. (16) at an arbitrary time will also satisfy the system (5) with fixed k. Since as $|x| \to \infty$

$$\frac{d}{dx}M\varphi(x) \sim +\frac{1}{4}\varphi(x),$$

we have from Eq. (16) (as $|x| \to \infty$)

$$\frac{\partial^2}{\partial t\,\partial r}f(x,k,t) \sim -\frac{1}{4}f(x,k,t),$$

from which we easily derive that $a(k)$ is independent of the time, while

$$b(k, t)=\exp(-it/2k)\,b(k, 0), \quad m_j(t)=\exp(-it/2\lambda_j)\,m_j(0). \tag{17}$$

The formulas (17) give us the general solution of Eq. (1).[1)]

We now note that the condition (A) is equivalent to the condition $a(0)=\pm 1$. It then follows from Eq. (17) that a solution of Eq. (1) satisfies condition (A), provided that it satisfies that condition at time $t=0$.

2. N-SOLITON SOLUTIONS (EXPLICIT FORMULA)

Let us consider the inverse scattering problem in the case $r(k, t)=0$. Then the Marchenko equation has a degenerate kernel and is easily solved. We note to begin with that owing to the condition (14) we have

$$K(x, y)=\alpha_1(x, y)\tau_0+\alpha_2(x, y)\tau_1.$$

We write the function $F(u+y)$ in the form

$$F(u+y)=i\sum_{j=1}^{N} m_j \exp(-i\lambda_j u-i\lambda_j y)=i(\varphi(u), \psi(y))$$

and shall look for α_1, α_2 in the form

$$\alpha_1(x, y)=(a_1(x), \psi(y)), \quad \alpha_2(x, y)=(a_2(x), \psi(y)).$$

Then after obvious transformations the Marchenko equation reduces to a linear algebraic system for the determination of the functions $a_1(x)$ and $a_2(x)$:

$$a_1(x)+iV(x)a_2(x)=0, \quad iV(x)a_1(x)-a_2(x)=-i\varphi(x),$$

where the matrix $V(x)$ is defined by the formula

$$V_{jk}(x)=\int_{-\infty}^{x}\varphi_j(x')\psi_k(x')\,dx'; \qquad j, k=1, \ldots, N.$$

This system, as will be shown in the Appendix, is nondegenerate, and, using Eq. (15), we get

$$\begin{aligned} A(x,t)&=4(a_2(x), \psi(x))=+4i\,\mathrm{Sp}\left((I-V^2(x))^{-1}V'(x)\right) \\ &=+2i\frac{d}{dx}\ln\frac{\det(I+V(x,t))}{\det(I-V(x,t))}, \end{aligned} \tag{18}$$

where in the last form we have included the dependence of m_j on t, and

$$V_{jk}(x,t)=\frac{im_j}{\lambda_j+\lambda_k}\exp\left(-i(\lambda_j+\lambda_k)x-\frac{i}{2\lambda_j}t\right).$$

Finally, we write the explicit formula for an N-soliton solution $\sigma(x, t)$ of Eq. (1)

$$\sigma(x,t)=+2i\ln\frac{\det(I+V(x,t))}{\det(I-V(x,t))} \tag{19}$$

We note that Eqs. (18) and (19) have appeared in a paper by Lamb, but without a sufficiently rigorous derivation and without explanation of the conditions imposed on the numbers λ_j, m_j, $j=1, \ldots, N$. From general arguments about the solubility of Marchenko equations it follows that the functions $A(x, t)$ and $\sigma(x, t)$ defined by Eqs. (18) and (19) take only real values. This, by the way, will also be proved in the Appendix.

We note that since $\mathrm{Im}\,\lambda_j > 0$, $j=1, \ldots, N$, it immediately follows that $A(x, t)$ goes to zero exponentially for $x \to \infty$. In the Appendix it will be shown that $A(x, t)$ also decreases exponentially for $x \to -\infty$.

We point out that for the function $A(x, t)$ defined by Eq. (18) condition (A) is satisfied. In fact, in this case it follows from (10) that on the real axis $|a(k)|^2=1$, from which we conclude that $a(k)$ can be continued analytically into the lower half-plane, and has poles there at the points λ_j^*, $j=1, \ldots, N$. Noting the asymptotic behavior of $a(k)$ for $|k| \to \infty$, we conclude that

$$a(k)=\prod_{j=1}^{N}\frac{k-\lambda_j}{k-\lambda_j^*},$$

from which we see that $a(0)=\pm 1$.

Let us now examine some special cases of Eq. (19).

1) $\lambda=ia$, $m=ib$, $\mathrm{Im}\,b=0$. We find then that

$$\sigma(x,t)=-4\,\mathrm{arctg}\left(\frac{b}{2a}\exp\left(2ax-\frac{1}{2a}t\right)\right),$$

i.e., we have a soliton with velocity $v=1/4a^2$ and center coordinate

$$x_0=\frac{1}{2a}\ln\frac{|b|}{2a}.$$

In our present case

$$\int_{-\infty}^{\infty} A(x,t)\,dx=\sigma(\infty,t)-\sigma(-\infty)=-2\pi\,\mathrm{sign}\,b.$$

That is, we have a 2π pulse.

2) $\lambda_1=ia_1$, $\lambda_2=ia_2$, $a_1, a_2>0$, $m_1=ib_1$, $m_2=ib_2$, $\mathrm{Im}\,b_1=\mathrm{Im}\,b_2=0$. After some simple calculations we find that

$$\sigma(x,t)=4\,\mathrm{arctg}\left(\left|\frac{a_1+a_2}{a_1-a_2}\right|\mathrm{sign}\,b_1\frac{\mathrm{ch}(^1/_2(w_1-w_2))}{\mathrm{sh}(^1/_2(w_1+w_2+2\alpha))}\right),\quad b_1b_2>0,$$

$$\sigma(x,t)=4\,\mathrm{arctg}\left(\left|\frac{a_2+a_1}{a_2-a_1}\right|\mathrm{sign}\,b_1\frac{\mathrm{sh}(^1/_2(w_1-w_2))}{\mathrm{ch}(^1/_2(w_1+w_2+2\alpha))}\right),\quad b_1b_2<0;$$

$$w_j=2a_j(x+x_{0j})-\frac{t}{2a_j},\quad x_{0j}=\frac{1}{2a_j}\ln\frac{|b_j|}{2a_j},\quad j=1,2;$$

$$\alpha=\ln\left|\frac{a_1-a_2}{a_1+a_2}\right|,$$

i.e., we have a two-soliton solution with velocities $v_1=1/4a_1^2$, $v_2=1/4a_2^2$, and center coordinates x_{01}, x_{02}. As will be shown further on, for $t\to\pm\infty$ this two-soliton state decays into two solitons with these same velocities and when changed center coordinates $x_{01}^{\pm}$, $x_{02}^{\pm}$.

We note that for $b_1b_2>0$ we have a 4π pulse, for $b_1b_2<0$ a 0π pulse.

3) $\lambda_1=c+ia$, $\lambda_2=-\lambda_1^*$, $m_1=d+ib$, $m_2=-m_1^*$. After elementary transformations we find that

$$\sigma(x,t)=-4\,\mathrm{arctg}\left(\frac{a}{|c|}\frac{\cos(2c(x+ct/4|\lambda|^2-\beta))}{\mathrm{ch}(2a(x-at/4|\lambda|^2+x_0))}\right),$$

$$\beta=\frac{1}{2c}\arg\frac{m_1}{2\lambda_1},\quad x_0=\frac{1}{2a}\ln\left|\frac{mc}{2\lambda a}\right|,$$

i.e., we have a double soliton with velocity $v=1/4|\lambda|^2$, phase β, center coordinate x_0, and amplitude $a/|c|$. For $|t|\to\infty$ the double soliton does not decay, but moves as a whole; that is, the double soliton is an elementary object, as is the soliton. For $|t|\to\infty$, as can be seen from the formula as written, the main contribution to the double soliton comes from the part periodic in t.

We also point out that the double soliton is a 0π pulse, i.e.,

$$\int_{-\infty}^{\infty} A(x,t)\,dx=\sigma(\infty,t)-\sigma(-\infty,t)=0.$$

3. N-SOLITON SOLUTIONS (ASYMPTOTIC BEHAVIOR AS $|t|\to\infty$)

Let us study the behavior of an N-soliton solution for large $|t|$. We shall confine ourselves to the case in which all the velocities v_j, v_p are different. Then the N-soliton solution decays for $|t|\to\infty$ into solitons and double solitons diverging from each other. To verify this, we arrange the v_j, v_p in decreasing order: $v_1>\ldots>v_{k_1+k_2}$, and consider the limit of $\sigma(x+v_1t,t)$, $1\le l\le k_1+k_2$, as $t\to\infty$. For definiteness we investigate the case when $\lambda_l=-\lambda_l^*$, i.e., $\lambda_l=ia_l$, $a_l>0$.

To begin with we note that

$$\det(I+V(x,t))=\det(I+W^+(x,t));$$

$$W_{jk}(x,t)=\frac{2(\lambda_j\lambda_k)^{1/2}}{\lambda_j+\lambda_k}\exp(\zeta_j^+(x,t)+\zeta_k^+(x,t)),$$

$$\zeta_j^+(x,t)=-i\lambda_jx-\frac{i}{4\lambda_j}t+\frac{1}{2}\ln\frac{m_j}{2\lambda_j}+i\frac{\pi}{4},$$

and use the well known formula (cf. [20])

$$\det(I+W)=1+\sum_{n=1}^{N}\sum_{NC_n}\theta(j_1,\ldots,j_n)\exp(2(\zeta_{j_1}^++\ldots+\zeta_{j_n}^+)),$$
$$\theta(j_1,\ldots,j_n)=\prod_{k<l}^{n}\theta(j_k,j_l),\quad \theta(j_k,j_l)=\frac{(\lambda_{j_k}-\lambda_{j_l})^2}{(\lambda_{j_k}+\lambda_{j_l})^2},\tag{20}$$

where NC_n is the set of all choices of n out of N elements. We then find that

$$\mathrm{Re}\,\zeta_j^+(x+v_lt,t)\to-\infty\ \text{as}\ t\to\infty\ (1\le j<l),$$
$$\mathrm{Re}\,\zeta_j^+(x+v_lt,t)\to\infty\ \text{as}\ t\to\infty\ (l<j\le N).$$

We now use the following factorization (cf. [21])

$$\det(I+W^+)=\exp(2(\zeta_{l+1}^++\ldots+\zeta_N^+))f_l^+,$$
$$f_l^+=A\exp(2(\zeta_1^++\ldots+\zeta_l^+))$$
$$+B\exp(2\zeta_l^+)+C+\ldots+D\exp(-2(\zeta_{l+1}^++\ldots+\zeta_N^+)).$$

We note that there are analogous formulas for det(I $-$V)x, t)), in which ζ_j^+ is everywhere replaced with $\zeta_j^-=\zeta_j^+-i\pi/2$. Accordingly we find that

$$\lim_{t\to\infty}\sigma(x+v_lt,t)$$
$$=+2i\ln\left[(-1)^{N-l}\frac{C+(im_l/2\lambda_l)B\exp(-2i\lambda_lx)}{C-(im_l/2\lambda_l)B\exp(-2i\lambda_lx)}\right],$$

and, using the fact that Eq. (20) gives the relation

$$\frac{B}{C}=\prod_{j=l+1}^{N}\theta(l,j),$$

we conclude that

$$\lim_{t\to\infty}\sigma(x+v_lt,t)=\lim_{t\to\infty}\sigma(x+v_lt,t;v_l,x_{0l}^+)\ (\mathrm{mod}\ 2\pi),$$

where $\sigma(x,t;v_l,x_{0l}^+)$ is a soliton with the parameters v_1 and x_{01}^+:

$$x_{0l}^+=x_{0l}+\Delta^+x_{0l},\quad x_{0l}=\frac{1}{2a_l}\ln\frac{|m_l|}{2a_l},$$
$$\Delta^+x_{0l}=\frac{1}{a_l}\sum_{j=l+1}^{N}\ln\frac{|\lambda_j-\lambda_l|}{|\lambda_j+\lambda_l|}.\tag{21}$$

In the case $\lambda_l^*=-\lambda_{l+1}$ we find similarly, using Eq. (20) and the corresponding factorization, that

$$\lim_{t\to\infty}\sigma(x+v_lt,t)=\lim_{t\to\infty}\sigma(x+v_lt,t;v_l,x_{0l}^+,c_l,\beta_l^+)\ (\mathrm{mod}\ 2\pi),$$

where $\sigma(x,t;v_l,x_{0l}^+,c_l,\beta_l^+)$ is a double soliton with the parameters

$$v_l=1/4|\lambda_l|^2,\quad x_{0l}^+=x_{0l}+\Delta^+x_{0l},\quad \beta_l^+=\beta_l+\Delta^+\beta_l$$

and with the amplitude $a_l/|c_l|$, where

$$\lambda_l=c_l+ia_l,\quad x_{0l}=\frac{1}{2a_l}\ln\left|\frac{m_lc_l}{2\lambda_la_l}\right|,\quad \beta_l=\frac{1}{2c_l}\arg\frac{m_l}{2\lambda_l},$$

$$\Delta^+x_{0l}=\frac{1}{a_l}\sum_{j=l+2}^{N}\ln\frac{|\lambda_j-\lambda_l|}{|\lambda_j+\lambda_l|},\tag{22}$$

$$\Delta^+\beta_l=\frac{1}{c_l}\sum_{j=l+2}^{N}\arg\frac{\lambda_j-\lambda_l}{\lambda_j+\lambda_l}.$$

Similar formulas corresponding to Eqs. (21) and (22) hold as $t\to-\infty$. To obtain these formulas all we have to do is to take the sums indicated in Eqs. (21) and (22) from 1 to $l-1$. Also if v is a number such that $v\ne v_j$, $j=1,\ldots,k_1+k_2$, then just as before we find that as $|t|\to\infty$, $\sigma(x+vt,t)$ goes to zero exponentially, and this proves that the N-soliton solution decays into solitons and double solitons for $|t|\to\infty$. For $t\to\infty$ the fastest one of the solitons and double solitons is ahead, and the slowest is behind. As $t\to-\infty$ the arrangement is reversed.

Equations (21) and (22) enable us to describe the process of scattering of solitons and double solitons.

As the time t varies from $-\infty$ to ∞ there is a change of the coordinate of the center of the l-th soliton:

$$\Delta x_{0l}=\Delta^+x_{0l}-\Delta^-x_{0l}=\frac{1}{a_l}\Big(\sum_{j=l+1}^{N}\ln\frac{|\lambda_j-\lambda_l|}{|\lambda_j+\lambda_l|}-\sum_{j=1}^{l-1}\ln\frac{|\lambda_j-\lambda_l|}{|\lambda_j+\lambda_l|}\Big)$$

and a change of the center coordinate and the phase of the p-th double soliton:

$$\Delta x_{0p}=\Delta^+x_{0p}-\Delta^-x_{0p}=\frac{1}{a_p}\Big(\sum_{j=p+2}^{N}\ln\frac{|\lambda_j-\lambda_p|}{|\lambda_j+\lambda_p|}-\sum_{j=1}^{p-1}\ln\frac{|\lambda_j-\lambda_p|}{|\lambda_j+\lambda_p|}\Big)$$

$$\Delta\beta_p=\Delta^+\beta_p-\Delta^-\beta_p=\frac{1}{c_p}\Big(\sum_{j=p+2}^{N}\arg\frac{\lambda_j-\lambda_p}{\lambda_j+\lambda_p}-\sum_{j=1}^{p-1}\arg\frac{\lambda_j-\lambda_p}{\lambda_j+\lambda_p}\Big).$$

These formulas can be interpreted by supposing that the solitons and double solitons collide with each other and among themselves in pairs. In each binary collision of a soliton with a soliton the faster is shifted ahead by the amount

$$\frac{1}{a_l}\ln\frac{|\lambda_j+\lambda_l|}{|\lambda_j-\lambda_l|},\qquad v_l>v_j,$$

and the slower is shifted back by the amount

$$\frac{1}{a_j}\ln\frac{|\lambda_j+\lambda_l|}{|\lambda_j-\lambda_l|}.$$

In a collision between double solitons the faster is shifted ahead by the amount

$$\frac{1}{a_l}\ln\frac{|\lambda_j+\lambda_l|\,|\lambda_{j+1}+\lambda_l|}{|\lambda_j-\lambda_l|\,|\lambda_{j+1}-\lambda_l|},\qquad v_l>v_j,$$

and the slower is shifted back by the amount

$$\frac{1}{a_j}\ln\frac{|\lambda_j+\lambda_l|\,|\lambda_{j+1}+\lambda_l|}{|\lambda_j-\lambda_l|\,|\lambda_{j+1}-\lambda_l|}.$$

Finally, in a collision between a soliton and a double soliton the faster—for definiteness suppose it is the soliton—is shifted ahead by the amount

$$\frac{1}{a_l}\ln\frac{|\lambda_j+\lambda_l|\,|\lambda_{j+1}+\lambda_l|}{|\lambda_j-\lambda_l|\,|\lambda_{j+1}-\lambda_l|}$$

and the slower—the double soliton—back by the amount

$$\frac{1}{a_j}\ln\frac{|\lambda_j+\lambda_l|}{|\lambda_j-\lambda_l|},\qquad v_l>v_j.$$

We note that if we take the "mass" of the l-th soliton to be a_l, and the "mass" of a double soliton to be $2a_l$, then a law of conservation of "momentum" holds in binary collisions. We also point out that the total displacement of a soliton or double soliton is equal to the algebraic sum of its shifts in the binary collisions, so that there is absolutely no effect of many-particle collisions. There is a similar situation with the phases of double solitons.

We note that the existence of binary collisions only for the KdV equation and for Eq. (3) was established in papers by Zakharov and Shabat.[13,14,18]

4. BOUND STATES AND MULTIPLE ZEROES. CONSERVATION LAWS

The velocity of separation of a pair of solitons (or a pair of double solitons, or a soliton-double soliton pair) is proportional to the difference of their parameters v_j. For identical velocities the objects do not separate for large $|t|$, but form a bound state. Let us consider the bound state of a soliton and k_2 double solitons, $1+2k_2=N$, where $\lambda_j\neq\lambda_k$ for $j\neq k$; $j,k=1,\ldots,N$ and $v_j=v$, $j=1,\ldots,k_2+1$. Then it can be seen immediately from the general formula (19) that the main term in the bound state will come from the conditionally periodic part of the solution, which is characterized by k_2 frequencies $\omega_j=\operatorname{Re}2\lambda_j v$, where $\lambda_j=\lambda^*_{j+1}$, $j=2,\ldots,N$.

We have so far been considering the case in which all the zeroes of the function a(k) are simple. We can deal similarly with the case of multiple zeroes if we change the term in the kernel of the Marchenko equation which is obtained by the theorem of residues, since the residues of the function $1/a(k)$ will be of different form. In all other respects the scheme described above is still valid.

Equation (1) has an infinite set of conservation laws; this follows simply from the fact that the function a(k) is independent of the time. A characteristic feature of equations solvable by the method of the inverse problem is the presence of so-called polynomial conservation laws, i.e., functionals of the form

$$I_n(\sigma)=\int_{-\infty}^{\infty}P_n(\sigma,\sigma_x,\ldots)\,dx,\qquad n=1,2,\ldots,$$

where P_n is a polynomial in its arguments (cf.[13,14,22]). These conservation laws are also due to the fact that a(k) is constant in time. There exists a regular method for expressing such conservation laws, based on trace identities for the operator L (cf. [13-15]). We give a recurrence formula for the polynomials P_n (cf. [13]):

$$P_{n+1}=\sigma_x\Big(\frac{1}{\sigma_x}P_n\Big)_x+\sum_{\substack{j+k=n\\ j,k>0}}P_jP_k,\qquad n>1,$$

$$P_1={}^1/_4\sigma_x^2.$$

We note that one can obtain a different series of conservation laws by starting from the expansion of a(k) in a Taylor's series in the neighborhood of zero:

$$a(k)=\sum_{n=0}^{\infty}a_nk^n.$$

Because of lack of space we give only the expressions for the first two coefficients:

$$a_0=\pm1,\qquad a_1=\frac{a_0}{i}\int_{-\infty}^{\infty}(1-\cos\sigma)\,dx.$$

APPENDIX

1. Let us consider the solution of the homogeneous system $+a_1(x)+iV(x)a_2(x)=0$, $iV(x)a_1(x)-a_2(x)=0$. Introducing $\alpha_{1x}(y)=(a_1(x),\psi(y))$ and $\alpha_{2x}(y)=(a_2(x),\psi(y))$ and using the definition of the matrix V(x), we find that

$$+\alpha_{1x}(y)+i\int_{-\infty}^{x}\alpha_{2x}(u)F(u+y)\,du=0,$$

$$i\int_{-\infty}^{x}\alpha_{1x}^*(u)F(u+y)\,du-\alpha_{2x}^*(y)=0.$$

We multiply the first equation by $\alpha^*_{1x}(y)$ and the second by $\alpha_{2x}(y)$, and integrate over y from $-\infty$ to x. Subtracting the first equation from the second and using the formula for $F(u+y)$, we get

$$\int_{-\infty}^{x}(|\alpha_{1x}(y)|^2+|\alpha_{2x}(y)|^2)\,dy=0,$$

i.e., $\alpha_{1x}(y)=\alpha_{2x}(y)=0$ for $y\leq x$, from which we conclude that $a_1(x)=a_2(x)=0$.

2. Let $\lambda_j=-\lambda_j^*$ and $m_j=-m_j^*$ for $1\leq j\leq K$, and $\lambda_j=\lambda^*_{j+1}$ and $m_j=-m^*_{j+1}$ for $K<j\leq N$. This can always be achieved by renumbering the λ_j and m_j, $j=1,\ldots,N$.

It is now to be noted that the matrix $I+V^*(x)$ is obtained from the matrix $I-V(x)$ by successive binary interchanges of column number $K+1$ with the columns numbered $K+2, \ldots, N-1, N$, and of row number $K+1$ with rows $K+2, \ldots, N-1, N$; that is, after $N-K$ (sic) interchanges we obtain from $I-V(x)$ the matrix $I+V^*(x)$. Owing to a well-known property of determinants we find that

$$\det(I+V^*(x))=\det(I+V(x))^*=\det(I-V(x)),$$

since $N-K$ (sic) is even.

3. We note that

$$\det(\lambda I+V(x))=\lambda^N+a_1(x)\lambda^{N-1}+\ldots+a_N(x),$$

where $a_k(x)$ is the sum of the k-th order minors of the matrix $V(x)$ in which the rows and columns have the same set of indices. Since each such minor is of the form of the matrix $V(x)$, on setting $\lambda=1$ in the equation just written, we find that

$$\det(I+V(x))=1+\sum_{n=1}^{N}\sum_{NC_n}a(j_1,\ldots,i_n)\exp(-2i(\lambda_{j_1}+\ldots+\lambda_{j_n})x). \quad \text{(A.1)}$$

Denoting $\det(I+V(x))$ by $\Delta(x)$, we get

$$A(x)=+2i\frac{d}{dx}\ln\frac{\Delta(x)}{\Delta^*(x)}=+2i\frac{\Delta'(x)\Delta^*(x)-\Delta(x)\Delta^{*\prime}(x)}{|\Delta(x)|^2}.$$

We note that for $x\to\infty$ the largest contributions to numerator and denominator come from their terms containing

$$\exp\left(-4i\sum_{j=1}^{N}\lambda_j x\right).$$

However, using Eq. (A.1), we can verify that in the numerator this term comes in with the coefficient

$$|a(1,\ldots,N)|^2\sum_{j=1}^{N}2(-\lambda_j^*)+|a(1,\ldots,N)|^2\sum_{j=1}^{N}2(-\lambda_j)=0,$$

whereas in the denominator its coefficient is $|\det V|^2$, where

$$V_{jk}=\frac{m_j}{\lambda_j+\lambda_k},\quad j,k=1,\ldots,N.$$

The fact that $\det V\neq 0$ is well known (cf. e.g., [23]). Now from the fact that

$$-4i\sum_{j=1}^{N}\lambda_j>0,$$

we conclude that $A(x)$ decreases exponentially for $x\to\infty$.

[1)] The formulas (17) are also contained in a paper which has recently appeared: M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, Phys. Rev. Letters 30, 1262 (1973). (Note added by authors, September 21, 1973.)

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